

Dimension of finite free complexes over commutative Noetherian rings

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To Roger and Sylvia Wiegand on the occasion of their aggregate 150th birthday.

ABSTRACT. Foxby defined the (Krull) dimension of a complex of modules over a commutative Noetherian ring in terms of the dimension of its homology modules. In this note it is proved that the dimension of a bounded complex of free modules of finite rank can be computed directly from the matrices representing the differentials of the complex.

Introduction

This short note concerns certain homological invariants—specifically, dimension and depth—of complexes of modules over commutative Noetherian local rings. The concepts of depth and dimension for modules, introduced by Krull and by Auslander and Buchsbaum, respectively, need no recollection. Both concepts were extended to complexes of modules by Foxby [5], and also by Iversen [9]. Their extensions agree up to a normalization; in what follows we work with Foxby’s definitions, recalled further below, for they are better suited to computations in the derived category. The depth and dimension of a complex depend only on the quasi-isomorphism class of the complex; said differently, they are defined on the derived category of the ring.

To compute these invariants one can usually reduce to the case where the complex is finite free, for they are independent of the domain. Indeed, if $Q \rightarrow R$ is a surjective map of rings with Q a regular local ring, then the depth and dimension of an R -complex M coincide with the corresponding invariants of M viewed as a complex over Q . And, at least when M is homologically finite, it is quasi-isomorphic, over Q , to a finite free complex. Thus, in what follows we consider a complex over a local ring R of the form:

$$F := 0 \longrightarrow F_b \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_a \longrightarrow 0$$

where each F_i is a free R -module of finite rank. We assume that F is minimal, in that $\partial(F) \subseteq \mathfrak{m}F$, where \mathfrak{m} is the maximal ideal of R .

For such a complex F , the depth can be read off easily: The equality of Auslander and Buchsbaum for modules of finite projective dimension applies equally

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to complexes—this was proved by Foxby [4]—and yields that the depth of F equals depth $R - b$, provided that $F_b \neq 0$. In this note we establish a formula that expresses the dimension of F in terms of the ranks of the modules F_i and the Fitting ideals of the differentials; see Theorem 1. We were lead to it in an attempt to relate the codimension, in the sense of Bruns and Herzog [3], to other homological invariants. It turns out that the codimension of F equals $\dim R - \dim_R \operatorname{Hom}_R(F, R)$; see Remark 3. This observation gives a different perspective on, and different proofs of, certain results in [3] related to the homological conjectures; see Proposition 6 and Theorem 8.

Dimension

Let R be a ring. By an R -complex we mean a complex of R -modules, with lower grading:

$$X := \cdots \longrightarrow X_n \xrightarrow{\partial_n} X_{n-1} \longrightarrow \cdots$$

A graded R -module, such as the homology $H(X)$ of X , is viewed as an R -complex with zero differentials; in particular, we use the same grading convention for such objects.

Dimension. Let R be a commutative Noetherian ring. In [5, Section 3] Foxby introduced the *dimension* of an R -complex X to be

$$(1) \quad \dim_R X := \sup\{\dim(R/\mathfrak{p}) - \inf H(X)_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

By [5, Proposition 3.5] this invariant can be computed in terms of the homology:

$$(2) \quad \dim_R X = \sup\{\dim_R H_n(X) - n \mid n \in \mathbb{Z}\}.$$

The convention is that the dimension of the zero module is $-\infty$.

Finite free complexes. By a *finite free* R -complex we mean a bounded R -complex

$$(3) \quad F := 0 \longrightarrow F_b \xrightarrow{\partial_b} F_{b-1} \longrightarrow \cdots \longrightarrow F_{a+1} \xrightarrow{\partial_{a+1}} F_a \longrightarrow 0$$

where each F_i is a free R -module of finite rank. For such a complex F we set

$$(4) \quad s_n = \sum_{i \leq n} (-1)^{n-i} \operatorname{rank}_R F_i \quad \text{for each } n \in \mathbb{Z}.$$

Given a map φ between finite free modules we write $I_s(\varphi)$ for the ideal generated by the $s \times s$ minors of a matrix representing φ ; see, for example, [3, p. 21]. It is convenient to adopt the convention that the determinant and all minors of the empty matrix is 1; in particular, for $s \leq 0$ an $s \times s$ minor of any matrix is 1. For $s \geq 1$ an $s \times s$ minor of a non-empty matrix is 0 if s exceeds the number of rows or columns. For the differentials of the complex (3) this means that $I_{s_n}(\partial_{n+1}) = R$ holds for integers n outside $[a, b]$.

THEOREM 1. *With F and s_n as above, there is an equality*

$$\dim_R F = \sup\{\dim(R/I_{s_n}(\partial_{n+1})) - n \mid n \in \mathbb{Z}\}.$$

PROOF. To prove the inequality “ \geq ” we verify that

$$\dim(R/I_{s_n}(\partial_{n+1})) \leq \dim_R F + n$$

holds for every integer n in $[a, b]$. Fix such an n . The inequality above holds if and only if one has $I_{s_n}(\partial_{n+1})_{\mathfrak{p}} = R_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Spec } R$ with $\dim(R/\mathfrak{p}) > \dim_R F + n$. For such a prime ideal and any integer $i \leq n$ one gets

$$\dim_R H_i(F) \leq \dim_R F + i < \dim(R/\mathfrak{p})$$

where the first inequality holds by (2). This yields

$$H_i(F)_{\mathfrak{p}} = 0 \quad \text{for all } i \leq n,$$

which implies that the homology of the complex

$$(5) \quad (F_{n+1})_{\mathfrak{p}} \xrightarrow{(\partial_{n+1})_{\mathfrak{p}}} (F_n)_{\mathfrak{p}} \longrightarrow \cdots \longrightarrow (F_{a+1})_{\mathfrak{p}} \xrightarrow{(\partial_{a+1})_{\mathfrak{p}}} (F_a)_{\mathfrak{p}} \longrightarrow 0$$

is zero in degrees $\leq n$. It follows that the image of $(\partial_{n+1})_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module of rank s_n . Hence one has $I_{s_n}(\partial_{n+1})_{\mathfrak{p}} = R_{\mathfrak{p}}$.

To prove the opposite inequality, “ \leq ”, we show that

$$\dim_R H_n(F) - n \leq \sup\{\dim(R/I_{s_i}(\partial_{i+1})) - i \mid a \leq i \leq b\}$$

holds for each integer n in $[a, b]$. Let t be the supremum above. One needs to verify that $H_n(F)_{\mathfrak{p}} = 0$ holds for primes \mathfrak{p} with $\dim(R/\mathfrak{p}) > t + n$. Fix such a \mathfrak{p} ; for every $i \leq n$ one has

$$\dim(R/I_{s_i}(\partial_{i+1})) \leq t + i \leq t + n < \dim(R/\mathfrak{p})$$

so that $I_{s_i}(\partial_{i+1})_{\mathfrak{p}} = R_{\mathfrak{p}}$. We now argue by induction on i that the homology of the complex (5) is zero in degrees $\leq n$; in particular, one has $H_n(F)_{\mathfrak{p}} = 0$, as desired. With $f_i = \text{rank}_R F_i$ and $K_i = \ker \partial_i$ the argument goes as follows: In the base case $i = a$ one applies [3, Lemma 1.4.9] to the presentation of the image of ∂_{a+1} afforded by (5), and one concludes that it is a free submodule of F_a of rank f_a , i.e. the whole thing. One also notices that a free module contained in K_{a+1} has rank at most s_{a+1} . In the induction step one applies *op.cit* to the presentation of the image of ∂_{i+1} and concludes that it is a free module of rank $f_{i+1} - s_{i+1} = s_i$. By the induction hypothesis a free module contained in K_i has rank at most s_i , so the complex is exact at $(F_i)_{\mathfrak{p}}$. \square

Codimension. Let R be a commutative Noetherian ring and F a finite free R -complex as in (3). For each integer n set

$$r_n := \sum_{i \geq n} (-1)^{i-n} \text{rank}_R(F_i).$$

For n in $[a + 1, b]$ this is the *expected rank* of the map ∂_n ; see [3, p. 24].

COROLLARY 2. *With F and r_n as above there is an equality*

$$\dim_R \text{Hom}_R(F, R) = \sup\{\dim(R/I_{r_n}(\partial_n^F)) + n \mid n \in \mathbb{Z}\}.$$

PROOF. Set $G := \text{Hom}_R(F, R)$. This too is a finite free complex, concentrated in degrees $[-b, -a]$, with differentials $\partial_n^G = \text{Hom}_R(\partial_{1-n}^F, R)$ for each n . It is now easy to check that the expected ranks r_n of F and the invariants s_n of G , from (4), determine each other:

$$s_n(G) = r_{-n}(F) \quad \text{for each } n.$$

Whence one gets equalities

$$\begin{aligned} \dim_R G &= \sup\{\dim(R/I_{s_n}(\partial_{n+1}^G)) - n \mid n \in \mathbb{Z}\} \\ &= \sup\{\dim(R/I_{r_n}(\partial_{-n}^F) - n \mid n \in \mathbb{Z}\} \\ &= \sup\{\dim(R/I_{r_n}(\partial_n^F)) + n \mid n \in \mathbb{Z}\}. \end{aligned}$$

□

REMARK 3. Bruns and Herzog [3, Section 9.1] have introduced a notion of “codimension” for finite free complexes. This is perhaps a misnomer: Applied to the minimal free resolution of a module, the codimension does not equal the usual codimension of the module. In fact, Corollary 2 yields that the codimension, in their sense, of any finite free R -complex F , is precisely $\dim R - \dim_R \operatorname{Hom}_R(F, R)$.

Foxby also has a notion of codimension for an R -complex X , namely the invariant

$$\begin{aligned} \operatorname{codim}_R X &:= \inf\{\dim R_{\mathfrak{p}} + \inf H(X)_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \inf\{\operatorname{codim}_R H_n(X) + n \mid n \in \mathbb{Z}\}; \end{aligned}$$

see [5, Lemma 5.1] and the definition preceding it. From the definitions one immediately gets $\operatorname{codim}_R X + \dim_R X \leq \dim R$; equality holds if R is local, catenary, and equidimensional. For a finite free complex F over such a ring one thus has

$$\operatorname{codim}_R \operatorname{Hom}_R(F, R) = \dim R - \dim_R \operatorname{Hom}_R(F, R).$$

In particular, the codimension of F in the sense of [3] is the codimension of the dual complex, $\operatorname{Hom}_R(F, R)$, in the sense of [5].

In Bruns and Herzog’s [3] treatment of the homological conjectures—most of which are now theorems thanks to André [2]—their notion of codimension of a finite free complex is key. Per Remark 3 this suggests that estimates on the dimension of $\operatorname{Hom}_R(F, R)$ are useful, and that motivates the development below.

Support. Let R be a commutative Noetherian ring. The *large support* of an R -complex X is the support of the graded module $H(X)$, i.e.

$$\operatorname{Supp}_R X := \{\mathfrak{p} \in \operatorname{Spec} R \mid H_n(X)_{\mathfrak{p}} \neq 0 \text{ for some } n\}.$$

Foxby [5, Section 2] also introduced the (small) *support* of X to be the set

$$\operatorname{supp}_R X := \{\mathfrak{p} \in \operatorname{Spec} R \mid H(\kappa(\mathfrak{p}) \otimes_R^L X) \neq 0\};$$

as usual, $\kappa(\mathfrak{p})$ denotes the residue field of the local ring $R_{\mathfrak{p}}$. Support is connected to the finiteness of the depth of X :

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} < \infty\}.$$

We recall that the *depth* of a complex X over local ring R with residue field k is

$$\operatorname{depth}_R X := \inf\{n \in \mathbb{Z} \mid \operatorname{Ext}_R^n(k, X) \neq 0\}.$$

This invariant can also be computed in terms of the Koszul homology, and the local cohomology, of X ; see [6].

PROPOSITION 4. *Let R be a commutative noetherian ring. For every finite free R -complex F one has*

$$\dim_R \operatorname{Hom}_R(F, R) \leq \dim R + \sup H(F).$$

PROOF. For every prime ideal \mathfrak{p} the complex $F_{\mathfrak{p}}$ has finite projective dimension, so the Auslander–Buchsbaum Formula combines with standard (in)equalities between invariants to yield

$$\begin{aligned} -\inf H(\mathrm{Hom}_R(F, R))_{\mathfrak{p}} &= \sup\{m \in \mathbb{Z} \mid \mathrm{Ext}_{R_{\mathfrak{p}}}^m(F_{\mathfrak{p}}, R_{\mathfrak{p}}) \neq 0\} \\ &= \mathrm{proj. dim}_{R_{\mathfrak{p}}} F_{\mathfrak{p}} \\ &= \mathrm{depth} R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} F_{\mathfrak{p}} \\ &\leq \dim R_{\mathfrak{p}} + \sup H(F)_{\mathfrak{p}}. \end{aligned}$$

From the definition (1) one now gets

$$\begin{aligned} \dim_R \mathrm{Hom}_R(F, R) &\leq \sup\{\dim(R/\mathfrak{p}) + \dim R_{\mathfrak{p}} + \sup H(F)_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Spec} R\} \\ &\leq \dim R + \sup H(F). \end{aligned}$$

□

REMARK 5. For the minimal free resolution of a finitely generated module of finite projective dimension, the codimension considered in [3] is non-negative by the Buchsbaum–Eisenbud acyclicity criterion; see the comment before [3, Lemma 9.1.8]. This compares to the inequality in Proposition 4, rewritten as

$$\dim R - \dim_R \mathrm{Hom}_R(F, R) \geq -\sup H(F).$$

Balanced big Cohen–Macaulay modules. Let (R, \mathfrak{m}) be local and M a big Cohen–Macaulay module; that is, a module with $\mathrm{depth}_R M = \dim R$ and $\mathfrak{m}M \neq M$. Hochster [7, 8] proved that such a module exists for every equicharacteristic local ring, and André [1] proved their existence over local rings of mixed characteristic. A big Cohen–Macaulay R -module M is called *balanced* if every system of parameters for R is an M -regular sequence. The \mathfrak{m} -adic completion of any big Cohen–Macaulay module is balanced; see [3, Theorem 8.5.3]. Sharp [10] demonstrated that these modules behave much like maximal Cohen–Macaulay modules. Of interest here is the fact that for a balanced big Cohen–Macaulay module M one has

$$(6) \quad \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \dim R - \dim(R/\mathfrak{p}) \quad \text{for each } \mathfrak{p} \in \mathrm{supp}_R M;$$

this is part (iii) in [10, Theorem 3.2]. Note that what Sharp calls the supersupport of M is the support of M , in the sense above; this follows from comparison of [5, Remark 2.9] and part (v) in *op.cit.*

PROPOSITION 6. *Let R be a local ring, F a finite free R -complex, and M a balanced big Cohen–Macaulay module. One has*

$$\sup H(F \otimes_R M) = \dim_R \mathrm{Hom}_R(F, R) - \dim R.$$

PROOF. Set $G := \mathrm{Hom}_R(F, R)$. There is an isomorphism $F \otimes M \cong \mathrm{Hom}_R(G, M)$. In the computation below, the first equality holds by [5, Proposition 3.4]. The second equality follows from (6) and the fourth one follows from (1).

$$\begin{aligned} \sup H(\mathrm{Hom}_R(G, M)) &= -\inf\{\mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \inf H(G)_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Spec} R\} \\ &= -\inf\{\dim R - \dim(R/\mathfrak{p}) + \inf H(G)_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Spec} R\} \\ &= \sup\{\dim(R/\mathfrak{p}) - \inf H(G)_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Spec} R\} - \dim R \\ &= \dim_R G - \dim R. \end{aligned}$$

□

Given Remark 3, the theorem above recovers [3, Lemma 9.1.8]:

COROLLARY 7. *Let R be a local ring and $F := 0 \rightarrow F_b \rightarrow \cdots \rightarrow F_0 \rightarrow 0$ a finite free R -complex. If $\dim R - \dim_R \operatorname{Hom}_R(F, R) \geq 0$ holds, then for any balanced big Cohen–Macaulay module M one has $H_i(F \otimes_R M) = 0$ for all $i \geq 1$.*

In [3, Section 9.4] it is shown how to derive various intersections theorems, including the New Intersection Theorem, from Corollary 7. The latter sheds further light on the invariant $\dim R - \dim_R \operatorname{Hom}_R(F, R)$.

THEOREM 8. *Let R be a local ring and F a finite free R -complex. One has*

$$\dim R + \inf H(F) \leq \dim_R \operatorname{Hom}_R(F, R) \leq \dim R + \sup H(F).$$

PROOF. The right-hand inequality holds by Proposition 4. Since R is local, one can apply the version of the New Intersection Theorem recorded by Foxby [5, Lemma 4.1] to the complex $\operatorname{Hom}_R(F, R)$ to get

$$\dim R - \dim_R \operatorname{Hom}_R(F, R) \leq \operatorname{proj. dim}_R \operatorname{Hom}_R(F, R).$$

As $\operatorname{Hom}_R(F, R)$ is also a finite free complex one has

$$\operatorname{proj. dim}_R \operatorname{Hom}_R(F, R) = -\inf H(\operatorname{Hom}_R(\operatorname{Hom}_R(F, R), R)) = -\inf H(F).$$

□

REMARK 9. Let R be a local ring and M a nonzero finitely generated R -module of finite projective dimension. Applying Theorem 8 to a finite free resolution of M yields the equality

$$\max\{\dim_R \operatorname{Ext}_R^n(M, R) + n \mid n \in \mathbb{Z}\} = \dim R.$$

Notice that with $p := \operatorname{proj. dim}_R M$ one gets inequalities

$$\dim R - \dim_R M \leq \operatorname{proj. dim}_R M \leq \dim R - \dim_R \operatorname{Ext}_R^p(M, R);$$

the inequality on the left is the version of the New Intersection Theorem that went into the proof of Theorem 8.

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