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Four-dimensional homogeneous Kähler Ricci solitons

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Abstract. We show that four-dimensional homogeneous Kähler Ricci solitons are Einstein, rigid or an algebraic Ricci soliton on the only 3-symmetric space.

1. Introduction

A Riemannian manifold (M, g) is a *Ricci soliton* if there is a vector field X on M so that

$$\mathcal{L}_X g + \rho = \lambda g, \tag{1.1}$$

where $\mathcal{L}$ is the Lie derivative, ρ denotes the Ricci tensor and $\lambda \in \mathbb{R}$. Ricci solitons are self-similar solutions of the Ricci flow $\frac{\partial}{\partial t} g(t) = -2\rho(g(t))$, i.e., they are fixed points of the flow up to diffeomorphisms and rescaling. While Ricci flat metrics are the genuine fixed points of the flow, Einstein metrics remain constant up to scaling under the flow. Indeed, if $g(0)$ is an Einstein metric satisfying $\rho(0) = \lambda g(0)$, then $g(0)$ evolves with the flow as $g(t) = (1 - 2\lambda t)g(0)$. A Ricci soliton (M, g, X) is called *shrinking*, *steady* or *expading* if $\lambda > 0$, $\lambda = 0$ or $\lambda > 0$, respectively. We refer to [**6, 7**] for more information and references on Ricci solitons.

If the vector field $X = \nabla f$ is a gradient, then (1.1) becomes $\mathrm{Hes}_f + \rho = \lambda g$ for some potential function f and (M, g, f) is called a *gradient Ricci soliton.* A gradient Ricci soliton is called *rigid* if (M, g) splits as a product $\mathbb{R}^k \times N$, where N is Einstein with $\rho = \lambda g$ and the potential function $f(\,\cdot\,) = \frac{\lambda}{4}\|\pi_{\mathbb{R}^k}(\,\cdot\,)\|^2$ is determined by the projection on the Euclidean factor [**16**]. While homogeneous gradient Ricci solitons are rigid [**15**], there are irreducible homogeneous manifolds admitting generic Ricci solitons. Many of them are constructed by using the notion of algebraic Ricci soliton.

Based on the fact that Ricci solitons are self-similar solutions of the Ricci flow, Lauret [**12**] considered Lie groups and searched for fixed points of the flow up to automorphisms of the Lie group instead of diffeomorphisms. Let G be a Lie group with Lie algebra $\mathfrak{g}$. A left-invariant metric $\langle\,\cdot\,,\,\cdot\,\rangle$ on G is called an *algebraic Ricci soliton* if

$$\mathfrak{D} = \mathrm{Ric} - \lambda\, \mathrm{id} \tag{1.2}$$

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is a derivation of the Lie algebra, where Ric denotes the Ricci operator $\langle \operatorname{Ric} X, Y\rangle = \rho(X,Y)$. Let $\mathfrak{D}$ be a derivation given by (1.2) and let φ_t denote the one-parameter family of automorphisms of G determined by $d\varphi_t|_e = \exp \frac{t}{2}\mathfrak{D}$. Then the vector field X given by $X(p) = \frac{d}{dt}\varphi_t(p)|_{t=0}$ satisfies (1.1), thus defining a Ricci soliton on G (see [**10**, **12**, **13**] for more information on homogeneous Ricci solitons).

Our first purpose on this note is to prove the following result which gives an explicit description of four-dimensional homogeneous Kähler manifolds.

THEOREM 1.1. *A complete and simply connected four-dimensional Riemannian Kähler manifold is homogeneous if and only if it is symmetric or isometric to the* 3*-symmetric space.*

A consequence of the previous result provides an explicit description of four-dimensional homogeneous Kähler Ricci solitons as follows.

COROLLARY 1.2. *A four-dimensional complete and simply connected Riemannian homogeneous Kähler Ricci soliton is either Einstein, rigid or the* 3*-symmetric space, which is itself an algebraic soliton.*

We fix some notation on Kähler surfaces, homogeneous spaces and generalized symmetric spaces in Section 2. By using previous work of Bérard-Bergery [**2**] and Ovando [**14**] we analyze the homogeneous Kähler surfaces in Section 3 to proof Theorem 1.1. Finally the proof of Corollary 1.2 is given in Section 4.

2. Preliminaries

2.1. Four-dimensional geometry. We work at the purely algebraic level. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of an inner product vector space of Riemannian signature $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$ and let $\{e^1, e^2, e^3, e^4\}$ be the associated dual basis. Further, let $e^{ij} = e^i \wedge e^j$ and consider the induced metric on $\Lambda^2(V)$ given by $\langle\langle x\wedge y, z\wedge w\rangle\rangle = \langle x,z\rangle\langle y,w\rangle - \langle y,z\rangle\langle x,w\rangle$. In dimension four the Hodge star operator $\star$ is an endomorphism of the space of 2-forms which satisfies $\star^2 = \mathrm{id}_{\Lambda^2}$. Hence it induces a splitting $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, where $\Lambda^\pm = \{\alpha \in \Lambda^2 : \star\alpha = \pm\alpha\}$ denote the spaces of self-dual and anti-self-dual 2-forms. Furthermore,

$$E_1^\pm = (e^{12} \pm e^{34})/\sqrt{2}, \quad E_2^\pm = (e^{13} \mp e^{24})/\sqrt{2}, \quad E_3^\pm = (e^{14} \pm e^{23})/\sqrt{2}$$

is an orthonormal basis of $\Lambda^\pm = \operatorname{span}\{E_1^\pm, E_2^\pm, E_3^\pm\}$.

Since Λ^2 splits under the action of the Hodge star operator so does the curvature operator acting on the space of 2-forms as

$$R = \frac{\tau}{12}\,\mathrm{id}_{\Lambda^2} + \rho_0 + W^+ + W^-$$

where $W^\pm = \frac{1}{2}(W \pm \star W)$ denote the self-dual and the anti-self-dual Weyl curvature operators, τ is the scalar curvature and ρ_0 denotes the trace-free Ricci tensor.

2.2. Almost Hermitian four-manifolds. An *almost Hermitian manifold* is a Riemannian manifold (M, g) equipped with an orthogonal almost complex structure (i.e., a $(1,1)$-tensor field J satisfying $J^2 = -\,\mathrm{id}$ and $J^*g = g$). Any almost Hermitian structure (g, J) gives rise to a non-degenerate 2-form $\Omega(X,Y) = g(JX,Y)$ inducing an orientation which agrees with the one defined by the almost complex structure. Hence, orienting M by the almost complex structure, the Kähler form Ω is self-dual. Hence Ω defines a section of Λ^+ and conversely, any section of

Λ^+ (resp., Λ^-) defines an almost Hermitian structure on M (resp., on the manifold M with the opposite orientation).

For every almost Hermitian four-dimensional manifold $\Lambda^+ = \mathbb{R}\Omega \oplus \langle \mathbb{R}\Omega \rangle^\perp$ and thus the self-dual part of the Weyl curvature operator can be written as

$$W^+ = \left(\begin{array}{c|c} \dfrac{\kappa}{6} & W_2^+ \\ \hline {}^t(W_2^+) & W_3^+ - \dfrac{\kappa}{12}\,\mathrm{id}_{\Lambda^2} \end{array} \right)$$

where κ is the conformal scalar curvature, W_2^+ corresponds to the part of W^+ that interchanges the two factors of the splitting of Λ^+ and W_3^+ is a trace-free, self-adjoint endomorphism of $\langle \mathbb{R}\Omega \rangle^\perp$. Moreover, the traceless part of the Ricci tensor (which is not necessarily J-invariant) decomposes into its invariant and anti-invariant components $\rho_0 = \rho_0^i + \rho_0^a$, where $\rho_0^i(X,Y) = \frac{1}{2}(\rho_0(X,Y) + \rho_0(JX,JY))$ and $\rho_0^a(X,Y) = \frac{1}{2}(\rho_0(X,Y) - \rho_0(JX,JY))$ are J-invariant and J-anti-invariant, respectively.

(M,g,J) is said to be *almost Kähler* if the 2-form Ω is closed and (M,g,J) is said to be *Kähler* if, in addition, the almost complex structure is integrable (or, equivalently, the Kähler form Ω is parallel). Curvature identities for Kähler manifolds show that $\rho_0^a = 0$ and $W_2^+ = W_3^+ = 0$ in the Kähler setting. Apostolov, Armstrong and Drăghici showed in [**1**] that there is only one non-Kähler, almost Kähler four-dimensional manifold for which both the Ricci and the Weyl curvatures have the same algebraic symmetries as they have for a Kähler metric (see Theorem 2.2).

2.3. Homogeneous spaces. A connected Riemannian manifold (M,g) is said to be *homogeneous* if the group of isometries acts transitively on M. Further a Kähler manifold (M,g,J) is homogeneous if there is a group of J-preserving isometries which acts transitively on M. Sekigawa showed in [**17**] that any three-dimensional complete and simply connected homogeneous Riemannian manifold is a symmetric space or it is isometric to a Lie group with left-invatiant metric. Bérard-Bergery extended this result to the four-dimensional setting so that one has

THEOREM 2.1. [**2**] *Let (M,g) be a four-dimensional complete and simply connected homogeneous Riemannian manifold. Then (M,g) is either symmetric or it is isometric to a Lie group with a left-invariant metric.*

In particular, M is one of the groups $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$, $SU(2) \times \mathbb{R}$ or it is a solvable Lie group. Four-dimensional solvable Lie algebras are obtained as extensions of the three-dimensional unimodular Lie algebras: the abelian Lie algebra $\mathfrak{r}^3$, the Heisenberg algebra $\mathfrak{h}_3$, the Poincaré algebra $\mathfrak{e}(1,1)$ and the Euclidean algebra $\mathfrak{e}(2)$. It is an important observation that four-dimensional Lie groups which admit a left-invariant non-degenerate closed 2-form are necessarily solvable [**8**]. In particular any four-dimensional Kähler Lie group is solvable thus excluding the cases $\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$ and $SU(2) \times \mathbb{R}$.

2.4. 3-symmetric spaces. Let (M,g,J) be an almost Hermitian manifold. Then $\Theta = -\frac{1}{2}\,\mathrm{id} + \frac{\sqrt{3}}{2}J$ satisfies $\Theta^3 = \mathrm{id}$ and it determines a family of local cubic diffeomorphisms, i.e., a differentiable function $p \in M \mapsto \vartheta_p$ where ϑ_p is the diffeomorphism defined in a suitable small neighborhood of p so that $d\vartheta_p = \Theta$ and

p is the only fixed point of ϑ_p. Conversely given a family $p \mapsto \vartheta_p$ of local cubic diffeomorphisms, there exists an almost complex structure $\mathfrak{J}$ on M determined by $d\vartheta_p = -\frac{1}{2}\,\mathrm{id} + \frac{\sqrt{3}}{2}\mathfrak{J}_p$ (see [**9**]). A Riemannian *locally* 3*-symmetric space* is a manifold (M, g) equipped with a family of local cubic diffeomorphisms which are $\mathfrak{J}$-holomorphic isometries. Whenever the local cubic diffeomorphisms are globally defined one says that M is a 3*-symmetric space*.

There is only one Riemannian 3-symmetric space in dimension four, which is realized as the following metric on $\mathbb{R}^4$ with coordinates (t, x, y, z) (see [**11**]):

$$\begin{aligned} g \;&= \big((\sqrt{1+t^2+x^2} - t)dy^2 + (\sqrt{1+t^2+x^2} + t)dz^2 - 2x\,dydz\big) \\ &\quad +\beta\left((1+x^2)dt^2 + (1+t^2)dx^2 - 2tx\,dtdx\right)/(1+t^2+x^2), \end{aligned}$$

where β is a non-zero constant. Viewed as a homogeneous space G/K, the four-dimensional 3-symmetric space corresponds to the matrix groups (see [**11**])

$$G = \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } K = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1.$$

A straightforward calculation shows that the self-dual and anti-self-dual Weyl curvature operators satisfy $W^\pm = \pm\,\mathrm{diag}[\frac{1}{2\beta}, -\frac{1}{4\beta}, -\frac{1}{4\beta}]$ and thus $W^\pm$, have a distinguished eigenvalue. Hence any four-dimensional 3-symmetric space is naturally equipped with an almost complex structure J_+ and an opposite almost complex structure J_- determined by the one-dimensional eigenspaces $\ker(W^\pm \mp \frac{1}{2\beta}\,\mathrm{id}_{\Lambda^\pm})$. Moreover $(\mathbb{R}^4, g, J_+, J_-)$ is almost Kähler and opposite Kähler (i.e., $d\Omega_+ = 0$ and $\nabla J_- = 0$) and the curvature of the almost Kähler structure has the algebraic symmetries of the Kähler case (see, for example [**5**]). Now the previously mentioned rigidity result of Apostolov, Armstrong and Drăghici can be stated as follows.

THEOREM 2.2. [**1**] *Any strictly almost Kähler four-dimensional manifold whose curvature satisfies* $\rho_0^a = 0$*, and* $W_2^+ = W_3^+ = 0$ *is locally isometric to the unique four-dimensional 3-symmetric space.*

3. Homogeneous Kähler surfaces. The proof of Theorem 1.1

By Theorem 2.1 and the results in [**8**], a complete and simply connected Riemannian homogeneous Kähler four-manifold (M, g, J) is either symmetric or isometric to a solvable Lie group with a left-invariant Kähler structure. Since Ovando classified left-invariant indefinite Kähler structures on four-dimensional Lie groups, in what remains of this section we will examine the different possibilities in [**14**]. The proof of Theorem 1.1 follows after a case by case analysis of Ovando's classification. Following the notation in [**14**], the Kähler structures corresponding to the Lie algebras $\mathfrak{rh}_3$, $\mathfrak{r}'_2$, $\mathfrak{r}_{4,-1,-1}$ and $\mathfrak{d}_{4,1}$ are of neutral signature (cf. [**14**, Corollary 3.12]). Therefore, we omit those cases in what follows and consider separately the other posibilities.

3.1. The Lie algebra $\mathfrak{rr}_{3,0}$. The Lie algebra $\mathfrak{rr}_{3,0}$ is described, with respect to a suitable basis $\{e_1, e_2, e_3, e_4\}$, by the non-zero bracket (up to the usual symmetries) given by $[e_1, e_2] = e_2$. $\mathfrak{rr}_{3,0}$ admits a left-invariant Kähler structure determined by the complex structure $Je_1 = e_2$, $Je_3 = e_4$ and Kähler form $\Omega = ae^{12} + be^{34}$, where $ab \neq 0$. Then the associated metric is given by

$$g = a(e^1 \circ e^1 + e^2 \circ e^2) + b(e^3 \circ e^3 + e^4 \circ e^4).$$

Hence, the metric is either Riemannian or of neutral signature depending on the sign of ab. The Ricci operator takes the form $\operatorname{Ric} = -\frac{1}{a}\operatorname{diag}[1,1,0,0]$ with respect to the orthogonal basis $\{e_1, e_2, e_3, e_4\}$ and it is parallel. Hence the underlying manifold is isometric to the product of the Euclidean plane and a surface with constant Gauss curvature $G = \mathbb{R}^2 \times N(-\frac{1}{a})$, and therefore symmetric.

3.2. The Lie algebra $\mathfrak{rr}'_{3,0}$. The Lie algebra $\mathfrak{rr}'_{3,0}$ is described, with respect to a suitable basis $\{e_1, e_2, e_3, e_4\}$, by the non-zero brackets (up to the usual symmetries) given by $[e_1, e_2] = -e_3$, and $[e_1, e_3] = e_2$. $\mathfrak{rr}'_{3,0}$ admits a left-invariant Kähler structure determined by the complex structure $Je_1 = e_4$, $Je_2 = e_3$ and Kähler form $\Omega = ae^{14} + be^{23}$, where $ab \neq 0$. Then the associated metric is given by

$$g = a(e^1 \circ e^1 + e^4 \circ e^4) + b(e^2 \circ e^2 + e^3 \circ e^3).$$

Hence, the metric is Riemannian or of neutral signature. Moreover, the curvature tensor vanishes and the homogeneous space is flat.

3.3. The Lie algebra $\mathfrak{r}_2\mathfrak{r}_2$. The Lie algebra $\mathfrak{r}_2\mathfrak{r}_2$ is described, with respect to a suitable basis $\{e_1, e_2, e_3, e_4\}$, by the non-zero brackets (up to the usual symmetries) given by $[e_1, e_2] = e_2$, and $[e_3, e_4] = e_4$. $\mathfrak{r}_2\mathfrak{r}_2$ admits a left-invariant Kähler structure determined by the complex structure $Je_1 = e_2$, $Je_3 = e_4$ and Kähler form $\Omega = ae^{12} + be^{34}$, where $ab \neq 0$. Then the associated metric is given by

$$g = a(e^1 \circ e^1 + e^2 \circ e^2) + b(e^3 \circ e^3 + e^4 \circ e^4).$$

A straightforward calculation shows that the Ricci operator, when expressed on the orthogonal basis $\{e_1, e_2, e_3, e_4\}$, takes the form $\operatorname{Ric} = -\operatorname{diag}[\frac{1}{a}, \frac{1}{a}, \frac{1}{b}, \frac{1}{b}]$. Moreover, the Ricci tensor is parallel and hence the homogeneous space is the product of two surfaces of constant Gauss curvature $G = N_1(-\frac{1}{a}) \times N_2(-\frac{1}{b})$, and thus symmetric.

3.4. The Lie algebra $\mathfrak{r}'_{4,0,\lambda}$. The Lie algebra $\mathfrak{r}'_{4,0,\lambda}$ is described, with respect to a suitable basis $\{e_1, e_2, e_3, e_4\}$, by the non-zero brackets

$$[e_1, e_4] = -e_1, \quad [e_2, e_4] = \lambda e_3, \quad [e_3, e_4] = -\lambda e_2.$$

In this case, there exist two possible Kähler structures.

3.4.1. *Case 1.* $\mathfrak{r}'_{4,0,\lambda}$ admits a left-invariant Kähler structure determined by the complex structure $J_1 e_4 = e_1$, $J_1 e_2 = e_3$ and Kähler form $\Omega_1 = ae^{14} + be^{23}$, where $ab \neq 0$. Then the associated metric is given by

$$g_1 = -a(e^1 \circ e^1 + e^4 \circ e^4) + b(e^2 \circ e^2 + e^3 \circ e^3).$$

Hence, the metric is Riemannian or of neutral signature. A straightforward calculation shows that the Ricci operator, when expressed in the orthogonal basis $\{e_1, e_2, e_3, e_4\}$, takes the form $\operatorname{Ric} = \frac{1}{a}\operatorname{diag}[1,0,0,1]$ and it is parallel. Hence, the homogeneous space is the product of the Euclidean plane and a surface with constant Gauss curvature $G = \mathbb{R}^2 \times N(\frac{1}{a})$ and thus symmetric.

3.4.2. *Case 2.* $\mathfrak{r}'_{4,0,\lambda}$ admits a left-invariant Kähler structure determined by the complex structure $J_2 e_4 = e_1$, $J_2 e_2 = -e_3$ and Kähler form $\Omega_2 = ae^{14} + be^{23}$, where $ab \neq 0$. Then the associated metric is given by

$$g_2 = -a(e^1 \circ e^1 + e^4 \circ e^4) - b(e^2 \circ e^2 + e^3 \circ e^3).$$

Proceeding exactly as in 3.4.1, the homogeneous space is the product of the Euclidean plane and a surface with constant Gauss curvature $G = \mathbb{R}^2 \times N(\frac{1}{a})$, and thus symmetric.

3.5. The Lie algebra $\mathfrak{d}_{4,2}$. The Lie algebra $\mathfrak{d}_{4,2}$ is described, with respect to a suitable basis $\{e_1, e_2, e_3, e_4\}$, by the non-zero brackets

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -2e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = -e_3.$$

In this case, there exist two Kähler structures as follows. An important fact is that none of the Kähler structures on $\mathfrak{d}_{4,2}$ is symmetric.

3.5.1. *Case 1.* $\mathfrak{d}_{4,2}$ admits a left-invariant Kähler structure determined by the complex structure $J_1 e_4 = -e_2$, $J_1 e_1 = e_3$ and Kähler form $\Omega_1 = a(e^{14} + e^{23}) + be^{24}$, where $a \neq 0$. Then the associated metric is given by

$$g_1 = b(e^2 \circ e^2 + e^4 \circ e^4) + 2a(e^1 \circ e^2 + e^3 \circ e^4).$$

Hence, the metric is of neutral signature and we therefore omit this case.

3.5.2. *Case 2.* $\mathfrak{d}_{4,2}$ admits a left-invariant Kähler structure determined by the complex structure $J_2 e_4 = -2e_1$, $J_2 e_2 = e_3$ and Kähler form $\Omega_2 = ae^{14} + be^{23}$, where $ab \neq 0$. Then the associated metric is given by

$$g_2 = \frac{a}{2} e^1 \circ e^1 + 2ae^4 \circ e^4 + b(e^2 \circ e^2 + e^3 \circ e^3).$$

Hence, the metric is Riemannian or of neutral signature.

A straightforward calculation shows that the Ricci operator, when expressed in the orthogonal basis $\{e_1, e_2, e_3, e_4\}$, takes the form $\mathrm{Ric} = -\frac{3}{a}\,\mathrm{diag}[1, 0, 0, 1]$. Furthermore the self-dual and anti-self-dual Weyl curvature operators are given by

$$W^+ = \mathrm{sign}(ab) \begin{pmatrix} \frac{1}{2a} & 0 & 0 \\ 0 & \frac{1}{2a} & 0 \\ 0 & 0 & -\frac{1}{a} \end{pmatrix}, \quad W^- = \mathrm{sign}(ab) \begin{pmatrix} -\frac{1}{2a} & 0 & 0 \\ 0 & -\frac{1}{2a} & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix},$$

where $\mathrm{sign}(ab)$ denotes the sign of ab, which determines the signature of the metric. Then $\ker(W^\pm \pm \mathrm{sign}(ab)\frac{1}{a}\,\mathrm{id}_{\Lambda^2})$ define one-dimensional subspaces in $\Lambda^\pm$. The corresponding 2-forms $\Omega_\pm = \sqrt{2}E_3^\pm$ define an almost Hermitian structure (g, J_+) and an opposite almost Hermitian structure (g, J_-) where $J_\pm e_4 = -2e_1$, $J_\pm e_2 = \pm e_3$.

Now, an easy calculation shows that the opposite almost Hermitian structure (g, J_-) is almost Kähler (i.e., $dE_3^- = 0$) and the Ricci tensor is $J_\pm$-invariant ($J_+ = J_2$). Moreover, since the anti-self-dual Weyl curvature operator W^- satisfies the identities $W_2^- = W_3^- = 0$, Theorem 2.2 shows that the underlying structure is isometric to the 3-symmetric space in the Riemannian setting.

3.6. The Lie algebra $\mathfrak{d}_{4,\frac{1}{2}}$. The Lie algebra $\mathfrak{d}_{4,\frac{1}{2}}$ is described, with respect to a suitable basis $\{e_1, e_2, e_3, e_4\}$, by the non-zero brackets

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -\frac{1}{2}e_1, \quad [e_2, e_4] = -\frac{1}{2}e_2, \quad [e_3, e_4] = -e_3.$$

In this case, there exist two Kähler structures (isometric to the complex hyperbolic plane) as follows.

3.6.1. *Case 1.* $\mathfrak{d}_{4,\frac{1}{2}}$ admits a left-invariant Kähler structure determined by the complex structure $J_1 e_4 = e_3$, $J_1 e_1 = e_2$ and Kähler form $\Omega_1 = a(e^{12} - e^{34})$, where $a \neq 0$. Then the associated metric is given by

$$g_1 = a(e^1 \circ e^1 + e^2 \circ e^2 + e^3 \circ e^3 + e^4 \circ e^4).$$

A straightforward calculation shows that the Ricci operator is a multiple of the identity $\mathrm{Ric} = -\frac{3}{2a}\,\mathrm{id}$ and thus Einstein. Furthermore $W^- = 0$ and $W^+ =$

$\operatorname{diag}[-\frac{1}{a}, \frac{1}{2a}, \frac{1}{2a}]$ and thus it corresponds to the complex hyperbolic space of constant holomorphic sectional curvature $H = -\frac{1}{a}$.

3.6.2. *Case 2.* $\mathfrak{d}_{4,\frac{1}{2}}$ admits a left-invariant Kähler structure determined by the complex structure $J_2e_4 = e_3$, $J_2e_1 = -e_2$ and Kähler form $\Omega_2 = a(e^{12} - e^{34})$ where $a \neq 0$. Then the associated metric, given by

$$g_2 = -a(e^1 \circ e^1 + e^2 \circ e^2 - e^3 \circ e^3 - e^4 \circ e^4)$$

is of neutral signature. Hence we avoid this case, which corresponds to a neutral signature Kähler manifold of constant holomorphic sectional curvature $H = -\frac{1}{a}$.

3.7. The Lie algebra $\mathfrak{d}'_{4,\lambda}$. The Lie algebra $\mathfrak{d}'_{4,\lambda}$ is described, with respect to a suitable basis $\{e_1, e_2, e_3, e_4\}$, by the non-zero brackets ($\lambda \geq 0$)

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = e_2 - \frac{\lambda}{2}e_1, \quad [e_2, e_4] = -(e_1 + \frac{\lambda}{2}e_2), \quad [e_3, e_4] = -\lambda e_3.$$

In this case, there exist four Kähler structures whose geometry reduces to the following two cases. Each of them is isometric to a suitable complex hyperbolic plane.

3.7.1. *Case 1.* $\mathfrak{d}'_{4,\lambda}$ admits a left-invariant Kähler structure determined by the complex structure $J_1e_4 = e_3$, $J_1e_1 = e_2$ and Kähler form $\Omega_1 = a(e^{12} - \lambda e^{34})$, where $a \neq 0$ and $\lambda > 0$. Then the associated metric is given by

$$g_1 = a\,(e^1 \circ e^1 + e^2 \circ e^2 + \lambda\, e^3 \circ e^3 + \lambda e^4 \circ e^4).$$

Hence, the metric is Riemannian and a straightforward calculation shows that the Ricci operator takes the form $\operatorname{Ric} = -\frac{3\lambda}{2a}\operatorname{id}$. Moreover $W^- = 0$ and $W^+ = \frac{\lambda}{2a}\operatorname{diag}[-2, 1, 1]$ depending on the signature of the metric. Hence it corresponds to the Riemannian or the neutral signature complex hyperbolic plane of constant holomorphic sectional curvature $H = -\frac{\lambda}{a}$, and thus symmetric.

Proceeding in a completely analogous way the opposite Kähler structure J'_1 given by $J'_1e_4 = e_3$, $J'_1e_1 = -e_2$ is isometric to the complex hyperbolic space.

3.7.2. *Case 2.* $\mathfrak{d}'_{4,\lambda}$ admits a left-invariant Kähler structure determined by the complex structure $J_2e_4 = -e_3$, $J_2e_1 = e_2$ and Kähler form $\Omega_2 = a(e^{12} - \lambda e^{34})$, where $a \neq 0$ and $\lambda > 0$. Then the associated metric is given by

$$g_2 = -a\,(e^1 \circ e^1 + e^2 \circ e^2 - \lambda\, e^3 \circ e^3 - \lambda e^4 \circ e^4).$$

Hence the metric is of neutral signature and it is of constant holomorphic sectional curvature $H = \frac{\lambda}{a}$.

Proceeding in a completely analogous way the opposite Kähler structure J'_2 given by $J'_2e_4 = -e_3$, $J'_2e_1 = -e_2$ is isometric to the complex hyperbolic space.

4. The proof of Corollary 1.2

Let (M, g, J) be a four-dimensional homogeneous Kähler manifold. We distinguish the two cases in Theorem 1.1. If (M, g, J) is the 3-symmetric space corresponding to 3.5.2, then a straightforward calculation shows that $\mathfrak{D} = \operatorname{Ric} + \frac{3}{a}\operatorname{id}$ is a derivation of the Lie algebra $\mathfrak{d}_{4,2}$ and thus it defines an algebraic Ricci soliton (see also [**13**] for a classification of four-dimensional solvsolitons).

If (M, g, J) is symmetric then it is either Einstein or the product of two oriented surfaces of constant Gauss curvatures $N_1(c_1) \times N_2(c_2)$. It was shown in [**10**] that homogeneous Ricci solitons admitting a transitive semi-simple group of isometries are necessarily Einstein. Hence the only posible non-Einsteinian cases occur when

M is isometric to $\mathbb{R}^2 \times N(c)$. Then it is a rigid gradient Ricci soliton with potential function $f = \frac{c}{2}\|\pi_{\mathbb{R}^2}\|^2$ where $\pi_{\mathbb{R}^2}$ is the projection on the Euclidean factor, which completes the proof of Corollary 1.2.

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