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A generalization of coefficient ideals

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Abstract. In this paper we give a generalization of the coefficient ideals of an $\mathfrak{m}$-primary ideal I in a quasi-unmixed local ring R with infinite residue field.

1. Introduction

Let $(R, \mathfrak{m})$ be a d-dimensional quasi-unmixed local ring with infinite residue field $R/\mathfrak{m}$. Let I be an $\mathfrak{m}$-primary ideal of R. Shah proved that for each $k \in \{1, \ldots, d\}$, there exists an unique largest ideal I_k containing I such that $e_i(I_k) = e_i(I)$ for $i = 0, ..., k$. The ideals I_k are called *coefficient ideals* and form a chain $I \subseteq I_d \subseteq \cdots \subseteq I_1 \subseteq \overline{I}$, where $\overline{I}$ denotes the integral closure of I. Let $I^* = \bigcup_{N \geq 1}(I^{N+1} : I^N)$ be the Ratliff-Rush closure of the ideal I. If I is a regular ideal then by [**S**, Corollary 1(E)] we have $I_d = I^*$. Coefficient ideals have been studied in many paper and have many generalizations (see for instance, [**HPV**] [**C**], [**C2**], [**C3**], [**CPF**], [**HJLS**], [**HLS**], [**HL**], [**LP**], [**LP2**]). In [**S**, Theorem 2 and Theorem 3], Shah provide structures for these ideals as follows. Once a k is fixed, he proved that

$$I_k = \bigcup_{N,\ \underline{x}} (I^{N+1} : x_1, \ldots, x_k), \tag{1.1}$$

where $N \geq 1$ is an integer and $\underline{x} = (x_1, \ldots, x_k, \ldots, x_d)$ is a minimal reduction of I^N. Also, there exists $N \geq 1$ and a minimal reduction $(x_1, \ldots, x_k, \ldots, x_d)$ of I^N such that

$$I_k = (I^{N+1} : x_1, \ldots, x_k) \text{ for all } k \in \{1, \ldots, d\}.$$

As an application, Shah [**S**] used these ideals to control the height of the associated prime ideals of the associated graded ring $G_I(R) = \bigoplus_{n \geq 0} \frac{I^n}{I^{n+1}}$. More precisely, he showed that $\operatorname{ht}(P) < k$ for every $P \in \operatorname{Ass}(G_I(R))$ if and only if $I^n = (I^n)_k$ for all $n \geq 1$ (see [**S**, Theorem 4]).

Our aim is to generalize the coefficient ideals, and consequently, the above results. Let K be an ideal such that $I^{n+1} \subseteq KI^n$ for some $n \geq 0$. Let $H_K(I, n) = \ell(\frac{R}{KI^n})$ be the Hilbert function with respect to I and K. Since $H_K(I, n) = \ell(R/I^n) + \ell(\frac{I^n}{KI^n})$, one can obtain that $H_K(I, n)$ is a polynomial of degree d for

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$n \gg 0$, which we may be denoted by

$$P_K(I,n) = g_0(I)\binom{n+d-1}{d} - g_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d g_d(I).$$

We prove in Theorem 2.5 that there exists an unique largest ideals $I_{[k]} \supseteq I$, with $1 \le k \le d$, such that

(a) $g_i(I) = g_i(I_{[k]})$ for $0 \le i \le k$;

(b) $I \subseteq I_{[d]} \subseteq \cdots \subseteq I_{[1]} \subseteq \overline{I}$.

The ideal $I_{[k]}$ will be called *the kth coefficient ideal* of I *with respect to the ideal* K. It coincides with the coefficient ideal I_k defined by Shah in [**S**] by taking $K = R$. Also, if $\operatorname{grade}(I,K) > 0$ we obtain in Corollary 2.6 that $I_{[d]} = r(I,K)$, where the ideal $r(I,K)$, defined by Puthenpurakal and Zulfeqarr in [**PZ**], is given by

$$r(I,K) = \bigcup_{N \ge 1} (KI^{N+1} : KI^N).$$

Next, we find the first structure of $I_{[k]}$ to be

$$I_{[k]} = \bigcup_{N,\ \underline{x}} (KI^{N+1} : K(x_1, ..., x_k))$$

where $N \ge 1$ and $\underline{x} = (x_1, \ldots, x_k, \ldots, x_d$ is a minimal reduction of I^N (see Theorem 2.8). A second and better structure is obtained in Theorem 2.9 by proving that there exists $N \ge 1$ and a minimal reduction $(x_1, \ldots, x_d)$ of I^N such that

$$I_{[k]} = (KI^{N+1} : K(x_1, \ldots, x_k)) \text{ for all } k \in \{1, \ldots, d\}.$$

Now let $G_I(K) = \bigoplus_{n \ge 0} \frac{KI^n}{KI^{n+1}}$. It is in a natural way a graded module over $G_I(R)$. Its main properties may be found in section 4.4 from [**BH**]. In conclusion, it can be shown (Theorem 2.10) that if $KI^n = (I^n)_{[k]}$ for all $n \ge 1$ then

$$\operatorname{ht}(P) < k \text{ for all } P \in \operatorname{Ass}_{G_I(R)}(G_I(K)). \tag{1.2}$$

Also, conversely, if (1.2) holds then

$$KI^n : K = (I^n)_{[k]} \text{ for all } n \ge 1.$$

If $K = R$, these two above statements provide [**S**, Theorem 4].

2. Coefficient ideals

Let R be a Noetherian ring, let I be an ideal of R and let M be a finitely generated R-module. In [**PZ**], Puthenpurakal and Zulfeqarr defined an ideal in R called the *Ratliff-Rush ideal I with respect M* to be

$$r(I,M) = \bigcup_{N \ge 1} (I^{N+1}M : I^N M).$$

THEOREM 2.1. *Let M be a finitely generated R-module. If* $\operatorname{grade}(I,M) > 0$ *then*

$$r(I,M)^n M = I^n M \quad \textit{for} \quad n \gg 0.$$

Moreover, if $J^n M = I^n M$ for $n \gg 0$ then $J \subseteq r(I,M)$.

PROOF. In [**PZ**, Corollary 2.6], it is proved that $r(I,M)^n M = I^n M$ for $n \gg 0$. By hypothesis, there exists a positive integer q such that $J^n M = I^n M$ for $n \geq q$. Let $n \geq 2q$. Then

$$K(I+J)^n = (KI^n + KI^{n-1}J + \cdots + KI^{n-q+1}J^{q-1})$$
$$+(KI^{n-q}J^q + KI^{n-q-1}J^{q+1} + \cdots + KIJ^{n-1} + KJ^n).$$
$$= KJ^n + KI^n$$
$$= KJ^n = KI^n$$

Thus $JKI^{n-1} \subseteq KI^n$. Therefore $J \subseteq r(I,M)$. □

From now on, let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and let I be an $\mathfrak{m}$-primary ideal of R. Also, assume $R/\mathfrak{m}$ is infinite.

Let $H(I,n) = \ell(R/I^n)$ denote the Hilbert function of the ideal I. It is known that for $n \gg 0$ it coincides with a polynomial of degree d, which we denote by

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I).$$

Let K be an ideal such that $I^{n+1} \subseteq KI^n$ for some $n \geq 0$. Let $H_K(I,n) = \ell(\frac{R}{KI^n})$ be the Hilbert function with respect to I and K. Since $H_K(I,n) = \ell(R/I^n) + \ell(\frac{I^n}{KI^n})$, it follows that $H_K(I,n)$ is a polynomial of degree d for $n \gg 0$, which we may be denoted by

$$P_K(I,n) = g_0(I)\binom{n+d-1}{d} - g_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d g_d(I).$$

Let $\overline{I}$ be the integral closure of the ideal I. We recall the following Theorem due to Rees.

THEOREM 2.2. (Rees) *Let $(R,\mathfrak{m})$ be a quasi-unmixed local ring, and let $I \subseteq J$ be two ideals $\mathfrak{m}$-primary ideals. Then $J \subseteq \overline{I}$ if and only if $e_0(I) = e_0(J)$.*

By convention, a polynomial of degree ≤ -1 is the zero polynomial.

REMARK 2.3. Let $(R,\mathfrak{m})$ be a Noetherian local ring and $\dim R \geq 1$. Suppose $I \subseteq J$ are $\mathfrak{m}$-primary ideals and fix k such that $1 \leq k \leq d$. Let K be an ideal such that $I^{n+1} \subseteq KI^n$ for some $n \geq 0$. Then for all n large, $g_i(I) = g_i(J)$ with $0 \leq i \leq k$ if and only if $\deg \ell(KJ^n/KI^n) \leq d-(k+1)$ for $n \gg 0$.

PROOF. It suffices to observe that, for n large,

$$\ell\left(\frac{KJ^n}{KI^n}\right) = \ell\left(\frac{R}{KI^n}\right) - \ell\left(\frac{R}{KJ^n}\right) = \sum_{i=0}^{d}(-1)^i[g_i(I) - g_i(J)]\binom{n+d-i-1}{d-i}.$$

□

REMARK 2.4. Let $(R,\mathfrak{m})$ be a Noetherian local ring and $\dim R \geq 1$. Suppose $I \subseteq I' \subseteq J$ are $\mathfrak{m}$-primary ideals and fix k such that $1 \leq k \leq d$. Then $g_i(I) = g_i(J)$ with $0 \leq i \leq k$ if and only if $g_i(I) = g_i(I') = g_i(J)$ with $0 \leq i \leq k$.

PROOF. We just use that $\ell(K(I')^n/KI^n) \leq \ell(KJ^n/KI^n)$ and apply Remark 2.3. □

THEOREM 2.5. *Let* $(R,\mathfrak{m})$ *be a quasi-unmixed local ring. Assume that* $R/\mathfrak{m}$ *is infinite and* $\dim R = d \geq 1$. *Let* I *be* $\mathfrak{m}$*-primary ideals and let* K *be an ideal such that* $I^{n+1} \subseteq KI^n$ *for some* $n \geq 0$. *Then there exists an unique largest ideals* $I_{[k]} \supseteq I$, *with* $1 \leq k \leq d$, *such that*

(a) $g_i(I) = g_i(I_{[k]})$ *for* $0 \leq i \leq k$;

(b) $I \subseteq I_{[d]} \subseteq \cdots \subseteq I_{[1]} \subseteq \overline{I}$.

The ideal $I_{[k]}$ *will be called* the kth coefficient ideal *of* I with respect to the ideal K.

PROOF. For n large, the lengths $\ell(\frac{R}{KI^n}), \ell(\frac{R}{I^n}), \ell(\frac{I^n}{KI^n})$ may be written as polynomials in n, denoted by $P_K(I,n)$, $P_I(n)$ and $P(F_K(I),n)$ respectively, as follows:

$$\begin{array}{rcl} P_K(I,n) & = & g_0\binom{n+d-1}{d} - g_1\binom{n+d-2}{d-1} + \cdots + (-1)^d g_d \\ P_I(n) & = & e_0\binom{n+d-1}{d} - e_1\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d \\ P(F_K(I),n) & = & f_0\binom{n+d-2}{d-1} - f_1\binom{n+d-3}{d-2} + \cdots + (-1)^{d-1} f_{d-1} \end{array}$$

Since $P_K(I,n) = P(F_K(I),n) + P_I(n)$ one may conclude that $g_0 = e_0$ and $g_i = e_i + f_{i-1}$ for $1 \leq i \leq d$.

For each $k = 1,...,d$, consider the set

$$V_k = \{L \mid L \text{ is an ideal of } R \text{ such that } L \supseteq I \text{ and } g_i(I) = g_i(L) \text{ for every } 0 \leq i \leq k\ \}.$$

Firstly note that if $L \in V_k$ then in particular, $g_0(I) = g_0(L)$, and by Theorem 2.2 we get $L \subseteq \overline{I}$.

Since $I \in V_k$ and R is Noetherian there exists a maximal element $J \in V_k$. Now we prove that J is the unique maximal element in V_k. In fact, let $L \in V_k$ and $x \in L$. Since $I \subseteq (I,x) \subseteq L$, by Remark 2.4 we have $g_i(I) = g_i(I,x) = g_i(J)$ for $0 \leq i \leq k$. In particular, by Rees' theorem, $I \subseteq (I,x)$ is a reduction so that $x \in \overline{I}$. Since $I \subseteq J$ one gets $\overline{I} \subseteq \overline{J}$. Hence $(J,x) \subseteq \overline{J}$, so as $(J,x)^{t+1} = (J,x)^t J$ for some t and then $(J,x)^n = (J,x)^t J^{n-t}$ for $n \geq t$. We have, for all $n \geq t$,

$$\begin{array}{rl} \ell(K(J,x)^n/KJ^n) & = \ell\big(K(J,x)^t J^{n-t}/KJ^n\big) = \ell\big(K(J^n,\ J^{n-1}x,...,J^{n-t}x^t)/KJ^n\big) \\ & \leq \sum_{i=1}^t \ell\big(\frac{K(J^{n-i}x^i)+KJ^n}{KJ^n}\big) \leq \sum_{i=1}^t \ell\big(\frac{K(J^{n-i}x^i)+KJ^n}{KI^n}\big) \\ & \leq \sum_{i=1}^t \left[\ell\big(\frac{K(J^{n-i}x^i)+KI^n}{KI^n}\big) + \ell\big(\frac{KJ^n}{KI^n}\big)\right] \\ & = \sum_{i=1}^t \left[\ell\left(\frac{KJ^{n-i}x^i+KI^n}{KI^{n-i}x^i+KI^n}\right) + \ell\big(\frac{KI^{n-i}x^i+KI^n}{KI^n}\big) + \ell\left(\frac{KJ^n}{KI^n}\right)\right] \\ & \leq \sum_{i=1}^t \left[\ell\left(\frac{KJ^{n-i}}{KI^{n-i}}\right) + \ell\left(\frac{K(I,x)^n}{KI^n}\right) + \ell\left(\frac{KJ^n}{KI^n}\right)\right]. \end{array}$$

The first term after the last sum was obtained through a surjective map (multiplication by x^i). Once $g_i(I) = g_i(I,x)$ and $g_i(I) = g_i(J)$ holds for $0 \leq i \leq k$, one can use Remark 2.3 to obtain $\ell(K(J,x)^n/KJ^n) \leq P(n)$ where $P(n)$ is a polynomial in n of degree at most $d-(k+1)$. This implies, by the same remark, that $g_i(J) = g_i(J,x)$ for $0 \leq i \leq k$. However, J is maximal in V_k, so that we must have $J = (J,x)$. Therefore, $L \subseteq J$ and thus J is the unique maximal in V_k. We denote this ideal by $I_{[k]}$. □

COROLLARY 2.6. *If* $\operatorname{grade}(I,K) > 0$ *then*

$$I_{[d]} = r(I,K).$$

PROOF. See Theorem 2.1. □

For our purposes we will use a slightly more general version of [**CPF**, Lemma 2.3]. To prove it we use Rees valuations, see [**S**] for a summary of the main properties.

LEMMA 2.7. *Let* $(R, \mathfrak{m})$ *be a quasi-unmixed local ring of dimension* $d \geq 1$. *Assume that* $R/\mathfrak{m}$ *is infinite. Let* I *be an* $\mathfrak{m}$*-primary ideal in* R *and let* K *be an ideal in* R *which is not contained in any minimal prime of* R. *Let* $(x_1, \ldots, x_d)$ *be a minimal reduction of* I^N *for some* $N \geq 1$. *If* $yx_iK \subseteq KI^{N+1}$ *for some* i *then* $y \in \overline{I}$.

PROOF. Let $\hat{R}$ be the completion of R with respect to $\mathfrak{m}$. By hypothesis we get $yx_iK\hat{R} \subseteq KI^{N+1}\hat{R}$, so that $y\hat{x_i}\hat{K}\hat{R} \subseteq \hat{K}\hat{I}^{N+1}$. Also, $\hat{x_1}, \ldots, \hat{x_s}$ is a minimal reduction of $\hat{I}^N$. If we assume the lemma holds for $\hat{R}$, we would conclude that $y \in \overline{\hat{I}} \cap R = \overline{I}$ by [**SH**, Proposition 1.6.2]. Hence we may assume R is complete. Further, we may assume R is a domain by using [**SH**, Proposition 1.1.5(2)]. Under these assumptions, R is an analytically unramified, universally catenary domain. In this way, by [**S**, Lemma 2(B)] we have $v(x_i) = v(I^N)$ for all Rees valuation v of I^N. The inclusion $yx_iK \subseteq KI^{N+1}$ implies $v(KI^{N+1}) \leq v(yx_iz) = v(y) + v(x_i) + v(z)$ for all $z \in K$, so that $v(KI^{N+1}) \leq v(y) + v(I^N) + v(K)$. As $v(K) + v(I) + v(I^N) \leq v(KI^{N+1})$ it follows that $v(y) \geq v(I)$. Thus $v(y) \geq \frac{1}{N} v(I^N)$ for all Rees valuation v of I^N.

Now set $v_I(x) = \sup\{N \,:\, x \in I^N\}$, $v_I(0) = \infty$, and set $\overline{v}_I(x) = \lim_{N\to\infty} \frac{v_I(x^N)}{N}$ for $x \in R$. By [**S**, (V.9)] we get $\overline{v}_{I^N}(y) \geq \frac{1}{N}$. Also, by [**S**, (V.10)] one gets $\overline{v}_I(y) \geq 1$, and by [**S**, (V.8)] we conclude that $y \in \overline{I}$. □

Next result is a generalization of [**S**, Theorem 2].

THEOREM 2.8. *Assume the hypothesis of Theorem* 2.5. *Then*

$$I_{[k]} = \bigcup_{N,\ \underline{x}} (KI^{N+1} : K(x_1, \ldots, x_k))$$

where $N \geq 1$ *and* $\underline{x} = (x_1, \ldots, x_k, \ldots, x_d)$ *is a minimal reduction of* I^N.

PROOF. Let $y \in I_{[k]}$. For n large, we have

$$\ell\left(\frac{(I, yK)I^{n-1}K}{I^nK}\right) \leq \ell\left(\frac{(I, y)I^{n-1}K}{I^nK}\right) \leq \ell\left(\frac{(I, y)^nK}{I^nK}\right) \leq \ell\left(\frac{(I_k)^nK}{I^nK}\right),$$

where the last length is a polynomial in n of degree at most $d - (k+1)$. Let

$$L = \frac{(I, y)K}{IK} G_I(K) = \bigoplus_{n\geq 1} \frac{(I, yK)I^{n-1}K}{I^nK}$$

be the $G_I(R)$-submodule of $G_I(K)$ generated by $(I, y)K/IK$, in particular, a homogeneous ideal. Let $A := \operatorname{ann}_{G_I(R)}(L)$. By [**BH**, Theorem 4.1.3], the length $\ell(L_n)$ is eventually a polynomial in n of degree $\dim_{G_I(R)} L - 1 = \dim G_I(R)/A - 1 \leq d - (k+1)$, in a way that $\dim G_I(R)/A \leq d - k$. By using [**S**, Lemma 2(E)] there exist homogeneous elements $x_1^\circ, \ldots, x_d^\circ$ in $G_I(R)$ of the same degree, say N, satisfying $\ell(G_I(R)/(x_1', \ldots, x_d')) < \infty$ and $x_1', \ldots, x_k' \in A$. Let $x_1, \ldots, x_d \in I^N$ be such that their canonical images in $G_I(R)$ are $x_1', \ldots, x_d'$, respectively. By [**S**, Lemma 2(A)], the ideal $(x_1, \ldots, x_d)$ is a minimal reduction of I^N. Moreover, as $x_i' = x_i + I^{N+1} \in A$ for $1 \leq i \leq k$ we get $x_iyz + I^{N+1}K = (x_i + I^{N+1})(yz + IK) = 0'$ for all $z \in K$, so that $y \in (KI^{N+1} : K(x_1, \ldots, x_k))$.

For the converse inclusion, let $(x_1,\dots,x_d)$ be a minimal reduction of I^N where $N\geq 1$ is any fixed integer. By using Lemma 2.7 we obtain that $(KI^{N+1}:K(x_1,\dots,x_k))\subseteq \overline{I}$, so that $I\subseteq (KI^{N+1}:K(x_1,\dots,x_k))$ is a reduction. Set $L=(KI^{N+1}:K(x_1,\dots,x_k))$.

We now show that

$$\deg\ell\left(\frac{KL^n}{KI^n}\right)\leq d-(k+1). \tag{2.1}$$

By Remark 2.3, it will follow that $L=(KI^{N+1}:K(x_1,\dots,x_k))\subseteq I_{[k]}$ by maximality of $I_{[k]}$.

Let $R_J(K)=\oplus_{n\geq 0}KJ^nt^n$ denote the Rees algebra of an ideal J with respect to K. If $K=R$, we simply denote $R_J(K)=R(J)$. Since I is a reduction of the ideal $L=(KI^{N+1}:K(x_1,\dots,x_k))$, there exists an integer $n\geq 1$ such that $I^rL^n=L^{n+r}$ for all $r\geq 1$, so that $[R_L(K)]_{n+r}=KL^nt^nR(I)_r$. For $i=0,\dots,n$, let $c_{i1},\dots,c_{ir_i}$ be the generators of the R-module KL^i. Thus $R_L(K)=\sum c_{ij}t^iR(I)$, so that $R_L(K)$ is a finite $R(I)$-module. Let $M=\frac{R_L(K)}{R_I(K)}$, which is a finite $R(I)$-module. By [**HPV**, Corollary 4.7], the degree in (2.1) equals $\dim M-1$. To obtain (2.1), it suffices to get $\dim M\leq d-k$.

By [**HPV**, Proposition 4.6] we have

$$\dim\frac{R_L(K)}{R_I(K)}=\dim\frac{R_L(K)}{\mathfrak{m}R_L(K)+R_I(K)}.$$

Now consider the following ideal of $R(I)$:

$$H=\left(\mathfrak{m}R_L(K)+R_I(K):\ R_L(K)\right).$$

By definition of dimension of a module,

$$\dim\frac{R_L(K)}{\mathfrak{m}R_L(K)+R_I(K)}=\dim\frac{R(I)}{H}.$$

As $R_L(K)$ is a finite $R(I)$-module, we may write

$$R_L(K)=R(I)a_1t^{\alpha_1}+\cdots+R(I)a_mt^{\alpha_m},$$

where each $a_i\in K(KI^{N+1}:K(x_1,\dots,x_k))^{\alpha_i}$. Let $\alpha:=\max\{\alpha_i\}$. Now one can verify that $x_jt^N\cdot bt\in KI^{N+1}t^{N+1}$ for any $b\in K(KI^{N+1}:K(x_1,\dots,x_k))$ and $j\in\{1,\dots,k\}$, so that $(x_j)^\alpha t^{N\alpha}a_it^{\alpha_i}\subseteq R_I(K)$ for $i=1,\dots,m$ and $j=1,\dots,k$. Hence $((x_1)^\alpha t^{N\alpha},\dots,(x_k)^\alpha t^{N\alpha})\subseteq H$.

On the other hand, as $(x_1,\dots,x_d)$ is a reduction of I^N, we obtain that $((x_1)^\alpha,\dots,(x_d)^\alpha)$ is a reduction of $I^{N\alpha}$. By [**CPF**, Lemma 2.1], one gets

$$\dim\frac{F(I)}{(((x_1)^\alpha)^\circ,\dots,(x_k)^\alpha)^\circ)}=d-k,$$

where $F(I)$ denotes the fiber cone of I and $(((x_i)^\alpha)^\circ=(x_i)^\alpha+\mathfrak{m}I^{N\alpha}$.

In conclusion, since $((x_1)^\alpha t^{N\alpha},\dots,(x_k)^\alpha t^{N\alpha})+\mathfrak{m}R(I)\subseteq H$ and by using a surjective map

$$\frac{F(I)}{(\ ((x_1)^\alpha)^\circ,\dots,((x_k)^\alpha)^\circ\ )}\simeq\frac{R(I)}{((x_1)^\alpha t^{N\alpha},\dots,(x_k)^\alpha t^{N\alpha})+\mathfrak{m}R(I)}\to\frac{R(I)}{H},$$

one concludes that $\dim M\leq d-k$ as desired. □

THEOREM 2.9. *Assume the hypothesis of Theorem 2.5. There exists $N \geq 1$ and a minimal reduction $(x_1, \ldots, x_d)$ of I^N such that*

$$I_{[k]} = (KI^{N+1} : K(x_1, ..., x_k)) \text{ for all } k \in \{1, \ldots, d\}.$$

PROOF. Let $k \in \{1, \ldots, d\}$. By Remark 2.3, we have $\deg \ell(\frac{K(I_{[k]})^n}{KI^n}) \leq d-(k+1)$ for $n \gg 0$. Let

$$L_k = \frac{(I, I_{[k]})K}{IK} G_I(K) = \bigoplus_{n \geq 1} \frac{(I, I_{[k]}K)I^{n-1}K}{I^n K}$$

be the $G_I(R)$-submodule of $G_I(K)$ generated by $(I, I_{[k]})K/IK$. Note that $(I, I_{[k]}K)I^{n-1}K \subseteq K(I_{[k]})^n$, so $\deg \ell(\frac{(I, I_{[k]})I^{n-1}K}{KI^n}) \leq d-(k+1)$ for $n \gg 0$. Hence, by [**BH**, Theorem 4.1.3] the length $\ell((L_k)_n)$ is a polynomial in n of degree $\dim L_k - 1 \leq d-(k+1)$. Thus $\dim G_I(R)/A_k = \dim L_k \leq d-k$, where $A_k = \operatorname{ann}_{G_I(R)} L_k$. Since $I_{[d]} \subseteq \cdots \subseteq I_{[1]}$ we get $L_d \subseteq \cdots \subseteq L_1$, so that $A_1 \subseteq \cdots \subseteq A_d$. By [**S**, Lemma 2(F)], there exist $x'_1, \ldots, x'_d \in G_I(R)$, canonical images of $x_1, \ldots, x_d \in I^N$ for some $N \geq 1$, such that

$$\ell\left(\frac{G_I(R)}{(x'_1, \ldots, x'_d)}\right) < \infty \text{ and } x'_1, \ldots, x'_k \in A_k \text{ for } 1 \leq k \leq d.$$

Notice that $(x_1, \ldots, x_d)$ is a minimal reduction of I^N by [**S**, Lemma 2(A)]. Since $x'_1, \ldots, x'_k \in A_k$, it means $x'_i \frac{(I, I_{[k]})K}{IK} = 0'$ for $1 \leq i \leq k$, so that $x_i I_{[k]}K \subseteq KI^{N+1}$. Therefore $I_{[k]} \subseteq (KI^{N+1} : K(x_1, \ldots, x_k))$ for $1 \leq i \leq k$. In fact, the equality holds by Theorem 2.8. $\square$

In the next result, in case $K = R$, it provides [**S**, Theorem 4].

THEOREM 2.10. *Assume the hypothesis of Theorem 2.5. Fix $k \in \{1, \ldots, d\}$.*

(a) If $KI^n = (I^n)_{[k]}$ for all $n \geq 1$ then

$$\operatorname{ht}(P) < k \text{ for all } P \in \operatorname{Ass}_{G_I(R)}(G_I(K));$$

(b) If $\operatorname{ht}(P) < k$ for all $P \in \operatorname{Ass}_{G_I(R)}(G_I(K))$ then

$$KI^n : K = (I^n)_{[k]} \text{ for all } n \geq 1.$$

PROOF. Assume that $KI^n = (I^n)_{[k]}$ for all $n \geq 1$ and let $P \in \operatorname{Ass}_{G_I(R)}(G_I(K))$. Suppose that $\operatorname{ht}(P) \geq k$. We may write $P = (0' : y')$ where $y \in KI^{n-1} \backslash KI^n$ for some $n \geq 1$. Since $\dim G_I(R)/P \leq d-k$, by [**S**, Lemma 2(E)] there exist homogeneous elements $x'_1, \ldots, x'_d \in G_I(R)$ of the same degree, say $N \geq 1$, such that

$$\ell\left(\frac{G_I(R)}{(x'_1, \ldots, x'_d)}\right) < \infty \text{ and } x'_1, \ldots, x'_k \in P.$$

Thus $(x_1, \ldots, x_d)$ is a minimal reduction of I^N by [**S**, Lemma 2(A)]. Also, $(x_1^n, \ldots, x_d^n)$ is a minimal reduction of $(I^n)^N = I^{nN}$. Since $0' = (x_i^n)'y' = (x_i^n + (I^n)^{N+1})(y + KI^n)$ in $G_I(K)$ for $1 \leq i \leq k$ we get $yx_i^n \in KI^{nN+n-1+1}$. Hence $y \in (K(I^n)^{N+1} : x_1^n, \ldots, x_k^n)$ so that $y \in (K(I^n)^{N+1} : K(x_1^n, \ldots, x_k^n)) = KI^n$, which is a contradiction.

Now assume $\operatorname{ht}(P) < k$ for all $P \in \operatorname{Ass}_{G_I(R)}(G_I(K))$. Let $N, n \geq 1$ and let $x_1, \ldots, x_d$ be a minimal reduction of $(I^n)^N$. Let $x'_1, \ldots, x'_s$ denote their canonical image in $G_I(R)$ and let $x''_1, \ldots, x''_s$ denote their canonical image in $G_{I^n}(R)$. By

[**S**, Lemma 2(A)], we have $\operatorname{ht}(x'_1,\dots,x'_d)=d$. Also, since $G_I(R)$ is equidimensional (see [**SH**, Proposition 5.4.8]) it follows that $\operatorname{ht}(x'_1,\dots,x'_k)=k$. We claim that

$$\operatorname{grade}((x''_1,\dots,x''_k),G_{I^n}(K))>0. \tag{2.2}$$

In fact, if $\operatorname{grade}((x''_1,\dots,x''_k),G_I(K))=0$ then $(x''_1,\dots,x''_k)\subseteq P$ for some $P\in\operatorname{Ass}_{G_{I^n}(R)}(G_I(K))$. Then we may write $P=(0'':b'')$ for some $m\geq 0$ and some $b\in K(I^n)^m-K(I^n)^{m+1}=KI^{nm}-KI^{nm+n}$. There exists i with $0\leq i<n$ such that $b\in KI^{nm+i}-KI^{nm+i+1}$. Then note that $(x'_1,\dots,x'_k)\subseteq(0':b')$, where b' is the image of b in KI^{nm+i}/KI^{nm+i+1}. As $(0':b')\subseteq Q$ for some $Q\in\operatorname{Ass}_{G_I(R))}(G_I(K))$ (by avoidance lemma), we get $(x'_1,\dots,x'_k)\subseteq Q$, so that $k\leq\operatorname{ht}(Q)$, which is a contradiction. Now we claim that

$$(K(I^n)^{N+1}:K(x_1,\dots,x_k))\subseteq(KI^n:K).$$

In fact, let $y\in(K(I^n)^{N+1}:K(x_1,\dots,x_k))$. Then $\frac{yK+KI^n}{KI^n}(x_i+(I^n)^{N+1})=0''$ in $G_{I^n}(K)$ for $1\leq i\leq k$, so that $\frac{yK+KI^n}{KI^n}(x''_1,\dots,x''_k)=0''$, and from (2.2) one gets $yK\subseteq KI^n$, that is, $y\in(KI^n:K)$.

Moreover, note that if $z\in R$ is such that $zK\subseteq KI^n$ then $zx_iK\subseteq KI^n(I^n)^N=K(I^n)^{N+1}$ which means

$$(K(I^n)^{N+1}:K(x_1,\dots,x_k))=(KI^n:K)\text{ for all }n\geq 1.$$

From this we conclude that $(I^n)_{[k]}=(KI^n:K)$ for all $n\geq 1$ by Theorem 2.8 as $N,n\geq 1$ and the minimal reduction $x_1,\dots,x_d$ of $(I^n)^N$ were taken arbitrarily. □

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