CONTEMPORARY MATHEMATICS

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Geometric and Functional Inequalities and Recent Topics in Nonlinear PDEs

Virtual Conference
Geometric and Functional Inequalities and Recent Topics in Nonlinear PDEs
February 28–March 1, 2021
Purdue University, West Lafayette, IN

Emanuel Indrei
Diego Marcon
Levon Nurbekyan
Editors

AMERICAN MATHEMATICAL SOCIETY

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Preface

Historically, the most difficult problems are at the intersection of several fields of mathematics. The mathematical content of this volume is at the intersection of viscosity theory, Fourier analysis, mass transport theory, fractional elliptic theory, and geometric analysis. Recently, advanced tools have emerged which successfully and elegantly deal with long-standing problems. The most ubiquitous example in the last two decades is optimal transport theory which addressed sharp estimates in a concise manner through the regularity theory for the Monge-Ampère equation via the quadratic cost. Interestingly, the regularity theory for the classical 1781 Monge cost is still not fully developed. However, in several applications, other cost functions appear and a regularity theory to address this was developed via the Ma-Trudinger-Wang (MTW) condition at the start of the 21^{st} century.

The 19^{th} century witnessed the development of the theory of electromagnetism via Maxwell's equations inclusive of Ampère's law, Faraday's law, and Gauss' law. One of the features is the propagation of electromagnetic waves: light from the sun, radio waves, computer signals, etc. The central partial differential equation (PDE) in the theory is the wave equation and there are several still unresolved estimates today. The classical tool to address this is the Fourier transform.

In a first complex analysis class, Liouville's theorem appears: an entire and bounded function is constant. In modern PDE theory several Liouville-type theorems are proved for fully nonlinear elliptic equations and there are many open problems which naturally emerge. An interesting regularity theory which permits weak solutions in an elegant geometric way is encoded in viscosity theory. An application of the theory also includes isoperimetric-type inequalities.

A historically ubiquitous inequality is the classical isoperimetric inequality which is summed up via: under a fixed mass constraint the Euclidean ball minimizes the isotropic perimeter in the class of sets of finite perimeter. Ironically, if one includes a potential energy in the total energy, the isoperimetric problem is not fully understood although several advances appeared in recent years. One characteristic which emerges is convexity and the way a minimizer changes via the change in mass is of central interest. Supposing one considers the anisotropic perimeter without a potential energy, the Wulff shape emerges as the minimizer. The applications involve crystal formation and recently fractional elliptic regularity theory considered crystal defects via shearing forces and edge dislocations. A similar problem involves fixing the width and identifying shapes which maximize/minimize the volume. Historically, this was considered by individuals such as Lebesgue and Blaschke. Novel tools are therefore necessary to address these problems and of interest are geometric flows which generate a convergence into a well-known object, e.g. a ball.

A glimpse into these fields of mathematics appears in this volume:

- *A deterministic counterexample for high dimensional L^2L^∞ Strichartz estimates for the Wave equation* by Cristian Gavrus answers the question of homogeneous L^2L^∞ Strichartz estimates for the wave equation in dimensions $n \geq 4$ raised by Fang and Wang via a deterministic example.
- *On the Liouville property for fully nonlinear equations with superlinear first-order terms* by Marco Cirant and Alessandro Goffi addresses the classical Liouville property for various classes of fully nonlinear PDEs, emphasizing uniformly elliptic inequalities, in the framework of viscosity solutions, with superlinear growth in the gradient variable of the type $F(x, D^2u) \geq H_i(x, u, Du)$ in $\mathbb{R}^N$, for model cases $H_1(u, Du) = u^q + |Du|^\gamma$, $H_2(u, Du) = u^q|Du|^\gamma$, and $H_3(x, u, Du) = \pm u^q|Du|^\gamma - b(x) \cdot Du$, where $q \geq 0$, $\gamma > 1$, and b is a suitable velocity field. The paper contains an in-depth discussion of diverse techniques for studying Liouville properties and includes new sharp results, counterexamples, conjectures, and numerous challenging open problems.
- *Regularity of the solution to the principal-agent problem* by Shibing Chen proves the C^1 regularity of the solution to the principal-agent problem with general preference function satisfying structural conditions which involve the MTW condition in optimal transport theory via adapting a method of Carlier and Lachand-Robert.
- *A doubly monotone flow for constant width bodies in $\mathbb{R}^3$* by Ryan Hynd discusses a classical conjecture: a constant width body is a compact, convex subset of Euclidean space in which parallel supporting planes are separated by the same distance in every direction (this distance is called its width). In $\mathbb{R}^2$, it was proved independently by Lebesgue and Blaschke over a century ago that Reuleaux triangles enclose the least amount of area (a Reuleaux triangle is the intersection of three closed disks of radius 1 which are centered at the vertices of an equilateral triangle of side length one). There have been many subsequent proofs of the Lebesgue-Blasckhe theorem. Harrell showed that Reuleaux triangles are uniquely area-minimizing among constant width shapes. In dimensions larger than two, it is known that volume-minimizing constant width bodies exist via Blaschke's selection theorem. Surprisingly little is known about these shapes. However, there are conjectured volume-minimizing constant width bodies that Meissner and Schilling constructed which are based on a regular tetrahedron, analogous to how the Reuleaux triangle is based on an equilateral triangle. Hynd introduces a flow in the space of constant width bodies in $\mathbb{R}^3$ that simultaneously increases the volume and decreases the circumradius of the shape as time increases. Starting from any initial constant width figure, he shows that the flow exists for all positive times and converges to a closed ball as time tends to infinity. Therefore this generates a technique to study the shape as the volume decreases via a time-reversal.
- *A fractional glance to the theory of edge dislocations* by Serena Dipierro, Stefania Patrizi, and Enrico Valdinoci studies the edge dislocations problem for crystals guided by the classical Peierls-Nabarro model that describes the equilibrium configurations by a nonlocal fractional Laplacian equation $-\sqrt{-\Delta}u = W'(u)$ in $\mathbb{R}$, where u is the dislocation function

along the slip line, and W is a multiwell potential penalizing the dislocated atoms. The detailed analysis of the model reveals the emergence of chaotic stationary behaviors in the form of complex and unusual equilibrium patterns. Further results address the dislocation dynamics, collisions between dislocation points, Orowan's Law, and the homogenization problem related to the Peierls-Nabarro model.

Our hope is that you will enjoy the volume!

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The mathematical content of this volume is at the intersection of viscosity theory, Fourier analysis, mass transport theory, fractional elliptic theory, and geometric analysis. The reader will encounter, among others, the following topics: the principal-agent problem; Maxwell's equations; Liouville-type theorems for fully nonlinear elliptic equations; a doubly monotone flow for constant width bodies; and the edge dislocations problem for crystals that describes the equilibrium configurations by a nonlocal fractional Laplacian equation.

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