DIMACS
Series in Discrete Mathematics
and Theoretical Computer Science

Volume 60

Algorithmic and Quantitative Real Algebraic Geometry

DIMACS Workshop
Algorithmic and Quantitative Aspects of
Real Algebraic Geometry in Mathematics
and Computer Science
March 12–16, 2001
DIMACS Center

Saugata Basu
Laureano Gonzalez-Vega
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American Mathematical Society

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Foreword

This volume consists of papers resulting from the DIMACS Workshop on Algorithmic and Quantitative Aspects of Real Algebraic Geometry, held at DIMACS on March 12–16, 2001. We would like to express our appreciation to Saugata Basu and Laureano Gonzalez-Vega for putting together the volume and Ricky Pollack, Marie-Francoise Roy and Micha Sharir for organizing the workshop.

The workshop was part of the Special Focus on Mathematics and the Foundations of Computer and Information Science. We extend our thanks to Michael Fredman, Janos Komlos, and Fred Roberts for their work as special focus organizers.

The workshop brought together researchers in algorithmic and quantitative aspects of real algebraic geometry and other mathematicians and computer scientists who are concerned with related problems, aiming at promoting more productive interactions between them. The conference and this volume have succeeded in emphasizing such areas of commonality as interactions between real algebraic geometry and computational geometry, efficient recent algorithms in real algebraic geometry and their implementation, quantitative aspects of basic semi-algebraic sets, lower bounds and real algebraic geometry, combinatorial characterization of real algebraic varieties, combinatorial patchworking for constructing real algebraic curves and hypersurfaces, and real aspects of enumerative geometry.

DIMACS gratefully acknowledges the generous support that makes these programs possible. The National Science Foundation, through its Science and Technology Centers program, the New Jersey Commission on Science and Technology, DIMACS's partners at Rutgers, Princeton, AT&T Labs-Research, Bell Labs, NEC Laboratories America, and Telcordia Technologies, and DIMACS affiliates at Avaya Labs and Microsoft Research generously supported the special focus.

Fred S. Roberts
Director

Robert Tarjan
Co-Director for Princeton

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Preface

Algorithmic and quantitative aspects are becoming increasingly important areas of research in Real Algebraic Geometry because of their roles in other areas of mathematics and computer science. The papers in this volume collectively span several different areas of Real Algebraic Geometry which are of current research interest. These include deciding basic algebraic properties of real semi-algebraic sets, application of quantitative results in real algebraic geometry towards investigating the computational complexity of various problems, algorithmic and quantitative questions in real enumerative geometry, new approaches towards solving decision problems in semi-algebraic geometry as well as computing algebraic certificates, and applications of real algebraic geometry to concrete problems arising in robotics and computer graphics.

Subsets of finite dimensional real spaces, defined by Boolean formulas whose atoms are polynomial inequalities and equalities, are called semi–algebraic sets, and they are one of the most important geometric objects in Real Algebraic Geometry.

Semi–algebraic sets which are defined by a conjunction of strict inequalities are called basic open semi–algebraic sets. The paper by Carlos Andradas aims at characterizing the basic open semi–algebraic sets, by presenting procedures for recognizing whether a given open semi-algebraic set is basic or not and, if it is the case, finding a minimal (in the sense of shortest) description.

In the same vein, the paper by Clint McCrory gives a technique for proving that a given semi–algebraic set is not a real algebraic set in dimensions up to four, by constructing a list of local topological obstructions using the link operator on the ring of constructible functions (introduced by C. McCrory and A. Parusinski). These functions are used in the paper by Isabelle Bonnard to show how to reformulate a condition due to S. Akbulut and H. King, for a compact semi-algebraic set X to be homeomorphic to a real algebraic set when $\dim X \leq 3$, and use it to produce new necessary conditions in dimension 4.

The paper by Benoit Chevallier uses Viro's method with quadratic transforms, to solve several instances of the problem of deciding if there exists a real algebraic hyper-surface in $\mathbb{R}^n$ defined by a polynomial in $\mathbb{R}[x_1, \ldots, x_n]$ of degree d which is isotopic to a given smooth hyper-surface in $\mathbb{R}^n$.

Semi–Pfaffian sets are useful generalizations of semi–algebraic sets obtained by enlarging the family of polynomial functions by including Pfaffian functions which are analytic functions satisfying certain triangular systems of differential equations. Semi–Pfaffian sets share many of the finiteness properties of semi–algebraic sets, and this leads to the problem of determining uniform bounds on their topological invariants, such as the number of connected components, Betti numbers etc. The family of semi–Pfaffian sets are distinguished from semi–algebraic sets by the fact that it is not closed under projection. Recently, A. Gabrielov introduced the notion of relative closure in order to enlarge the family of semi-Pfaffian sets and obtain a family which is stable under projection. A different definition of the same family was given by A. J. Wilkie earlier, but the new definition makes it simpler to estimate the topological complexity of sets in the family. The paper by Andrei Gabrielov and Thierry Zell provides the first uniform bounds on the number of connected components of the relative closure of a pair of semi–Pfaffian sets, which is an important first step in this direction.

The problem of computing algebraic certificates is becoming an exciting new research area in Real Algebraic Geometry: if S is the basic closed semi-algebraic set in $\mathbb{R}^n$ defined by $f_1 \geq 0, \ldots, f_k \geq 0$ and a polynomial p can be written as a finite sum of products of the f_i's and sums of squares in $\mathbb{R}[x_1, \ldots, x_n]$ then clearly p is non–negative on S. A "representation theorem" or an algebraic certificate is a converse to this fact. Namely, such a theorem would state that if a polynomial p is positive on S, then there is a representation of p in terms of the f_i's and (possibly) sums of squares making evident the positivity of p on S. The paper by Dionne Bailey and Victoria Powers presents a constructive approach to the Kadison–Dubois Theorem (giving an algebraic certificate for the non–negativity of p over S under one additional hypothesis implying the compactness of S), with degree bounds and an algorithm for finding such a representation. This is then used to give a constructive representation theorem in certain non-compact cases.

The paper by Bruce Reznick is devoted to another quantitative aspect of Real Algebraic Geometry: if $F = \{f_1, \ldots, f_r\}$ is a family of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ and $T(F)$ denote the set of integers m such that $\{f_1^m, \ldots, f_r^m\}$ is linearly dependent, then it is proven that $|T(F)| \leq \binom{r-1}{2}$. Many concrete examples are also discussed in the paper. The elements in $T(F)$ are called the tickets of F and they generalize several classical constructions such as Fermat's Last Theorem for univariate polynomials ($r = 3$) and have connections with Nevanlinna Theory and with Waring's Problem for polynomials.

The paper by Pablo Parrilo and Bernd Sturmfels describes a new approach towards global optimization of polynomial functions in many variables by using sums of squares and a convex relaxation to semi–definite programming. The resulting algorithm seems to (heuristically) work very well relative to other exact methods and it is the first practical algorithm for computing algebraic certificates for the Positivstellensatz.

The critical point method provided the first single exponential algorithm for solving the problem of deciding emptiness of a given semi–algebraic set. The paper by Fabrice Rouillier describes recent advances in the efficient implementations of critical point method in solving real polynomial systems. The theoretically best algorithms are designed from the point of view of asymptotic complexity. Efficient practical implementation requires important modifications to the theoretically optimal algorithms, and several approaches in this direction are described in Rouillier's paper.

The problem of proving lower and upper bounds on the number of *real* (as opposed to complex) solutions of algebraic problems is often considerably more difficult than the problem of bounding (or even counting) complex solutions. The classical tool in this area is that of Schubert calculus, which provides the answer to complex enumerative problems. The paper by Frank Sottile is an exhaustive survey of results from enumerative Real Algebraic Geometry showing that the upper bounds regarding real solutions and arising from Schubert calculus are usually sharp. It also describes how a new notion of degree for real Grassmann varieties, introduced by A. Eremenko and A. Gabrielov, helps to prove new improved lower bounds for some enumerative problems.

The paper by Thorsten Theobald presents one application of Real Algebraic Geometry to computer graphics and computational geometry: the investigation of visibility computations with moving viewpoints. The considered problems are of discrete and algorithmic nature, but even for simple classes of objects (such as balls and polytopes), they lead to interesting and difficult problems from Enumerative Real Algebraic Geometry. In particular, a new sweep algorithm in dimension 2 is introduced, together with the extension of several algebraic–geometric results regarding the tangency problems involving lines and spheres in $\mathbb{R}^3$.

An important application of Real Algebraic Geometry in Theoretical Computer Science has been in proving lower bounds in certain models of computations such as algebraic computation trees or decision trees. Upper bounds on the geometric and topological complexity of semi–algebraic sets gets translated into lower bounds in these models of computation. Often, these lower bounds are tight and thus the exact computational complexity of certain problems gets determined. The paper by Peter Bürgisser provides a comprehensive survey of these results.

The paper by Ileana Streinu deals with the algorithmic problem of motion planning of a planar robot arm with many links (open or closed polygonal chains). Translating this problem into the language of configurations spaces, one obtains algorithmic problems in Real Algebraic Geometry mainly related with the computation of real solutions of a polynomial system of equations. Connections with pseudo–triangulations are also discussed.

The papers appearing in this volume are based on the talks given at the *DIMACS Workshop on Algorithmic and Quantitative Aspects of Real Algebraic Geometry* held at DIMACS Center (Rutgers University), March 12–16, 2001. The editors of this volume would like to thank Ricky Pollack, Marie–Françoise Roy and Micha Sharir for providing us with an optimal set of initial conditions to start with.

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