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Examples of multiplicities and mixed multiplicities of filtrations

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This paper is dedicated to Roger and Sylvia Wiegand on the occasion of their combined 150th birthday.

Abstract. In this paper we construct examples of irrational behavior of multiplicities and mixed multiplicities of divisorial filtrations. The construction makes essential use of anti-positive intersection products.

1. Introduction

In this paper, we begin by giving an overview of the theory of multiplicities and mixed multiplicities of (not necessarily Noetherian) filtrations, including an interpretation of multiplicities and mixed multiplicities of divisorial filtrations as anti-positive intersection multiplicities. Using this interpretation, we construct a resolution of singularities of a normal three dimensional local ring and compute the multiplicities and mixed multiplicities of its divisorial filtrations, showing essentially irrational behavior.

1.1. Overview of multiplicities and mixed multiplicities of filtrations.

The study of mixed multiplicities of m_R-primary ideals in a Noetherian local ring R with maximal ideal m_R was initiated by Bhattacharya [**1**], Rees [**30**] and Teissier and Risler [**36**]. In [**12**] the notion of mixed multiplicities is extended to arbitrary, not necessarily Noetherian, filtrations of R by m_R-primary ideals. It is shown in [**12**] that many basic theorems for mixed multiplicities of m_R-primary ideals are true for filtrations.

A report on the development of the subject of mixed multiplicities can be found in [**17**]. A survey of the theory of mixed multiplicities of ideals can be found in [**35**, Chapter 17].

Let R be a Noetherian local ring of dimension d with maximal ideal m_R. Let $\ell_R(M)$ denote the length of an R-module M.

A filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of a ring R is a descending chain

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots$$

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of ideals such that $I_iI_j \subset I_{i+j}$ for all $i, j \in \mathbb{N}$. A filtration $\mathcal{I} = \{I_n\}$ of a local ring R by m_R-primary ideals is a filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of R such that I_n is m_R-primary for $n \ge 1$. A filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of a ring R is said to be Noetherian if $\bigoplus_{n\ge 0} I_n$ is a finitely generated R-algebra.

The nilradical $N(R)$ of R is

$$N(R) = \{x \in R \mid x^n = 0 \text{ for some positive integer } n\}.$$

We have that $\dim N(R) = d$ if and only if there exists a minimal prime P of R such that $\dim R/P = d$ and R_P is not reduced. Let $\hat{R}$ be the m_R-adic completion of R.

In [**6**, Theorem 1.1] and [**7**, Theorem 4.2] we have shown that the limit

$$\lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d} \tag{1.1}$$

exists for any filtration $\mathcal{I} = \{I_n\}$ of R by m_R-primary ideals if and only if dim $N(\hat{R}) < d$. We observe that the condition $\dim N(\hat{R}) < d$ holds if R is analytically unramified; that is, $\hat{R}$ is reduced.

The problem of existence of such limits (1.1) has been considered by Ein, Lazarsfeld and Smith [**15**] and Mustaţă [**28**]. When the ring R is a domain and is essentially of finite type over an algebraically closed field k with $R/m_R = k$, Lazarsfeld and Mustaţă [**25**] showed that the limit exists for all filtrations of R by m_R-primary ideals. Cutkosky proved it in the complete generality stated above in [**6**] and [**7**]. These proofs use the theory of volumes of cones of Okounkov [**29**], Kaveh and Khovanskii [**22**] and Lazarsfeld and Mustaţă [**25**].

We now impose the necessary condition that the dimension of the nilradical $N(\hat{R})$ of the completion $\hat{R}$ of R is less than the dimension of R, to insure the existence of limits. We define the multiplicity of R with respect to a filtration $\mathcal{I} = \{I_n\}$ of mR-primary ideal to be

$$e_R(\mathcal{I}; R) = \lim_{n\to\infty} \frac{\ell_R(R/I_n)}{n^d/d!}.$$

In the case that $\mathcal{I} = \{I^n\}_{n\in\mathbb{N}}$ is the filtration of powers of a fixed m_R-primary ideal I, the filtration $\mathcal{I}$ is Noetherian, and we have that

$$e_R(\mathcal{I}; R) = e_R(I; R)$$

is the ordinary multiplicity of R with respect to the ideal R (here we use the notation $e_R(I; R)$ of [**35**]).

Mixed multiplicities of filtrations are defined in [**12**]. Let M be a finitely generated R-module where R is a d-dimensional Noetherian local ring with $\dim N(\hat{R}) < d$. Let $\mathcal{I}(1) = \{I(1)_n\}, \ldots, \mathcal{I}(r) = \{I(r)_n\}$ be filtrations of R by m_R-primary ideals. In [**12**, Theorem 6.1] and [**12**, Theorem 6.6], it is shown that the function

$$P(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\ell_R(M/I(1)_{mn_1} \cdots I(r)_{mn_r} M)}{m^d} \tag{1.2}$$

is a homogeneous polynomial of total degree d with real coefficients for all $n_1, \ldots, n_r \in \mathbb{N}$. The mixed multiplicities of M are defined from the coefficients of P, generalizing the definition of mixed multiplicities for m_R-primary ideals. Specifically, we write

$$P(n_1, \ldots, n_r) = \sum_{d_1+\cdots+d_r=d} \frac{1}{d_1! \cdots d_r!} e_R(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; M) n_1^{d_1} \cdots n_r^{d_r}. \tag{1.3}$$

We say that $e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M)$ is the mixed multiplicity of M of type $(d_1,\ldots,d_r)$ with respect to the filtrations $\mathcal{I}(1),\ldots,\mathcal{I}(r)$. Here we are using the notation

$$e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M) \tag{1.4}$$

to be consistent with the classical notation for mixed multiplicities of M with respect to m_R-primary ideals from [**36**]. The mixed multiplicity of M of type $(d_1,\ldots,d_r)$ with respect to m_R-primary ideals $I_1,\ldots,I_r$, denoted by

$$e_R(I_1^{[d_1]},\ldots,I_r^{[d_r]};M)$$

([**36**], [**35**, Definition 17.4.3]) is equal to the mixed multiplicity

$$e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};M),$$

where the Noetherian I-adic filtrations $\mathcal{I}(1),\ldots,\mathcal{I}(r)$ are defined by

$$\mathcal{I}(1)=\{I_1^i\}_{i\in\mathbb{N}},\ldots,\mathcal{I}(r)=\{I_r^i\}_{i\in\mathbb{N}}.$$

We have that

$$e_R(\mathcal{I};M)=e_R(\mathcal{I}^{[d]};M) \tag{1.5}$$

if $r=1$, and $\mathcal{I}=\{I_i\}$ is a filtration of R by m_R-primary ideals. Thus

$$e_R(\mathcal{I};M)=\lim_{m\to\infty}\frac{\ell_R(M/I_mM)}{m^d/d!}.$$

In [**12**], it is shown that many classical theorems about mixed multiplicities of m_R-primary ideals continue to hold for filtrations. For instance, the four "Minkowski inequalities" for mixed multiplicities are proven in [**12**, Theorem 6.3].

Another important property is that the mixed multiplicities of R with respect to filtrations of m_R-primary ideals are nonnegative real numbers ([**14**, Proposition 1.3]). In contrast, the mixed multiplicities of R with respect to m_R-primary ideals are always strictly positive integers ([**36**] or [**35**, Corollary 17.4.7]).

1.2. Divisorial filtrations. An important category of filtrations is that of the divisorial filtrations. Theorems about inequalities of multiplicities and mixed multiplicities of m_R-primary ideals tend to be true for arbitrary m_R-filtrations. Theorems characterizing when equality holds in these inequalities for m_R-primary ideals generally fail for arbitrary filtrations of m_R-primary ideals but tend to be true for divisorial filtrations. Before defining divisorial filtrations, we give a couple of examples. We do not make the assumptions on rings explicit in these examples, although in all these statements the required assumptions are very mild (the most restrictive is "excellent local domain"). Let R have dimension d.

Rees's Theorem [**30**] or [**35**, Proposition 11.3.1], shows that if $I'\subset I$ are m_R-primary ideals, then $e_R(I',R)=e_R(I,R)$ if and only if the R-algebras $\sum_{n\ge 0}(I')^nt^n$ and $\sum_{n\ge 0}I^nt^n$ have the same integral closure in $R[t]$. This characterization is true for divisorial filtrations of m_R-primary ideals ([**10**, Theorem 1.4]; a slightly less general statement is in [**9**, Theorem 3.5]). However, only the "if" direction of this characterization is true for arbitrary filtrations of m_R-primary ideals ([**12**, Theorem 6.9] and the following example and [**9**, Appendix]).

The Minkowski inequalities of m_R-primary ideals were proven by Teissier [**36**], [**37**] and Rees and Sharp [**33**]. These inequalities also hold for arbitrary m_R-filtrations, as shown in [**12**, Theorem 6.3]. It is shown by Teissier [**38**], Rees and

Sharp [**33**] and Katz [**19**] that if $I(1)$ and $I(2)$ are m_R-primary ideals, then the Minkowski equality

$$e(I(1)I(2),R)^{\frac{1}{d}} = e(I(1),R)^{\frac{1}{d}} + e(I(2),R)^{\frac{1}{d}} \tag{1.6}$$

holds if and only if

(1.7)
There exist positive integers a and b such that $\sum_{n\ge 0}(I')^{na}t^n$ and $\sum_{n\ge 0} I^{nb}t^n$ have the same integral closure in $R[t]$.

This characterization holds for divisorial filtrations of m_R-primary ideals ([**10**, Theorem 13.2]). However, only the direction (1.7) implies (1.6) is true for arbitrary m_R-filtrations (Example in [**12**]).

We now define divisorial valuations and filtrations, and discuss some of their basic properties.

Suppose that R is a d-dimensional excellent local domain, with quotient field K. A valuation ν of K is called an m_R-valuation if ν dominates R ($R \subset V_\nu$ and $m_\nu \cap R = m_R$ where V_ν is the valuation ring of ν with maximal ideal m_ν) and $\mathrm{trdeg}_{R/m_R} V_\nu/m_\nu = d-1$.

Suppose that I is an ideal in R. Let X be the normalization of the blowup of I, with projective birational morphism $\phi : X \to \mathrm{Spec}(R)$. Let $E_1,\ldots,E_t$ be the irreducible components of $\phi^{-1}(V(I))$ (which necessarily have dimension $d-1$). The Rees valuations of I are the discrete valuations ν_i for $1 \le i \le t$ with valuation rings $V_{\nu_i} = \mathcal{O}_{X,E_i}$. If R is normal, then X is equal to the blowup of the integral closure $\overline{I^s}$ of an appropriate power I^s of I.

Every Rees valuation ν that dominates R is an m_R-valuation and every m_R-valuation is a Rees valuation of an m_R-primary ideal by [**32**, Statement (G)].

Associated to an m_R-valuation ν are valuation ideals

$$I(\nu)_n = \{f \in R \mid \nu(f) \ge n\} \tag{1.8}$$

for $n \in \mathbb{N}$. In general, the filtration $\mathcal{I}(\nu) = \{I(\nu)_n\}$ is not Noetherian. In a two-dimensional normal local ring R, the condition that the filtration of valuation ideals of R is Noetherian for all m_R-valuations dominating R is the condition (N) of Muhly and Sakuma [**26**]. It is proven in [**5**] that a complete normal local ring of dimension two satisfies condition (N) if and only if its divisor class group is a torsion group. An example is given in [**3**] of an m_R-valuation of a 3-dimensional regular local ring R such that the filtration is not Noetherian.

Definition 1.1. Suppose that R is an excellent local domain. We say that a filtration $\mathcal{I}$ of R by m_R-primary ideals is a divisorial filtration if there exists a projective birational morphism $\phi : X \to \mathrm{Spec}(R)$ such that X is the normalization of the blowup of an m_R-primary ideal and there exists a nonzero effective Weil divisor D on X with exceptional support for ϕ such that $\mathcal{I} = \{I(mD)\}_{m\in\mathbb{N}}$ where

$$I(mD) = \Gamma(X, \mathcal{O}_X(-mD)) \cap R. \tag{1.9}$$

We will write

$$\mathcal{I}(D) = \{I(mD)\}_{m\in\mathbb{N}}.$$

If R is normal, then $I(mD) = \Gamma(X, \mathcal{O}_X(-mD))$. If $D = \sum_{i=1}^t a_iE_i$ where the $a_i \in \mathbb{N}$ and the E_i are prime exceptional divisors of ϕ, with associated m_R-valuations ν_i, then

$$I(mD) = I(\nu_1)_{a_1m} \cap \cdots \cap I(\nu_t)_{a_tm}.$$

1.3. The main constructions and key ideas. Let R be an excellent, d-dimensional normal local ring with an isolated singularity, and suppose that $\phi : X \to \mathrm{Spec}(R)$ is a resolution of singularities which is the blowup of an m_R-primary ideal. Let $E_1, \ldots, E_r$ be the prime exceptional divisors of ϕ.

We consider three natural functions on effective exceptional divisors on X. We have the associated multiplicities

$$e_R(\mathcal{I}(D), R) = d! \lim_{m\to\infty} \frac{\ell_R/I(mD)}{m^d}. \tag{1.10}$$

where $D = a_1E_1 + \cdots + a_rE_r$ is an effective divisor.

We have the function

$$P(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\ell_R(R/I(mn_1E_1)\cdots I(mn_rE_r))}{m^d}. \tag{1.11}$$

By (1.3), this is a homogeneous polynomial in $n_1, \ldots, n_r$ which computes the mixed multiplicities $e_R(\mathcal{I}(E_1)^{[d_1]}, \ldots, \mathcal{I}(E_r)^{[d_r]}; R)$.

We also consider the function

$$G(n_1, \ldots, n_r) = \lim_{m\to\infty} \frac{\ell_R(R/I(mn_1E_1 + \cdots + mn_rE_r))}{m^d}. \tag{1.12}$$

We will see that the function $G(n_1, \ldots, n_r)$ is given by the anti-intersection product (1.16) on X.

The situation in dimension $d = 2$ is relatively simple and well behaved. Suppose that R has dimension $d = 2$. Then all limits (1.10) are rational numbers, and the coefficients of (1.11) are rational numbers [**9**, Proposition 5.7]. Further, it is shown in [**12**, Section 7] (using some results from [**11**]) that the function $G(n_1, \ldots, n_r)$ of (1.12) is a piecewise rational polynomial function on an abstract complex of rational polyhedral sets whose union is $(\mathbb{Q}_{\geq 0})^r$. This holds, even though the multi graded filtration $\{I(n_1E_1 + \cdots + n_rE_r)\}$ is generally not Noetherian (as discussed earlier in Subsection 1.2). The anti-positive intersection product of (1.16) is in this case the ordinary intersection product of the Zariski decomposition of $-n_1E_1 - \cdots - n_rE_r$ ([**9**, Proposition 5.7] and [**9**, Theorem 8.3]).

In this paper we give examples showing that this kind of good behavior does not always hold in dimension $d = 3$ and higher. We compute a specific resolution of singularities from a three dimensional excellent local ring R which demonstrates irrational behavior in equations (1.10), (1.11) and (1.12).

Our example is on a resolution of singularities of a normal three dimensional excellent local ring with two prime exceptional divisors. The function $G(n_1, n_2)$ of (1.12), given in Theorem 1.2, is a piecewise polynomial function on an abstract complex of polyhedral sets, but the polynomials and the polyhedral sets are not rational.

An algebraic local ring is a local domain which is essentially of finite type over a field. Let R be a d-dimensional normal algebraic local ring and $\phi : X \to \mathrm{Spec}(R)$ be the blow up of an m_R-primary ideal of R such that X is normal. Let k be an algebraically closed field. We construct a 3-dimensional normal algebraic local ring R over k and the blow up $\phi : X \to \mathrm{Spec}(R)$ of an m_R-primary ideal such that X is nonsingular with two irreducible exceptional divisors, which we denote by $\overline{S}$ and F. Theorems 1.2 and 1.3 below refer to this example. The rest of this paper will be devoted to the proofs of these theorems.

The resolution of singularities of a three dimensional normal local ring which we construct is similar to the one constructed in [**13**, Example 6], which is used to give an example of a valuative filtration with irrational multiplicity. In [**13**, Example 6], no details of the construction or analysis of the example are given. We give complete details in this paper. We illustrate the application of anti-positive intersection products in the proof.

THEOREM 1.2. *Let $D = n\overline{S} + jF$ with $n, j \in \mathbb{N}$. Then*

$$\lim_{m\to\infty} \frac{\ell_R(R/I(mD))}{m^3} = \begin{cases} 33n^3 & \text{if } j < n \\ 78n^3 - 81n^2j + 27nj^2 + 9j^3 & \text{if } n \le j < n\left(3 - \frac{\sqrt{3}}{3}\right) \\ \left(\frac{2007}{169} - \frac{9\sqrt{3}}{338}\right) j^3 & \text{if } n\left(3 - \frac{\sqrt{3}}{3}\right) < j. \end{cases}$$

In particular,

$$e_R(\mathcal{I}(D); R) = \begin{cases} 198n^3 & \text{if } j < n \\ 468n^3 - 486n^2j + 162nj^2 + 54j^3 & \text{if } n \le j < n\left(3 - \frac{\sqrt{3}}{3}\right) \\ \left(\frac{12042}{169} - \frac{27\sqrt{3}}{169}\right) j^3 & \text{if } n\left(3 - \frac{\sqrt{3}}{3}\right) < j. \end{cases}$$

As a consequence, we have that

$$\lim_{m\to\infty} \frac{\ell_R(R/I(mF))}{m^3} = \left(\frac{2007}{169} - \frac{9\sqrt{3}}{338}\right) \tag{1.13}$$

and

$$e_R(\mathcal{I}(F); R) = \frac{12042}{169} - \frac{27\sqrt{3}}{169}, \tag{1.14}$$

giving an example of a divisorial valuation $\nu = \nu_F$ dominating R such that $e_R(\mathcal{I}(\nu); R) = e_R(\mathcal{I}(F); R)$ is an irrational number, where $\mathcal{I}(\nu) = \{I(\nu)_m\}$ and $I(\nu)_n$ is the valuation ideal $I(\nu)_m = \{f \in R \mid \nu(f) \ge m\}$.

THEOREM 1.3. *For $n, j \in \mathbb{N}$,*

$$\lim_{m\to\infty} \frac{\ell_R(R/I(mn\overline{S})I(mjF))}{m^3} = 33n^3 + \left(\frac{891}{26} + \frac{99\sqrt{3}}{26}\right) n^2 j + \left(\frac{6021}{169} - \frac{27\sqrt{3}}{338}\right) nj^2 + \left(\frac{2007}{169} - \frac{9\sqrt{3}}{338}\right) j^3.$$

In particular, the mixed multiplicities are

$$\begin{aligned} &e_R(\mathcal{I}(\overline{S})^{[3]}; R) = e_R(\mathcal{I}(\overline{S}); R) = 198 \\ &e_R(\mathcal{I}(\overline{S})^{[2]}, \mathcal{I}(F)^{[1]}; R) = \tfrac{891}{13} + \tfrac{99\sqrt{3}}{13} \\ &e_R(\mathcal{I}(\overline{S})^{[1]}, \mathcal{I}(F)^{[2]}; R) = \tfrac{12042}{169} - \tfrac{27\sqrt{3}}{169} \\ &e_R(\mathcal{I}(F)^{[3]}; R) = e_R(\mathcal{I}(F); R) = \tfrac{12042}{169} - \tfrac{27\sqrt{3}}{169}. \end{aligned}$$

In [**9**], anti-positive intersection products $\langle(-D_1)^{d_1} \cdot \ldots \cdot (-D_r)^{d_r}\rangle$ where $D_1, \ldots, D_r$ are effective Cartier divisors on X with exceptional support are defined, generalizing the positive intersection product of Cartier divisors defined on projective varieties in [**2**] over an algebraically closed field of characteristic zero and in [**8**] over an arbitrary field. The anti-positive intersection multiplicities have the property that if $d_1 + \cdots + d_r = d$, then $\langle(-D_1)^{d_1}\cdot, \ldots, \cdot(-D_r)^{d_r}\rangle$ is a non positive real number.

It is shown in [**9**, Theorem 8.3] that we have identities

$$e_R(\mathcal{I}(D_1)^{[d_1]}, \ldots, \mathcal{I}(D_r)^{[d_r]}; R) = -\langle(-D_1)^{d_1}\cdot, \ldots, \cdot(-D_r)^{d_r}\rangle.$$

In particular, $e_R(\mathcal{I}(D);R) = -\langle(-D)^d\rangle$. Thus by (1.3), we have that

$$\begin{aligned}(1.15)\qquad &\lim_{m\to\infty} \frac{\ell_R(R/I(mn_1D_1)\cdots\mathcal{I}(mn_rD_r))}{m^d}\\ &= -\sum_{d_1+\cdots+d_r=d} \frac{1}{d_1!\cdots d_r!}\langle(D_1)^{d_1}\cdot\ldots\cdot(-D_r)^{d_r}\rangle n_1^{d_1}\cdots n_r^{d_r}.\end{aligned}$$

From the case $r=1$, we obtain

$$(1.16)\qquad \lim_{m\to\infty} \frac{\ell_R(R/I(mD))}{m^d} = -\frac{\langle(-D)^d\rangle}{d!}.$$

The interpretation of mixed multiplicities as anti-positive intersection multiplicities is particularly useful in the calculation of examples. This is the method we use in the example constructed in this paper.

2. Anti-positive intersection products

2.1. The construction of anti-positive intersection products. In this subsection we review the construction of anti-positive intersection products in [**9**].

Anti-positive intersection products generalize the positive intersection products of Cartier divisors defined on projective varieties in [**2**] over an algebraically closed field of characteristic zero and in [**8**] over an arbitrary field.

Let K be an algebraic function field over a field k. An algebraic local ring of K is a local ring R that is a localization of a finitely generated k-algebra and is a domain whose quotient field is K with maximal ideal m_R. Let R be a d-dimensional algebraic normal local ring of K. Let $\mathrm{BirMod}(R)$ be the directed set of blowups $\phi: X \to \mathrm{Spec}(R)$ of an m_R-primary ideal I of R such that X is normal.

Suppose that $\phi: X \to \mathrm{Spec}(R)$ is in $\mathrm{BirMod}(R)$. Let $\{E_1,\ldots,E_t\}$ be the irreducible exceptional divisors of ϕ. We define $M^1(X)$ to be the subspace of the real vector space $E_1\mathbb{R}+\cdots+E_t\mathbb{R}$ that is generated by the Cartier divisors. An element of $M^1(X)$ will be called an $\mathbb{R}$-divisor on X. We will say that $D\in M^1(X)$ is a $\mathbb{Q}$-Cartier divisor if there exists $n\in\mathbb{Z}_+$ such that nD is a Cartier divisor.

In Section 6.1 of [**9**], we define a natural intersection product $(D_1\cdot D_2\cdot\ldots\cdot D_d)$ on X for $D_1,\ldots,D_d\in M^1(X)$. The intersection product is a restriction of the one defined in [**23**].

We will say that a divisor $F = a_1E_1+\cdots+a_tE_t\in M^1(X)$ is effective if $a_i\ge 0$ for all i, and anti-effective if $a_i\le 0$ for all i. This defines a partial order $\le$ on $M^1(X)$ by $A\le B$ if $B-A$ is effective. The effective cone $\mathrm{EF}(X)$ is the closed convex cone in $M^1(X)$ of effective $\mathbb{R}$-divisors. The anti-effective cone $\mathrm{AEF}(X)$ is the closed convex cone in $M^1(X)$ consisting of all anti-effective $\mathbb{R}$-divisors.

We will say that an anti-effective divisor $F\in M^1(X)$ is numerically effective (nef) if

$$(F\cdot C)\ge 0$$

for all closed curves C in $\phi^{-1}(m_R)$. The nef cone $\mathrm{Nef}(X)$ is the closed convex cone in $M^1(X)$ of all nef $\mathbb{R}$-divisors on X. There is an inclusion of cones $\mathrm{Nef}(X)\subset\mathrm{AEF}(X)$. We define a divisor $F\in M^1(X)$ to be ample if F is a formal sum $F=\sum a_iF_i$ where F_i are ample anti-effective Cartier divisors and a_i are positive real numbers. A divisor D is anti-ample if $-D$ is ample. We define the convex cone

$$\mathrm{Amp}(X) = \{F\in M^1(X)\mid F \text{ is ample}\}.$$

We have that $\mathrm{Amp}(X)\subset\mathrm{Nef}(X)$, the closure of $\mathrm{Amp}(X)$ is $\mathrm{Nef}(X)$, and the interior of $\mathrm{Nef}(X)$ is $\mathrm{Amp}(X)$, as in [**23**], [**24**, Theorem 1.4.23].

Suppose that $X \in \mathrm{BirMod}(R)$. Let $E_1, \ldots, E_r$ be the exceptional components of X for the morphism $X \to \mathrm{Spec}(R)$. For $0 < p \leq d$, we define $M^p(X)$ to be the direct product of $M^1(X)$ p times, and we define $M^0(X) = \mathbb{R}$. For $1 < p \leq d$, we define $L^p(X)$ to be the vector space of p-multilinear forms from $M^p(X)$ to $\mathbb{R}$, and define $L^0(X) = \mathbb{R}$.

The intersection product gives us p-multilinear maps

$$M^p(X) \to L^{d-p}(X) \tag{2.1}$$

for $0 \leq p \leq d$.

We have that $\mathrm{BirMod}(R)$ is a directed set by the R-morphisms $Y \to X$ for $X, Y \in \mathrm{BirMod}(R)$. There is at most one R-morphism $X \to Y$ for $X, Y \in \mathrm{BirMod}(X)$.

The set $\{M^p(Y_i) \mid Y_i \in \mathrm{BirMod}(R)\}$ is a directed system of real vector spaces, where we have a linear mapping $f_{ij}^* : M^p(Y_i) \to M^p(Y_j)$ if the natural birational map $f_{ij} : Y_j \to Y_i$ is an R-morphism. We define

$$M^p(R) = \lim_{\rightarrow} M^p(Y_i)$$

Anti-positive intersection products $\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle$ for anti-effective $\alpha_1, \ldots, \alpha_p \in M^1(R)$ are defined in [**9**, Definition 7.4], generalizing the positive intersection products defined on projective varieties in [**2**] over an algebraically closed field of characteristic zero, and in [**8**, Definition 4.4] over an arbitrary field. The anti-positive intersection product $\langle \alpha_1 \cdot \ldots \cdot \alpha_d \rangle$ of d anti-effective divisors $\alpha_1, \ldots, \alpha_d \in M^1(R)$ is always a non positive real number.

The proof of the following proposition is similar to that of [**8**, Proposition 4.12], replacing the reference to [**8**, Proposition 4.3] with [**9**, Proposition 7.3], and replacing the use of the continuity statement of [**8**, Proposition 4.7] with the continuity statement of [**9**, Proposition 7.5].

PROPOSITION 2.1. *Suppose that $\alpha_1, \ldots, \alpha_p \in M^1(R)$ are anti-effective. Then the anti-positive intersection product $\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle$ is the least upper bound of all ordinary intersection products $\beta_1 \cdot \ldots \cdot \beta_p$ in $L^{d-p}(R)$ with $\beta_i \in M^1(R)$ anti-effective and nef and $\beta_i \leq \alpha_i$.*

2.2. $\gamma_E(D)$ and anti-positive intersection products. Let $\phi : X \to \mathrm{spec}(R) \in \mathrm{BirMod}(R)$. Let $E_1, \ldots, E_t$ be the irreducible exceptional divisors of ϕ.

Suppose that $D = \sum a_i E_i$ is an effective $\mathbb{Q}$-Cartier divisor on X. If D is Cartier, then $\Gamma(X, \mathcal{O}_X(-D))$ is an m_R-primary ideal since R is normal. Write $I(D) = \Gamma(X, \mathcal{O}_X(-D))$.

Let ν_{E_i} be the natural discrete valuation with valuation ring $\mathcal{O}_{X,E_i}$.

Let r be a fixed positive integer such that rD is a Cartier divisor. Define

$$\tau_{rm,E_i}(rD) = \min\{\nu_{E_i}(f) \mid f \in \Gamma(X, \mathcal{O}_X(-mrD))\},$$

and $\gamma_{E_i}(D) = \inf_m \frac{\tau_{rm,E_i}(rD)}{rm}$. The real number $\gamma_{E_i}(D)$ is independent of r. We have that

$$\gamma_{E_i}(D) \geq a_i$$

for all i, and

$$I(mrD) = \Gamma(X, \mathcal{O}_X(-mrD)) = \Gamma(X, \mathcal{O}_X(-\lceil \sum mr\gamma_{E_i}(D)E_i \rceil)$$

for all $m \in \mathbb{N}$ (this is shown in [**9**, Lemma 3.1]). Here $\lceil x \rceil$ denotes the round up of a real number x.

Lemma 2.2. *Suppose that D is anti-nef. Then $\gamma_{E_i}(D) = a_i$ for all i.*

Proof. If $-D$ is ample, then $\mathcal{O}_X(-mrD)$ is generated by global sections if r is such that rD is Cartier and $m \gg 0$. Thus for all i, there exists $f \in \Gamma(X, \mathcal{O}_X(-rmD))$ such that $\nu_{E_i}(f) = mra_i$. Thus $\gamma_{E_i}(D) = a_i$.

Now suppose that $-D$ is nef. Given $\epsilon > 0$, there exists an anti-ample effective $\mathbb{Q}$-Cartier divisor A on X such that $A = \sum c_i E_i$ with $a_i \le c_i < a_i + \epsilon$ for all i. Let r be such that rA and rD are Cartier divisors. For $m \gg 0$, there exists $f \in \Gamma(X, \mathcal{O}_X(-mrA))$ such that $\nu_{E_i}(f) = mrc_i < mra_i + mr\epsilon$. Thus $\gamma_{E_i}(D) \le a_i + \epsilon$. Since this is true for all ϵ, we have that $\gamma_{E_i}(D) = a_i$. □

Lemma 2.3. *Suppose that $D \in M^1(X)$ is an effective $\mathbb{Q}$-Cartier divisor such that $-\sum \gamma_{E_i}(D)E_i$ is nef. Suppose that $Y \to \mathrm{Spec}(R) \in \mathrm{BirMod}(R)$ and there exists a factorization $\psi : Y \to X$. If $G \in M^1(Y)$ is an effective and anti-nef $\mathbb{Q}$-Cartier divisor such that $-G \le \psi^*(-D)$ then $-G \le \psi^*(-\sum \gamma_{E_i}(D)E_i)$.*

Proof. Let $F_1, \ldots, F_s$ be the irreducible exceptional divisors of $Y \to \mathrm{Spec}(R)$. Since $-\sum \gamma_{E_i}(D)E_i$ is nef, we have that $\sum \gamma_{F_i}(\psi^*(D))F_i = \psi^*(\sum \gamma_{E_i}(D)E_i)$. Write $G = \sum g_i F_i$. Since $-G$ is nef, we have that $\gamma_{F_i}(G) = g_i$ for all i. Since

$$\mathcal{O}_Y(-mrG) \subset \mathcal{O}_Y(-\psi^*(mrD))$$

whenever rG, rD are Cartier divisors and $m > 0$, we have that $\gamma_{F_i}(D) \le \gamma_{F_i}(G) = g_i$ for all i. Thus

$$-G \le -\sum \gamma_{F_i}(\psi^*(D))F_i = -\psi^*(\sum \gamma_{E_i}(D)E_i).$$

□

The following proposition is a consequence of Proposition 2.1 and 2.3.

Proposition 2.4. *Suppose that $D_1, \ldots, D_d \in M^1(X)$ are effective $\mathbb{Q}$-Cartier divisors such that the divisors $-\sum \gamma_{E_i}(D_j)E_i$ are nef for $1 \le j \le d$. Then the positive intersection product $\langle -D_1 \cdot \ldots \cdot -D_d \rangle$ is the ordinary intersection product $(-\sum \gamma_{E_i}(D_1)E_i \cdot \ldots \cdot -\sum \gamma_{E_i}(D_d)E_i)$.*

3. Intersection theory on projective varieties

In this section we review some material on intersection theory on Projective varieties. We refer to [**23**] and [**24**]. Let k be an algebraically closed field. Let T be a nonsingular projective surface over k. Then $(\mathrm{Pic}(T)/\equiv) \otimes \mathbb{R}$, where $\equiv$ denotes numerical equivalence, is a finite dimensional real vector space. We will often abuse notation, identifying the class of an invertible sheaf $\mathcal{O}_T(D)$ with the class of the divisor D.

We will denote the closure of the real cone in $(\mathrm{Pic}(T)/\equiv) \otimes \mathbb{R}$ generated by the classes of effective divisors by $\overline{\mathrm{Eff}}(T)$ and the closure of the real cone in $(\mathrm{Pic}(T)/\equiv) \otimes \mathbb{R}$ generated by the classes of numerically effective divisors by $\overline{\mathrm{Nef}}(T)$, and the closure of the real cone in $(\mathrm{Pic}(T)/\equiv) \otimes \mathbb{R}$ generated by the classes of ample divisors by $\overline{\mathrm{Amp}}(T)$.

If V is a nonsingular r-dimensional projective variety and $D_1, \ldots, D_r$ are divisors on V we will denote the intersection product of $D_1, \ldots, D_r$ on V by $(D_1 \cdot D_2 \cdot \ldots \cdot D_r)_V$. When there is no danger of confusion about the ambient variety, we will simply write $(D_1 \cdot D_2 \cdot \ldots \cdot D_r)$.

4. An example

In this section, we construct a resolution of singularities of a three dimensional normal local ring, and compute the multiplicities and mixed multiplicities of its divisorial filtrations. This resolution of singularities is similar to the one constructed in [**13**, Example 6], which is used to give an example of a filtration of a divisorial valuation with irrational multiplicity. In [**13**, Example 6], no details of the construction or analysis of the example are given. We give complete details in this section.

Let k be an algebraically closed field, Let W be an elliptic curve over k and S be the abelian surface $S = W \times W$. Let $\pi_1 : S \to W$ and $\pi_2 : S \to W$ be the two projections. Let $p \in W$ be a closed point, $A = \pi_1^*(p)$, $B = \pi_2^*(p)$ and $\Delta \subset W \times W = S$ be the diagonal. Let V be the real subspace of $(\mathrm{Pic}(S)/\equiv) \otimes \mathbb{R}$ generated by the classes of A, B and Δ. As shown in [**4**] and [**13**, Example 4], we have that V has dimension 3 and

$$(\Delta^2) = (A^2) = (B^2) = 0 \text{ and } (A \cdot B) = (A \cdot \Delta) = (B \cdot \Delta) = 1.$$

Further, $\overline{\mathrm{Amp}}(S) = \overline{\mathrm{Eff}}(S) = \overline{\mathrm{Nef}}(S)$, and $V \cap \overline{\mathrm{Eff}}(S)$ is the real cone which is the component of

$$\{xA + yB + z\Delta \mid (xA + yB + z\Delta)^2 \geq 0\}$$

which contains the ample divisor $A + B + \Delta$.

If $j, n \geq 0$ we have that

$$n(A + 2B + 3\Delta) - j(A + B + \Delta) \in \overline{\mathrm{Eff}}(S) \text{ if and only if } j < n\left(2 - \frac{\sqrt{3}}{3}\right). \tag{4.1}$$

The canonical divisor of the Abelian surface S is $K_S = 0$.

Let X be the projective bundle $X = \mathbb{P}(\mathcal{O}_S(-3(A + 2B + 3\Delta)) \oplus \mathcal{O}_S)$ with projection $\pi : X \to S$. Identify the section of π corresponding to the natural surjection of $\mathcal{O}_S$-modules

$$\mathcal{O}_S(-3(A + 2B + 3\Delta)) \oplus \mathcal{O}_S \to \mathcal{O}_S(-3(A + 2B + 3\Delta))$$

with S (c.f. [**18**, Proposition 7.12]). Then $\mathcal{O}_X(1) \cong \mathcal{O}_X(S)$ and

$$\mathcal{O}_X(S) \otimes \mathcal{O}_S \cong \mathcal{O}_S(-3(A + 2B + 3\Delta)).$$

A canonical divisor on X is $K_X = -2S + \pi^*(-3(A + 2B + 3\Delta))$ (this can be seen by applying adjunction on a fiber of π and then on the section S). The Picard group of X is

$$\mathrm{Pic}(X) = \mathcal{O}_X(1)\mathbb{Z} \oplus \pi^*\mathrm{Pic}(S).$$

Suppose that Γ is an effective divisor on X. Then $\Gamma \sim nS + \pi^*(L)$ for some divisor L on X. We have that $n = (\Gamma \cdot g) \geq 0$ for a fiber g of π. Since Γ is effective,

$$\begin{array}{rcl} 0 < h^0(X, \mathcal{O}_X(\Gamma)) & = & h^0(S, \mathrm{Sym}^n\left(\mathcal{O}_S(-3(A + 2B + 3\Delta)) \oplus \mathcal{O}_S\right) \otimes \mathcal{O}_S(L))) \\ & = & \sum_{i=0}^n h^0(S, \mathcal{O}_S(L - i3(A + 2B + 3\Delta))). \end{array}$$

Thus $L \in \overline{\mathrm{Eff}}(S)$.

Let T be the section of π corresponding to the surjection of $\mathcal{O}_S$-modules

$$\mathcal{O}_S(-3(A + 2B + 3\Delta)) \oplus \mathcal{O}_S \to \mathcal{O}_S.$$

Then $\mathcal{O}_X(1) \otimes \mathcal{O}_T \cong \mathcal{O}_S$ so that $T \cap S = \emptyset$. Further, $\mathcal{O}_X(T) \cong \mathcal{O}_X(S + 3(A + 2B + 3\Delta))$. Now $3(A + 2B + 3\Delta)$ is very ample on S (by the theorem in Section 17 [**27**]), so the complete linear system $|T|$ on X is base point free, and thus induces a projective morphism $X \to Z$ which contracts S. Suppose that γ is an irreducible curve on X which is not contained in S and is not a fiber of π. Let $\overline{\gamma}$ be the image of γ by π in S which is a curve. We have that

$$\begin{array}{rcl}(\gamma\cdot T)_X & = & (\gamma\cdot(S+\pi^*(3(A+2B+3\Delta))))_X\\ & = & \deg(\mathcal{O}_X(S+\pi^*(3(A+2B+3\Delta)))\otimes\mathcal{O}_\gamma)\\ & \ge & \deg(\mathcal{O}_X(\pi^*(3(A+2B+3\Delta)))\otimes\mathcal{O}_\gamma)\\ & = & \deg(\gamma/\overline{\gamma})(\overline{\gamma}\cdot 3(A+2B+3\Delta))_S>0\end{array}$$

by the projection formula ([**16**, Example 7.1.9]), and since $A+2B+3\Delta$ is ample. Thus $X\setminus S\to Z$ is finite to one. Let $\overline{Z}$ be the normalization of Z in the function field of X. Then there is an induced birational projective morphism $\lambda : X\to\overline{Z}$ such that S is contracted to a point q of $\overline{Z}$ and $X\setminus S\to\overline{Z}-q$ is an isomorphism. The divisor $-S$ is relatively ample for λ since $\mathcal{O}_X(-S)\otimes\mathcal{O}_S$ is ample on S. Thus there exists $n>0$ such that X is the blow up of the ideal sheaf $\lambda_*\mathcal{O}_X(-nS)$ of $\overline{Z}$, which has the property that the support of $\mathcal{O}_{\overline{Z}}/\lambda_*\mathcal{O}_X(-nS)$ is the point q.

The divisor $A+B+\Delta$ is ample on S so $3(A+B+\Delta)$ is very ample (Theorem in Section 17 [**27**]). Thus by Bertini's theorem ([**18**, Theorem II.8.18 and Remark III.7.9]) there exists an integral and nonsingular curve C on S such that $C\sim 3(A+B+\Delta)$. Let $\mathcal{I}_C$ be the ideal sheaf of C in X. Let $\tau : Y\to X$ be the blow up of C, so that $Y=\mathrm{Proj}(\oplus_{n\ge 0}\mathcal{I}_C^n)$. Let $F=\mathrm{Proj}(\oplus_{n\ge 0}\mathcal{I}_C^n/\mathcal{I}_C^{n+1})$ be the exceptional divisor and let $\overline{\tau} : F\to C$ be the induced morphism.

The composed morphism $Y\to\overline{Z}$ contracts F and the strict transform $\overline{S}$ of S to the point q of $\overline{Z}$ and is an isomorphism everywhere else, and is the blow up of an ideal sheaf $\mathcal{I}$ of $\mathcal{O}_Z$ such that the support of $\mathcal{O}_Z/\mathcal{I}$ is the point q. The map τ induces an isomorphism of $\overline{S}$ and S.

Let $G=\pi^*(C)$ which is an integral nonsingular surface in X. We have that C is the scheme theoretic intersection of G and S. Thus

$$(\mathcal{O}_X(-S)\otimes\mathcal{O}_C)\oplus(\mathcal{O}_X(-G)\otimes\mathcal{O}_C)\cong\mathcal{I}_C/\mathcal{I}_C^2,$$

and so $F=\mathbb{P}(\mathcal{O}_X(-S)\otimes\mathcal{O}_C)\oplus(\mathcal{O}_X(-G)\otimes\mathcal{O}_C))$. We have that $\mathcal{O}_Y(-F)=\mathcal{I}_C\mathcal{O}_Y=\mathcal{O}_Y(1)$, so $\mathcal{O}_Y(-F)\otimes\mathcal{O}_F\cong\mathcal{O}_F(1)$.

The Picard group of F is

$$\mathrm{Pic}(F)=\mathcal{O}_F(1)\mathbb{Z}\oplus\overline{\tau}^*\mathrm{Pic}(C).$$

A canonical divisor of Y is $K_Y=\tau^*K_X+F=-2\overline{S}-F+(\pi\tau)^*(-3(A+2B+3\Delta))$.

Since $\tau^*(S)=\overline{S}+F$ and C is (isomorphic) to the scheme theoretic intersection of F and $\overline{S}$, we have that

$$\mathcal{O}_Y(\overline{S})\otimes\mathcal{O}_{\overline{S}}\cong\mathcal{O}_S(-3(2A+3B+4\Delta)).$$

We have that

$$\mathcal{O}_Y(-n\overline{S}-jF)\otimes\mathcal{O}_{\overline{S}}\cong\mathcal{O}_X(-nS)\otimes\mathcal{O}_S(-(j-n)C)$$

so by (4.1),

$$\mathcal{O}_Y(-n\overline{S}-jF)\otimes\mathcal{O}_{\overline{S}}\in\overline{\mathrm{Eff}}(\overline{S})\text{ if and only if }j<n\left(3-\frac{\sqrt{3}}{3}\right).\tag{4.2}$$

We have that $F\cong\mathbb{P}(\mathcal{E})$ where $\mathcal{E}=\mathcal{O}_C\oplus(\mathcal{O}_X(H)\otimes\mathcal{O}_C)$ with $H=S-G$ (by [**18**, Proposition V.2.2]). Let C_0 be the section of $\overline{\tau}$ corresponding to the natural surjection of $\mathcal{O}_C$-modules $\mathcal{E} \to \mathcal{O}_X(H) \otimes \mathcal{O}_C$. Then $\mathcal{O}_F(C_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and since

$$\mathcal{O}_X(H) \otimes \mathcal{O}_C \cong \mathcal{O}_S(-3(2A+3B+4\Delta)) \otimes \mathcal{O}_C,$$

we have that

$$(C_0^2)_F = \deg \mathcal{O}_X(H) \otimes \mathcal{O}_C = (3(A+B+\Delta)\cdot 3(-2A-3B-4\Delta))_S = -9\times 18 = -162.$$

Let f be a fiber of $\overline{\tau}$ over a closed point of C.

Let K_F be a canonical divisor of F. By adjunction, we have

$$\mathcal{O}_F(K_F) \cong \mathcal{O}_Y(K_Y+F) \otimes \mathcal{O}_F \cong \mathcal{O}_Y(-2\overline{S}+(\pi\tau)^*(-3(A+2B+3\Delta))) \otimes \mathcal{O}_F. \tag{4.3}$$

By adjunction on f and C_0,

$$\mathcal{O}_F(K_F) \cong \mathcal{O}_F(-2C_0) \otimes \overline{\tau}^*(\mathcal{O}_C(K_C) \otimes \mathcal{O}_X(H)). \tag{4.4}$$

Let $C_{\overline{S}}$ be the scheme theoretic intersection of $\overline{S}$ and F, which is an integral curve which is a section over $\overline{\tau}$. Comparing (4.3) and (4.4), we have that

$$\mathcal{O}_F(-2C_{\overline{S}}) \cong \mathcal{O}_F(-2C_0) \otimes \overline{\tau}^*(\mathcal{O}_C(K_C) \otimes \mathcal{O}_X(H) \otimes \mathcal{O}_S(3(A+2B+3\Delta))) \cong \mathcal{O}_F(-2C_0)$$

since $\mathcal{O}_C(K_C) \cong \mathcal{O}_X(G) \otimes \mathcal{O}_C$ by adjunction, as $\mathcal{O}_X(G) \otimes \mathcal{O}_S \cong \mathcal{O}_S(C)$. Since $(C_0^2)_F < 0$, we have that $C_0 = C_{\overline{S}}$.

We have that

$$\overline{\tau}^*(\mathcal{O}_X(S) \otimes \mathcal{O}_C) \cong \tau^*\mathcal{O}_X(S) \otimes \mathcal{O}_F \cong \mathcal{O}_Y(\overline{S}+F) \otimes \mathcal{O}_F \cong \mathcal{O}_F(C_{\overline{S}}) \otimes \mathcal{O}_Y(F).$$

Thus

$$\mathcal{O}_Y(F) \otimes \mathcal{O}_F \cong \mathcal{O}_F(-C_0) \otimes \overline{\tau}^*(\mathcal{O}_X(S) \otimes \mathcal{O}_C),$$

where

$$\begin{aligned} \deg(\mathcal{O}_X(S) \otimes \mathcal{O}_C) &= \deg \mathcal{O}_S(-3(A+2B+3\Delta)) \otimes \mathcal{O}_C \\ &= (-3(A+2B+3\Delta)\cdot 3(A+B+\Delta)) \\ &= -9 \times 12 = -108. \end{aligned}$$

Thus $\mathcal{O}_Y(F) \otimes \mathcal{O}_F$ is represented in $(\mathrm{Pic}(F)/\equiv) \otimes \mathbb{R}$ by the class of $-C_0 - 108f$.

Suppose that γ is an irreducible curve on F which is not equal to C_0 and is not equal to a fiber over a closed point of C. There exists $n \in \mathbb{Z}$ and a divisor δ on C such that $\gamma \sim nC_0 + \overline{\tau}^*(\delta)$. Then $(\gamma \cdot f) > 0$ implies $n > 0$ and $(\gamma \cdot C_0) \geq 0$ implies $n(C_0^2) + \deg\delta \geq 0$. Thus $\deg\delta \geq -n(C_0)^2 = n162 > 0$.

We now compute

$$(\gamma^2) = n^2(C_0^2) + 2n\deg\delta \geq n^2(C_0^2) - 2n^2(C_0^2) = -n^2(C_0^2) > 0.$$

Thus C_0 is the only irreducible curve on F with negative intersection number.

It follows that $\overline{\mathrm{Eff}}(F) = \mathbb{R}_+C_0 + \mathbb{R}_+f$ and

$$\overline{\mathrm{Nef}}(F) = \{nC_0 + mf \mid n, m \geq 0 \text{ and } m \geq 162n\} = \mathbb{R}_+(C_0+162f) + \mathbb{R}_+f.$$

Let $j = n\left(3 - \frac{\sqrt{3}}{3}\right)$. On F, we have the numerical equivalence

$$(-n\overline{S} - jF)\cdot F \equiv (j-n)C_0 + 108jf = n\left(2 - \frac{\sqrt{3}}{3}\right)C_0 + 108n\left(3 - \frac{\sqrt{3}}{3}\right)f.$$

Letting $a = n\left(2 - \frac{\sqrt{3}}{3}\right)$ and $b = 108n\left(3 - \frac{\sqrt{3}}{3}\right)$, we have that

$$\frac{b}{a} = \frac{108}{33}\left(51 + 3\sqrt{3}\right) > 162.$$

Now suppose that $j = n$. Then

$$(-n\overline{S} - jF) \cdot F \equiv 108jf.$$

The nature of the sections of $\mathcal{O}_Y(-n\overline{S} - jF)$ for $n, j \in \mathbb{N}$ is determined by which of three separate regions of the positive quadrant of the plane contains the point (n, j). They are:

1) $j < n$
2) $n \leq j < n\left(3 - \frac{\sqrt{3}}{3}\right)$ and
3) $n\left(3 - \frac{\sqrt{3}}{3}\right) < j$.

In case 1),

$$\mathcal{O}_Y(-n\overline{S} - jF) \otimes \mathcal{O}_F \notin \overline{\mathrm{Eff}}(F). \tag{4.5}$$

In case 2),

$$\mathcal{O}_Y(-n\overline{S} - jF) \otimes \mathcal{O}_{\overline{S}} \in \overline{\mathrm{Nef}}(\overline{S}) \text{ and } \mathcal{O}_Y(-n\overline{S} - jF) \otimes \mathcal{O}_F \in \overline{\mathrm{Nef}}(F). \tag{4.6}$$

In case 3),

$$\mathcal{O}_Y(-n\overline{S} - jF) \otimes \mathcal{O}_{\overline{S}} \notin \overline{\mathrm{Eff}}(\overline{S}). \tag{4.7}$$

Let $R = \mathcal{O}_{\overline{Z},q}$ (q is the point on $\overline{Z}$ which $\overline{S}$ and F contract to) and $U = Y \times_{\overline{Z}} \mathrm{Spec}(R)$ with the natural projective morphism $U \to \mathrm{Spec}(R)$ induced by $Y \to \overline{Z}$. The morphism $U \to \mathrm{Spec}(R)$ is the blow up of the m-primary ideal $\mathcal{I}_q$. An effective Cartier divisor $D = n\overline{S} + jF$ on U is anti-nef on U if and only if $\mathcal{O}_Y(-D) \otimes \mathcal{O}_{\overline{S}} \in \overline{\mathrm{Nef}}(\overline{S})$ and $\mathcal{O}_Y(-D) \otimes \mathcal{O}_F \in \overline{\mathrm{Nef}}(F)$. If D is anti-nef on U, then $\gamma_{\overline{S}}(D) = n$ and $\gamma_F(D) = j$ by Lemma 2.2.

We deduce the following theorem.

THEOREM 4.1. *Let $D = n\overline{S} + jF$ with $j, n \in \mathbb{N}$, an effective exceptional divisor on U.*

1) *Suppose that $j < n$. Then $\gamma_{\overline{S}}(D) = n$ and $\gamma_F(D) = n$.*
2) *Suppose that $n \leq j < n\left(3 - \frac{\sqrt{3}}{3}\right)$. Then $\gamma_{\overline{S}}(D) = n$ and $\gamma_F(D) = j$.*
3) *Suppose that $n\left(3 - \frac{\sqrt{3}}{3}\right) < j$. Then $\gamma_{\overline{S}}(D) = \frac{3}{9-\sqrt{3}}j$ and $\gamma_F(D) = j$.*

In all three cases, $-\gamma_{\overline{S}}(D)\overline{S} - \gamma_F(D)F$ is nef on U.

PROOF. If D is in case 1), then $\Gamma(U, \mathcal{O}_U(-nm(\overline{S} + F))) = \Gamma(U, \mathcal{O}_U(-mD))$ for all $m \in \mathbb{N}$ by (4.5) and $\overline{S} + F$ is anti-nef on U by (4.6). If D is in case 2), then D is anti-nef on U by (4.6). If D is in case 3), then $\Gamma(U, \mathcal{O}_U(-\lceil \frac{m3j}{9-\sqrt{3}}\overline{S} + mjF\rceil)) = \Gamma(U, \mathcal{O}_U(-mD))$ for all $m \in \mathbb{N}$ by (4.7) and $\frac{3}{9-\sqrt{3}}\overline{S} + F$ is anti-nef on U by (4.6). $\square$

From Theorem 4.1, (1.16) and Proposition 2.4, we have that for $D = n\overline{S} + jF$ with $m, j \in \mathbb{N}$,

$$\lim_{m \to \infty} \frac{\ell_R(R/I(mD)}{m^3} = -\frac{((-\gamma_{\overline{S}}(D)\overline{S} - \gamma_F(D)F)^3)}{3!}, \tag{4.8}$$

and we have by (1.3), Theorem 4.1, (1.15) and Proposition 2.4 that
(4.9)

$$\lim_{m\to\infty}\frac{\ell_R(R/I(mn\overline{S})I(mjF))}{m^3}=\sum_{i_1+i_2=3}\frac{1}{i_1!i_2!}e_R(\mathcal{I}(1)^{[i_1]},\mathcal{I}(2)^{[i_2]})n^{i_1}j^{i_2}$$
$$=\sum_{i_1+i_2=3}\frac{-1}{i_1!i_2!}\left((-\gamma_{\overline{S}}(\overline{S})\overline{S}-\gamma_F(\overline{S})F)^{i_1}\cdot(-\gamma_{\overline{S}}(F)\overline{S}-\gamma_F(F)F\right)^{i_2})n^{i_1}j^{i_2}$$
$$=\sum_{i_1+i_2=3}\frac{-1}{i_1!i_2!}\left(\left(-\overline{S}-F\right)^{i_1}\cdot\left(-\frac{3}{9-\sqrt{3}}\overline{S}-F\right)^{i_2}\right)n^{i_1}j^{i_2}.$$

We now make equations (4.8) and (4.9) explicit. To compute the necessary intersection numbers, we use the facts that

$$\mathcal{O}_Y(\overline{S})\otimes\mathcal{O}_{\overline{S}}\cong\mathcal{O}_S(-3(2A+3B+4\Delta)),\ \mathcal{O}_Y(F)\otimes\mathcal{O}_{\overline{S}}\cong\mathcal{O}_S(3(A+B+\Delta)),$$
$$\mathcal{O}_Y(\overline{S})\otimes\mathcal{O}_F\cong\mathcal{O}_F(C_0),\ \mathcal{O}_Y(F)\otimes\mathcal{O}_F\equiv\mathcal{O}_F(-C_0-108f),$$

to calculate that

$$(\overline{S}^3)=(-3(2A+3B+4\Delta)\cdot-3(2A+3B+4\Delta))_S=9\times 52=468$$
$$(\overline{S}^2\cdot F)=(-3(2A+3B+4\Delta)\cdot 3(A+B+\Delta))_S=-9\times 18=-162$$
$$(\overline{S}\cdot F^2)=(3(A+B+\Delta)\cdot 3(A+B+\Delta))_S=9\times 6=54$$
$$(F^3)=((-C_0-108f)\cdot(-C_0-108f))_F=9\times 6=54.$$

The formulas of Theorems 1.2 and 1.3 of the introduction are now a consequence of Theorem 4.1, equations (4.8) and (4.9) and the above formulas computing intersection multiplicities.

References

[1] P. B. Bhattacharya, *The Hilbert function of two ideals*, Proc. Cambridge Philos. Soc. **53** (1957), 568–575. MR89835
[2] Sébastien Boucksom, Charles Favre, and Mattias Jonsson, *Differentiability of volumes of divisors and a problem of Teissier*, J. Algebraic Geom. **18** (2009), no. 2, 279–308, DOI 10.1090/S1056-3911-08-00490-6. MR2475816
[3] V. Cossart, C. Galindo, and O. Piltant, *Un exemple effectif de gradué non noethérien associé à une valuation divisorielle* (French, with English and French summaries), Ann. Inst. Fourier (Grenoble) **50** (2000), no. 1, 105–112. MR1762339
[4] Steven D. Cutkosky, *Zariski decomposition of divisors on algebraic varieties*, Duke Math. J. **53** (1986), no. 1, 149–156, DOI 10.1215/S0012-7094-86-05309-3. MR835801
[5] Steven Dale Cutkosky, *On unique and almost unique factorization of complete ideals. II*, Invent. Math. **98** (1989), no. 1, 59–74, DOI 10.1007/BF01388844. MR1010155
[6] Steven Dale Cutkosky, *Asymptotic multiplicities of graded families of ideals and linear series*, Adv. Math. **264** (2014), 55–113, DOI 10.1016/j.aim.2014.07.004. MR3250280
[7] Steven Dale Cutkosky, *Asymptotic multiplicities*, J. Algebra **442** (2015), 260–298, DOI 10.1016/j.jalgebra.2015.03.017. MR3395062
[8] Steven Dale Cutkosky, *Teissier's problem on inequalities of nef divisors*, J. Algebra Appl. **14** (2015), no. 9, 1540002, 37, DOI 10.1142/S0219498815400022. MR3368254
[9] Steven Dale Cutkosky, *Mixed multiplicities of divisorial filtrations*, Adv. Math. **358** (2019), 106842, 50, DOI 10.1016/j.aim.2019.106842. MR4018547
[10] S.D. Cutkosky, *The Minkowski equality of filtrations*, Advances in Mathematics 388 (2021).
[11] Steven Dale Cutkosky, Jürgen Herzog, and Ana Reguera, *Poincaré series of resolutions of surface singularities*, Trans. Amer. Math. Soc. **356** (2004), no. 5, 1833–1874, DOI 10.1090/S0002-9947-03-03346-4. MR2031043
[12] Steven Dale Cutkosky, Parangama Sarkar, and Hema Srinivasan, *Mixed multiplicities of filtrations*, Trans. Amer. Math. Soc. **372** (2019), no. 9, 6183–6211, DOI 10.1090/tran/7745. MR4024518
[13] S. D. Cutkosky and V. Srinivas, *On a problem of Zariski on dimensions of linear systems*, Ann. of Math. (2) **137** (1993), no. 3, 531–559, DOI 10.2307/2946531. MR1217347

[14] Steven Dale Cutkosky, Hema Srinivasan, and Jugal Verma, *Positivity of mixed multiplicities of filtrations*, Bull. Lond. Math. Soc. **52** (2020), no. 2, 335–348, DOI 10.1112/blms.12328. MR4171369

[15] Lawrence Ein, Robert Lazarsfeld, and Karen E. Smith, *Uniform approximation of Abhyankar valuation ideals in smooth function fields*, Amer. J. Math. **125** (2003), no. 2, 409–440. MR1963690

[16] William Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984, DOI 10.1007/978-3-662-02421-8. MR732620

[17] Kriti Goel, R. V. Gurjar, and J. K. Verma, *Minkowski inequality and equality for multiplicity of ideals of finite length in Noetherian local rings*, Contributions in algebra and algebraic geometry, Contemp. Math., vol. 738, Amer. Math. Soc., Providence, RI, 2019, pp. 33–56, DOI 10.1090/conm/738/14877. MR4014963

[18] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. MR0463157

[19] Daniel Katz, *Note on multiplicity*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1021–1026, DOI 10.2307/2047584. MR929434

[20] Daniel Katz and J. K. Verma, *Extended Rees algebras and mixed multiplicities*, Math. Z. **202** (1989), no. 1, 111–128, DOI 10.1007/BF01180686. MR1007742

[21] Kiumars Kaveh and Askold Khovanskii, *Convex bodies and multiplicities of ideals*, Proc. Steklov Inst. Math. **286** (2014), no. 1, 268–284, DOI 10.1134/S0081543814060169. Reprint of Tr. Mat. Inst. Steklova **286** (2014), 291–307. MR3482603

[22] Kiumars Kaveh and A. G. Khovanskii, *Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. of Math. (2) **176** (2012), no. 2, 925–978, DOI 10.4007/annals.2012.176.2.5. MR2950767

[23] Steven L. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. (2) **84** (1966), 293–344, DOI 10.2307/1970447. MR206009

[24] R. Lazarsfeld, Positivity in Algebraic Geometry Vol. 1, Springer Verlag, Berlin, 2004.

[25] Robert Lazarsfeld and Mircea Mustaţă, *Convex bodies associated to linear series* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 5, 783–835, DOI 10.24033/asens.2109. MR2571958

[26] H. T. Muhly and M. Sakuma, *Asymptotic factorization of ideals*, J. London Math. Soc. **38** (1963), 341–350, DOI 10.1112/jlms/s1-38.1.341. MR159837

[27] David Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970. MR0282985

[28] Mircea Mustaţă, *On multiplicities of graded sequences of ideals*, J. Algebra **256** (2002), no. 1, 229–249, DOI 10.1016/S0021-8693(02)00112-6. MR1936888

[29] Andrei Okounkov, *Why would multiplicities be log-concave?*, The orbit method in geometry and physics (Marseille, 2000), Progr. Math., vol. 213, Birkhäuser Boston, Boston, MA, 2003, pp. 329–347. MR1995384

[30] D. Rees, *$\mathfrak{a}$-transforms of local rings and a theorem on multiplicities of ideals*, Proc. Cambridge Philos. Soc. **57** (1961), 8–17, DOI 10.1017/s0305004100034800. MR118750

[31] D. Rees, *Multiplicities, Hilbert functions and degree functions*, Commutative algebra: Durham 1981 (Durham, 1981), London Math. Soc. Lecture Note Ser., vol. 72, Cambridge Univ. Press, Cambridge-New York, 1982, pp. 170–178. MR693635

[32] David Rees, *Izumi's theorem*, Commutative algebra (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 15, Springer, New York, 1989, pp. 407–416, DOI 10.1007/978-1-4612-3660-3_22. MR1015531

[33] D. Rees and R. Y. Sharp, *On a theorem of B. Teissier on multiplicities of ideals in local rings*, J. London Math. Soc. (2) **18** (1978), no. 3, 449–463, DOI 10.1112/jlms/s2-18.3.449. MR518229

[34] Irena Swanson, *Mixed multiplicities, joint reductions and quasi-unmixed local rings*, J. London Math. Soc. (2) **48** (1993), no. 1, 1–14, DOI 10.1112/jlms/s2-48.1.1. MR1223888

[35] Craig Huneke and Irena Swanson, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006. MR2266432

[36] Bernard Teissier, *Cycles évanescents, sections planes et conditions de Whitney* (French), Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972), Soc. Math. France, Paris, 1973, pp. 285–362. Astérisque, Nos. 7 et 8. MR0374482

[37] B. Teissier, *Sur une inégalité pour les multiplicitiés (Appendix to a paper by D. Eisenbud and H. Levine)*, Ann. Math. 106 (1977), 38 - 44.

[38] B. Teissier, *On a Minkowski-type inequality for multiplicities. II*, C. P. Ramanujam—a tribute, Tata Inst. Fund. Res. Studies in Math., vol. 8, Springer, Berlin-New York, 1978, pp. 347–361. MR541030

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