

CONTEMPORARY MATHEMATICS

783

Computational Aspects of Discrete Subgroups of Lie Groups

Virtual Conference

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June 14–18, 2021

Institute for Computational and Experimental Research
in Mathematics (ICERM)
Providence, Rhode Island

Alla Detinko
Michael Kapovich
Alex Kontorovich
Peter Sarnak
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Preface

This volume contains the proceedings of the workshop “Computational Aspects of Discrete Subgroups of Lie Groups”, held at the Institute for Computational and Experimental Research in Mathematics (ICERM), June 14–18, 2021. The workshop’s main theme interfaces algebra, geometry and computer science; it deals with the design, implementation, and application of algorithms based on matrix representations of groups and their geometric properties.

Lie groups and their subgroups have a long history, going back to the late 19th century. They impact many areas of mathematics, such as differential geometry, topology, number theory, algebraic geometry, differential equations, and combinatorics. The setting of *linear* Lie groups is suited to calculation and modeling transformations. At the same time, rapid technological development has realized the efficiency of matrix representation of groups and related algebraic structures in computers, leading to new ways of solving disparate problems via group-theoretical computer modeling.

The traditional theory of discrete subgroups of Lie groups (apart from Kleinian groups, discrete isometry groups of hyperbolic spaces) focused on *lattices*, i.e., discrete subgroups of finite covolume. Recently there has been significant progress in our understanding of “thin” discrete subgroups of matrix groups, which are discrete matrix groups that have infinite covolume in their Zariski closure. This was spurred by developments in several fields including geometry, dynamics, and number theory. The open problems encompass purely mathematical questions and those of more applied nature, relating to theoretical physics, quantum computing, and materials science. Many of the problems in the theory of discrete subgroups of Lie groups are, in principle, amenable to a computational approach.

While computer algebra systems provide a general computational framework, solution of particular problems requires building up foundations via new algorithms and software. This, in turn, necessitates collaboration between eclectic groups of scientists. Consequently a major goal of the workshop was to synergize and synthesize these independent strands. We aimed to facilitate solution of theoretical problems by means of recent advances in computational algebra, and stimulate development of computational algebra oriented to other mathematical disciplines.

The workshop featured four mini-courses and separate invited talks on major developments reflecting the workshop’s theme. The lectures were supplemented by software demonstrations and tutorials covering relevant MAGMA, GAP, and Snap-Pea packages. The program incorporated problem sessions, and multidisciplinary discussions on development of computational algebra and computer algebra systems. Further information, including videos of talks, is available at the workshop website: https://icerm.brown.edu/topical_workshops/tw-21-cads/.

We thank everyone who participated, especially the contributors and referees. We are grateful to ICERM for its generous support. Also, without the hard work of the editorial staff of the American Mathematical Society, this volume would not have seen print. Our thanks go to Christine M. Thivierge for her constant help.

The Editors

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This volume contains the proceedings of the virtual workshop on Computational Aspects of Discrete Subgroups of Lie Groups, held from June 14 to June 18, 2021, and hosted by the Institute for Computational and Experimental Research in Mathematics (ICERM), Providence, Rhode Island.

The major theme deals with a novel domain of computational algebra: the design, implementation, and application of algorithms based on matrix representation of groups and their geometric properties. It is centered on computing with discrete subgroups of Lie groups, which impacts many different areas of mathematics such as algebra, geometry, topology, and number theory. The workshop aimed to synergize independent strands in the area of computing with discrete subgroups of Lie groups, to facilitate solution of theoretical problems by means of recent advances in computational algebra.



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