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CRM PROCEEDINGS & LECTURE NOTES

Centre de Recherches Mathématiques
Montréal

New Perspectives and Challenges in Symplectic Field Theory

Miguel Abreu
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Dedicated to Yasha Eliashberg on the occasion of his 60th birthday

Contents

Preface vii

A Lagrangian Quantum Homology
Paul Biran and Octav Cornea 1

A Survey of Contact Homology
Frédéric Bourgeois 45

Minimal Atlases of Closed Contact Manifolds
Yuri Chekanov, Otto van Koert, and Felix Schlenk 73

The Role of String Topology in Symplectic Field Theory
Kai Cieliebak and Janko Latschev 113

A Morse Theoretic Description of String Topology
Ralph L. Cohen and Matthias Schwarz 147

A Version of Rational SFT for Exact Lagrangian Cobordisms in 1-Jet Spaces
Tobias Ekholm 173

Canonical Models of Filtered A_∞-Algebras and Morse Complexes
Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono 201

Constructing Stein Manifolds after Eliashberg
Robert E. Gompf 229

Floer Homology Groups in Hyperkähler Geometry
Sonja Hohloch, Gregor Noetzel, and Dietmar A. Salamon 251

The Embedded Contact Homology Index Revisited
Michael Hutchings 263

Positive Legendrian Regular Homotopies
François Laudenbach 299

Symplectic Birational Geometry
Tian-Jun Li and Yongbin Ruan 307

Heegaard Floer Homology, Double Points and Nice Diagrams
Robert Lipshitz 327

Preface

This Yashafest Volume and the Conference on New Challenges and Perspectives in Symplectic Field Theory, held at Stanford University on June 25 – 29 of 2007, celebrate the 60th birthday of Yakov Eliashberg, one of the founders of modern symplectic and contact topology. Eliashberg's mathematical career started at the famous Rokhlin seminar at Leningrad State University (former USSR) in the late sixties. At that time, Eliashberg started to collaborate with Gromov on homotopy principles for partial differential relations and made a fundamental contribution to this field. In a joint work (1971), Gromov and Eliashberg invented a powerful technique of removal of singularities, and in his dissertation (1972), Eliashberg introduced the surgery of singularities. In particular, he proved a fundamental theorem showing that the h-principle holds for maps with "sufficiently large" (but, yet, quite small!) singularities. Eliashberg has always remained one of the key researchers in the theory of the h-principle. His later contributions include h-principles for Stein manifolds (with Gromov) and a series of important works with Mishachev that culminated in their joint book (2002) *Introduction to the h-principle*, one of the major sources in the field nowadays.

After completing his dissertation, Eliashberg found himself in the northern Russian city of Syktyvkar which is, according to advertisements in some Russian newspapers after perestroika, "an ideal stop for those continuing on to the gulag sites farther north, or those wishing to continue into Siberia."[1] As a simple member and later as chairman of the Department of Pure Mathematics at Syktyvkar University (1972 – 1979), Eliashberg became the soul of the mathematical group. His colleagues, among those Viatcheslav Kharlamov, and all students still remember him with great warmth. It is common to think that it is the exceptional events of History that make great men, not the converse. And it is possible that the seven years spent at Syktyvkar revealed the Eliashberg that we all know: a mathematician with an exceptional sense of humanity and courage. What is sure is that Eliashberg has made a large part of the history of symplectic and contact topology and that the Syktyvkar period was transformed under his action to serve people and mathematics. Eliashberg's route then passed through the refusenik years in Leningrad (1980 – 1987) where he had to do software engineering in order to feed his family, and where he was virtually cut off from normal mathematical life. Nevertheless these 15 years in Syktyvkar and Leningrad were very fruitful: Eliashberg started to work on symplectic topology. He developed a combinatorial technique that led him to the first manifestation of symplectic rigidity: the group of symplectomorphisms is C^0-closed in the group of all diffeomorphisms. This fundamental result, proved

[1] Even in Canada, the coldest spells (say $-35\,^\circ$C) that are frequent, are still refered to by Quebecers as a "Siberian cold."

in a different way also by Gromov, is now called the Eliashberg-Gromov theorem. Let us emphasize that at that time, this statement was very far from being natural. Gromov recollects this discovery in the following way: "We spoke about this with Yasha, after we learned that Arnold and his school unquestionably believed in the existence of rigid symplectic geometry in dimensions > 2, motivated, I guess, by KAM-theory. We (at least myself) were skeptical, since everything (like Lagrangian immersions) indicated flexibility, except for the Kolmogorov – Arnold – Moser theory, but that seemed to us rather local. After some preliminary arguments by Yasha (who was first uncertain in what to believe) we arrived at the point that either there is a full flexibility or there is a nontrivial symplectic geometry. Eventually Yasha, despite my doubts, converged to the idea of rigidity and developed his combinatorial method." Another success of Eliashberg's combinatorial method was the proof of Arnold's conjecture on fixed points of Hamiltonian diffeomorphisms for surfaces (1979).

Rumors of Eliashberg's achievements reached the Western countries. He was invited to give an address at the ICM (Berkeley, 1986) on his combinatorial method, and though Soviet authorities did not let him go, the short paper appeared in the ICM volume. He also succeeded in publishing a paper in *Functional Analysis and its Applications* (1987) that provided more details. A MathSciNet review by Vaisman on this paper starts with a passage that eloquently reflects the atmosphere of the cold war: "It has been known for several years that the author made a breakthrough in symplectic geometry by establishing global rigidity properties for symplectic and contact manifolds..."

Soon after leaving the Soviet Union in 1988, Eliashberg wrote a number of very influential papers: in 1989 he developed a powerful technique consisting of filling by pseudo-holomorphic discs, that brings together Gromov's theory of pseudo-holomorphic curves and ideas of an earlier (1982) important work by Eliashberg and Kharlamov on surfaces in almost complex 4-manifolds, as well as earlier fundamental work by Bishop. Among many applications, Eliashberg obtained a far-reaching generalization of Bennequin's inequality. In 1990, he discovered a complete topological characterization of Stein manifolds of complex dimension > 2. In a series of papers (1989 – 1992), Eliashberg introduced and explored a fundamental dichotomy "tight vs. overtwisted" that shaped the face of modern contact topology. Using this dichotomy, he obtained the classification of contact structures on the 3-sphere (1992). In 1998, Eliashberg and Thurston developed a theory of confoliations that provides a unifying viewpoint on foliations and contact structures.

Towards the end of the nineties, Eliashberg (in collaboration with Hofer and Givental) started working on the foundation of the symplectic field theory (SFT). These results were presented in a seminal 2000 GAFA paper as well as in Eliashberg's two ICM talks (in an invited address in Berlin, 1998, and in a plenary lecture in Madrid, 2006). Nowadays the SFT is one of the most central and exciting directions in symplectic and contact topology.

Eliashberg is a master of informal mathematical communication. Hofer's paper on his famous metric on the group of Hamiltonian diffeomorphisms and Viterbo's paper on the generating function technique in symplectic topology were motivated by a stimulating question by Eliashberg. In the reverse direction, answering a question by Gay and Plamenevskaya, Eliashberg wrote a paper on embeddings of fillings of contact manifolds into closed symplectic manifolds (2004) which had a remarkable impact: it provided the last missing step in the program of Kronheimer and Mrowka for proving the long standing conjecture on Property P for knots, and enabled Ozsvath and Szabo to complete an important piece of their foundational work on Heegaard Floer homology.

As a professor at Stanford University since 1989, Eliashberg has supervised more than 20 graduate students, many of whom have become professional mathematicians actively working in symplectic and contact topology all over the world.

For his contributions, Eliashberg has received a number of prestigious awards, including the Guggenheim Fellowship in 1995 and the Oswald Veblen Prize in 2001. In 2003 he was elected to the National Academy of Sciences of the U.S.A.

The articles in this book survey, contain or announce some of the most fascinating developments in recent mathematics in symplectic, contact and gauge theories. They represent, above all else, the expression of the deep admiration that so many mathematicians around the world have for Yasha Eliashberg's works, influence and humanity. That admiration was also evident in the overwhelming response to the Yashafest Conference. The program of the conference included a set of introductory lectures on several aspects of SFT. Three of the articles in this volume are in that spirit. They are also based on lectures given by their authors at the 2006 SFT Leipzig workshop, an event that had Eliashberg as the main speaker. Bourgeois' survey of Contact Homology fills some missing parts in the literature and announces new results in a non-technical and unified way. It will be helpful for students and posdocs who want to enter the field. Cieliebak and Latschev outline their program for incorporating holomorphic curves with boundary into the general framework of SFT. The emphasis is on ideas, geometric intuition and a description of the resulting algebraic structures. Ekholm describes and computes in basic examples a version of rational SFT for exact Lagrangian cobordisms in symplectizations of 1-jet spaces. This gives rise to an invariant for Legendrian submanifolds of these contact manifolds.

With a similar introductory spirit, Li and Ruan present a survey of birational geometry in the symplectic category. This notion, introduced and studied by Ruan and others, can be seen as a first step in a classification program for symplectic manifolds.

Biran and Cornea review some of their work on Lagrangian quantum homology, presenting an accessible introduction to an effective computational technique which leads to several applications in the topological behaviour of Lagrangian submanifolds.

Chekanov, van Koert and Schlenk's article contains results on a question posed by R. Lutz twenty years ago, and the subject of previous work by Rudyak-Schlenk in the symplectic case: what is the minimal number of Darboux charts needed to cover a closed contact manifold?

Cohen and Schwarz, extending previous work, give a Morse theoretic description of the string topology operations introduced by Chas and Sullivan.

Fukuya, Oh, Ohta and Ono review the construction of the canonical models of filtered A_∞-bimodules and explain its implication in a geometric setting.

Gompf gives a unified summary of the existence theory of Stein manifolds in all dimensions, due to Yasha and himself, introducing also some important new concepts in the theory.

Hohloch, Noetzel and Salamon announce and outline their construction of a hyperkähler analogue of symplectic Floer homology.

Hutchings shows that the relative grading of embedded contact homology can be refined to an absolute grading. He then establishes refined index inequalities for this grading and presents a new useful computational tool: a relative filtration on the contact homology of 3-manifolds.

Laudenbach describes an application of the h-principle, giving a proof of the fact that, in contrast with what happens for Legendrian embeddings, positive loops of Legendrian immersions always exist.

Lipshitz constructs a new version of knot Floer homology, taking into account holomorphic curves with double points, and shows how to make it combinatorial using grid diagrams.

This introduction was written on the day celebrating another 60th birthday, that of the Universal Declaration of Human Rights signed in Paris on December 10, 1948. We think of its article 13 about freedom of movement.

Miguel Abreu, François Lalonde and Leonid Polterovich,

December 10, 2008.

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This volume, in honor of Yakov Eliashberg, gives a panorama of some of the most fascinating recent developments in symplectic, contact and gauge theories. It contains research papers aimed at experts, as well as a series of skillfully written surveys accessible for a broad geometrically oriented readership from the graduate level onwards. This collection will serve as an enduring source of information and ideas for those who want to enter this exciting area as well as for experts.

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