

CONTEMPORARY MATHEMATICS

764

Polytopes and Discrete Geometry

AMS Special Session
Polytopes and Discrete Geometry
April 21–22, 2018
Northeastern University, Boston, MA

Gabriel Cunningham
Mark Mixer
Egon Schulte
Editors

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Preface

This volume focuses on developments in the fields of discrete and convex geometry. It contains the proceedings of the Special Session on Polytopes and Discrete Geometry at the American Mathematical Society meeting held from April 21-22, 2018, at Northeastern University, Boston, Massachusetts. While this volume is aimed at researchers in discrete and convex geometry and researchers who work with abstract polytopes or C-groups, the editors believe that junior mathematicians, including graduate students and post-doctoral fellows could benefit greatly from a glimpse into these research areas. This volume offers access to various current topics and research problems in these fields. Specifically,

- The paper *The cd-index: a survey* by Margaret M. Bayer is a survey of the cd-index of Eulerian partially ordered sets; it discusses inequalities on the cd-index, connections with other combinatorial parameters, computation, and algebraic approaches.
- The paper *d-dimensional self-dual polytopes and Meissner polytopes* by Tibor Bisztriczky and Déborah Oliveros presents a construction of a class of convex self-dual d -polytopes for $d > 2$, and examines conditions under which they are involutory self-dual, and have metric embeddings. Such a polytope generates a Reuleaux polytope and the paper explores the relation between this Reuleaux polytope and Meissner polytopes (modified ball-polytopes of constant width).
- The paper *On the ranks of string C-group representations for symplectic and orthogonal groups* by Peter A. Brooksbank determines the ranks of abstract regular polytopes whose automorphism group is $PSp(4, F_q) \cong \Omega(5, F_q)$, and comments on the regular ranks of higher-dimensional symplectic and orthogonal groups.
- The paper *Perfect colorings of regular graphs* by Joseph Ray Clarence Damasco and Dirk Frettlöh characterizes the color adjacency matrices of perfect colorings of graphs, and in particular, connected graphs. Then, it determines the lists of all color adjacency matrices corresponding to perfect colorings of 3-regular, 4-regular and 5-regular graphs with two, three and four colors, and all perfect colorings of the edge graphs of the Platonic solids with two, three and four colors, respectively.
- The paper *Tverberg theorems over discrete sets of points* by J. A. De Loera, T. A. Hogan, F. Meunier, and N. H. Mustafa discusses Tverberg-type theorems with coordinate constraints and determines the m -Tverberg number, when $m \geq 3$, of any discrete subset \mathbb{R}^2 . It also presents improvements on the upper bounds for the Tverberg numbers of \mathbb{Z}^3 and $\mathbb{Z}^j \times \mathbb{R}^k$

and an integer version of the well-known positive-fraction selection lemma of J. Pach.

- The paper *The vertices of primitive zonotopes* by Antoine Deza, Lionel Pournin, and Rado Rakotonarivo provides geometric and combinatorial properties for primitive zonotopes, and shows that the logarithm of the complexity of convex matroid optimization is quadratic. It also gives a sharp asymptotic estimate for the number of vertices of a primitive zonotope that can be seen as an intermediate between the permutahedra of types A and B.
- The paper *Barycenters of points in polytope skeleta* by Michael Gene Dobbin and Florian Frick classifies n -tuples of dimensions (k_1, \dots, k_n) that sum to nk such that, for a given point p in an nk -polytope, there are n points from faces of these prescribed dimensions whose barycenter is p . It also investigates the weighted analogue of this question.
- The paper *Two families of locally toroidal regular 4-hypertopes arising from toroids* by Maria Elisa Fernandes, Dimitri Leemans, Claudio Alexandre Piedade, and Asia Ivić Weiss, presents two infinite families of locally toroidal hypertopes of rank 4 that are constructed from regular toroids of types $\{4, 3, 4\}_{(s,s,0)}$ and $\{3, 3, 4, 3\}_{(s,0,0,0)}$. The Coxeter diagram of the first of the two families is star-shaped and the diagram of the other is a square. In both cases the toroidal residues are regular toroidal maps of type $\{3, 6\}$.
- The paper *Self-polar polytopes* by Alatheia Jensen investigates the existence, construction, facial structure, and practical applications of self-polar polytopes, as well as the place of these polytopes within the broader class of self-dual polytopes.
- The paper *Isomorphisms of maps on the sphere* by Ken-ichi Kawarabayashi, Pavel Klavík, Bojan Mohar, Roman Nedela, and Peter Zeman describes a modified linear-time algorithm solving the isomorphism problem for spherical maps. The algorithm described can also be used to determine (in linear time) the group of orientation-preserving symmetries of a spherical map.
- The paper *Some enumeration relating to intervals in posets* by Jim Lawrence considers an iterative construction using the intervals of a poset, and shows that the functions giving the number of elements of a given rank in the k th iteration are polynomials in 2^k .
- The paper *String C-group representations of almost simple groups: A survey* by Dimitri Leemans aims at giving the state of the art in the study of string C-group representations of almost simple groups. It also suggests a series of problems and conjectures to the interested reader.
- The paper *Orientation-reversing symmetry of closed surfaces immersed in euclidean 3-space* by Undine Leopold and Thomas W. Tucker considers finite groups of isometries G of \mathbb{E}^3 and closed surfaces S such that a G -general position immersion of S realizes a restricted Riemann-Hurwitz equation for the orientation-preserving subgroup G^+ of G , extending results of the authors for the orientation-preserving case $G = G^+$.

- The paper *Realizations of the 120-cell* by Peter McMullen provides the realization spaces of the 120-cell $\{5, 3, 3\}$. This completes such a classification for all the classical regular polytopes.
- The paper *Prescribing symmetries and automorphisms for polytopes* by Egon Schulte, Pablo Soberón, and Gordon Ian Williams studies the groups for which it is possible to find a convex polytope with that group as automorphism group with additional geometric conditions on the action of the group or its subgroups. In particular, it proves that for every abelian group G of even order and an involution s of G , there is a centrally symmetric convex polytope whose automorphism group is G and such that s corresponds to the central symmetry.
- The paper *The versatile rhombic triacontahedron and crystallography* by Marjorie Senechal and Jean E. Taylor shows that subsets of the rhombic triacontahedron tile \mathbb{R}^3 and correspond to the combinatorial types of lattice Voronoi cells. By relaxing the hypothesis of convexity in the classification of parallelohedra, it provides a uniform description of periodic approximants to a large class of quasicrystals.
- The paper *Tilings with congruent edge coronae* by Mark D. Tomenes and Ma. Louise Antonette N. De Las Peñas discusses properties of a normal tiling of the Euclidean plane with congruent edge coronae, and proves that the congruence of the first edge coronae is enough to say that the tiling is isotoxal.

During our session, we also had the pleasure of having five high-school students give a talk. Their paper *The geometry of H_4 polytopes* by Tomme Denney, Da'Shay Hooker, De'Janeke Johnson, Tianna Robinson, Majid Butler and Sandernish Claiborne has been accepted for publication in another journal and will appear soon.

Please enjoy this volume.

Gabe Cunningham
Mark Mixer
Egon Schulte

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This volume contains the proceedings of the AMS Special Session on Polytopes and Discrete Geometry, held from April 21–22, 2018, at Northeastern University, Boston, Massachusetts.

The papers showcase the breadth of discrete geometry through many new methods and results in a variety of topics. Also included are survey articles on some important areas of active research. This volume is aimed at researchers in discrete and convex geometry and researchers who work with abstract polytopes or string C -groups. It is also aimed at early career mathematicians, including graduate students and postdoctoral fellows, to give them a glimpse of the variety and beauty of these research areas.

Topics covered in this volume include: the combinatorics, geometry, and symmetries of convex polytopes; tilings; discrete point sets; the combinatorics of Eulerian posets and interval posets; symmetries of surfaces and maps on surfaces; self-dual polytopes; string C -groups; hypertopes; and graph coloring.



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