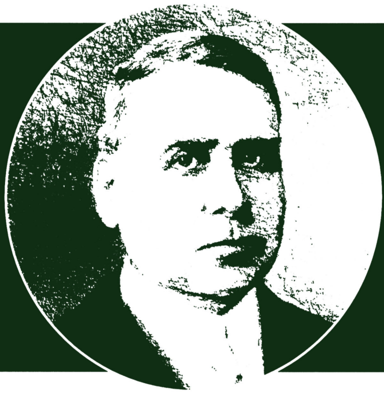

FIELDS INSTITUTE COMMUNICATIONS

THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES

Calabi-Yau Varieties and Mirror Symmetry

Noriko Yui
James D. Lewis
Editors

American Mathematical Society

http://dx.doi.org/10.1090/fic/038

FIELDS INSTITUTE COMMUNICATIONS

THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES

Calabi-Yau Varieties and Mirror Symmetry

Noriko Yui
James D. Lewis
Editors

American Mathematical Society
Providence, Rhode Island

The Fields Institute for Research in Mathematical Sciences

The Fields Institute is named in honour of the Canadian mathematician John Charles Fields (1863–1932). Fields was a visionary who received many honours for his scientific work, including election to the Royal Society of Canada in 1909 and to the Royal Society of London in 1913. Among other accomplishments in the service of the international mathematics community, Fields was responsible for establishing the world's most prestigious prize for mathematics research—the Fields Medal.

The Fields Institute for Research in Mathematical Sciences is supported by grants from the Ontario Ministry of Education and Training and the Natural Sciences and Engineering Research Council of Canada. The Institute is sponsored by McMaster University, the University of Toronto, the University of Waterloo, and York University, and has affiliated universities in Ontario and across Canada.

2000 *Mathematics Subject Classification.* Primary 14J32.

Library of Congress Cataloging-in-Publication Data

Calabi-Yau varieties and mirror symmetry / Noriko Yui, James D. Lewis, editors.
p. cm. — (Fields Institute communications, ISSN 1069-5265 ; 38)
Includes bibliographical references.
ISBN 0-8218-3355-3 (alk. paper)
1. Calabi-Yau manifolds. 2. Mirror symmetry. I. Yui, Noriko. II. Lewis, James Dominic, 1953– III. Fields Institute communications ; v. 38.

QC20.7.M24C35 2003
516.3′5–dc22 2003057820
CIP

♾ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
This publication was prepared by The Fields Institute.
Visit the AMS home page at http://www.ams.org/

10 9 8 7 6 5 4 3 2 1 08 07 06 05 04 03

Contents

Schedule of Workshops

July 23, Monday

9:00–9:30am Opening Ceremony and Information Seesion
Welcome by K. Davidson, the Director
9:30am–10:30am S.-T. Yau (Harvard)
Mirror Principle
11:00am–12:00 James D. Lewis (Alberta)
Regulators of higher Chow cycles on Calabi–Yau varieties
2:00pm–3:00pm V. Batyrev (Tuebingen)
Introduction to toric methods in mirror symmetry
3:30pm–4:30pm R. Schimmrigk (Georgia Southwestern U.)
Arithmetic aspects of Calabi–Yau varieites and conformal field theory
5:00pm–6:00pm C. Doran (Columbia)
Variation of the mirror map and algebro-geometric solutions to Garnier systems

July 24, Tuesday

9:30am–10:30am S. Hosono (Tokyo)
Introduction to the GKZ hypergeometric system
11:00am–12:00 noon X. de la Ossa (Oxford)
Counting $\mathbf{F}_p$*-rational points for Calabi–Yau manifolds*
2:00pm–3:00pm P. Candelas (Oxford)
Observations on the form of the local L*-function for the quintic threefold*
3:30pm–4:30pm Y. Goto (Hokkaido U. Education)
$K3$ *surfaces over finite fields with symplectic group actions*
5:00pm–6:00pm F. Rodriguez Villegas (University of Texas at Austin)
Arithmetic of some hypergeometric threefolds

July 25, Wednesday

9:30am–10:30am S. Müller-Stach (Essen)
Picard–Fuchs equations and algebraic K*-theory*
11:00am–12:00 noon I. Dolgachev (Michigan)
Calabi–Yau manifolds and Hermitian symmetric bounded domains
2:00pm–3:00pm S. Tankeev (Vladimir State U.)
On the Brauer group of a Calabi–Yau variety
3:30pm–4:30pm M. Saito (Kobe)
Relative Lefschetz action and BPS state counting
5:00pm–6:00pm P. Berglund (USC)
On hybrid phases in the extended Kähler moduli spaces of Calabi–Yau threefolds

July 26, Thursday

9:30am–10:30am S. Hosono (Tokyo)
Monodromy transforms of hypergeometric series in (local) mirror symmetry
11:00am–12:00 noon V. Batyrev (Tuebingen)
Local mirror symmetry and McKay correspondence
12:10 noon–1:10pm A. Todorov (UC Santa Cruz)
Moduli of Calabi–Yau manifolds

July 27, Friday

9:30am–10:30am B. Lian (Brandeis)
Monodromy of Calabi–Yau varieties
11:00am–12:00 noon J. Stienstra (Utrecht)
Ordinary Calabi–Yau threefolds
2:00pm–3:00pm N. Yui (Queen's)
The modularity of Calabi–Yau varieties
3:30pm–4:30pm H. Verrill (Hanover)
Intermediate Jacobians of certain rigid Calabi–Yau threefolds
5:00pm–6:00pm T. Ito (Tokyo)
Birational smooth minimal models have equal Hodge numbers in all dimensions

July 28, Saturday

9:30am–10:30am Pedro Luis del Angel (CIMAT)
On indecomposable higher cycles in a product of hyperelliptic curves
11:00am–12:00 A. Collino (Torino)
Indecomposable higher Chow cycles on hypersurfaces and on Jacobians
2:00pm–3:00pm R. Schimmrigk
Black hole attractor varieties and complex multiplication
3:30pm–4:30pm M. Saito (Kobe)
Deformations of Okamoto-Painlev'e pairs and Painlev'e equations

July 29, Sunday

9:30am–12:00 noon *Problem Session*

List of Participants

Shreya Amin	Brandeis University
Victor Batyrev	Universität Tübingen
Per Berglund	University of Southern California
Philip Candelas	University of Oxford
Jaydeep Chipalkatti	Queen's University
Alberto Collino	Universitá di Torino
Xenia de la Ossa	University of Oxford
Pedro Luis del Angel	Ctr de Invest en Matematicas
Igor Dolgachev	University of Michigan
Charles Doran	Columbia University
Yun Gao	York University
Yasuhiro Goto	Hokkaido University of Education
Michael Hoffman	U.S. Naval Academy
Shinobu Hosono	University of Tokyo
Tetsushi Ito	University of Tokyo
Shabnam Kadir	University of Oxford
Kenichiro Kimura	University of Chicago
James D. Lewis	University of Alberta
Bong H. Lian	Brandeis University
Ling Long	The Pennsylvania State University
Susan Hammond Marshall	University of Texas, Austin
David Marshall	McMaster University
Stephen Müller-Stach	Universität GH Essen
V. Kumar Murty	University of Toronto
Kyungho Oh	Harvard University
Fernando Rodriguez-Villegas	University of Texas, Austin
Masahiko Saito	Kobe University
Rolf Schimmrigk	Georgia Southwestern State University
Jan Stienstra	University of Utrecht
S. G. Tankeev	Vladimir State University
Andrey Todorov	University of California at Santa Cruz
Matei Toma	University of Oznabrück
Bert van Geemen	Universitá di Pavia
Helena Verrill	Universität Hannover
Shouhong Wang	Indiana University
Shing-Tung Yau	Harvard University
Noriko Yui	Queen's University
Bin Zhang	The Pennsylvania State University

This page intentionally left blank

Acknowledgments

The editors wish to express their appreciation to all the contributors for their effort in preparing their manuscripts for the Fields Communication Series.

We are deeply grateful to all our referees for their immense efforts and discipline in evaluating the papers.

The weeklong (7 days) workshop was supported by the Fields Institute. We thank the Fields Institute for their financial support and for their organizational help.

There will be a sequel to this workshop, to be held at the Banff International Research Station (BIRS) in the period December 6–11, 2003. We sincerely hope to see you there, and discuss recent developments since the Fields Workshop.

James D. Lewis and Noriko Yui, the Editors.

Introduction

One of the most significant developments in the last decade in theoretical physics (high energy) is, arguably, string theory and mirror symmetry. String theory proposes a model for the physical world in which the fundamental constituents are 1-dimensional mathematical objects "strings" rather than 0-dimensional objects "points". Mirror symmetry is a conjecture in string theory according to which certain "mirror pairs" of Calabi–Yau manifolds give rise to isomorphic physical theories. (A Calabi–Yau variety of dimension d is a complex manifold with trivial canonical bundle and vanishing Hodge numbers $h^{i,0}$ for $0 < i < d$. For instance, a 1-dimensional Calabi–Yau variety is an elliptic curve, a 2-dimensional Calabi–Yau variety is a K3 surface, and in dimension 3 we simply call it a Calabi–Yau threefold.) Calabi–Yau manifolds appear in string theory because in passing from the 10-dimensional space-time to a physically realistic description in four dimensions, string theory requires that the additional 6-dimensional space be a Calabi–Yau manifold.

Though the idea of mirror symmetry originated in physics, the field of mirror symmetry has exploded in recent years onto the mathematical scene. It has inspired many new developments in algebraic geometry, toric geometry, Riemann surfaces, infinite dimensional Lie algebras, among others. For instance, mirror symmetry has been used to tackle the problem of counting the number of rational curves on Calabi–Yau threefolds.

In the course of studying mirror symmetry, it has become apparent that Calabi–Yau varieties enjoy tremendously rich arithmetic properties. For instance, arithmetic objects such as modular forms, modular functions of one and more variables, algebraic cycles, L-functions, and p-adic L-functions, have popped up onto the scene. Moreover special classes of Calabi–Yau manifolds, e.g., Fermat type hypersurfaces, or their deformations pertinent to mirror symmetry, offer a promising testing ground for physical predictions as well as rigorous mathematical analysis and computations.

The goal of the workshop was to bring together to the Fields Institute experts, recent Ph.D.s and graduate students, working in or studying Calabi–Yau varieties and mirror symmetry in physics, geometry and arithmetic, and to exchange ideas and learn the subjects first-hand while mingling with researchers with different expertise. We expected these interactions to lead to progress in solving open problems in mathematics and physics as well as to pave the way to new developments.

The organizing and scientific committee of the workshop consisted of Victor Batyrev (University of Tübingen), Shinobu Hosono (University of Tokyo), James D. Lewis (University of Alberta), Bong H. Lian (Brandeis University), Noriko Yui (Queen's University), and S.-T. Yau (Harvard University) (who served as a scientific advisor to the committee).

There were about 40 registered participants – mathematicians and theoretical physicists – from ten countries. The workshop was enormously successful in fulfilling our goal. All participants were very happy with the workshop. Many expressed their satisfaction by saying that they learned a lot from this workshop. S.-T. Yau (Harvard) (one half of the workshop's namesake "Calabi–Yau") actually took part in the workshop for two days, in spite of his busy schedule. P. Candelas (Oxford) expressed his appreciation saying that "it was a very stimulating workshop". S. Müller-Stach (Essen) wrote that he appreciated the synergetic power which came from the variety of background that the participants had, and this workshop was much more valuable than a conference consisting of specialists in a given area XYZ.

The problem session on the last day of the workshop turned out to be a real hit. Many participants proposed future problems and research directions on Calabi–Yau varieties and mirror symmetry, and engaged in very lively discussions which made us almost forget the closing time of the workshop.

One of the significant outcomes of the workshop might be that we are finally beginning to understand the mirror symmetry phenomena of Calabi–Yau threefolds from the arithmetic point of view, namely in terms of the relations between zeta- and L-functions of mirror pairs of Calabi–Yau manifolds. We also formulated some future directions of research endeavors on mirror symmetry, e.g., further studies of algebraic cycles in connection with D-branes.

Here is a brief summary of the workshop schedule (for a full schedule, see below). After welcoming address of the new director, Kenneth Davidson, the kick-off lecture of the workshop was delivered by Professor S.-T. Yau on the *Mirror principle.* His lecture served to set the tone of the entire workshop. The schedule was arranged so that invited speakers delivered introductory lectures in the mornings and most of the afternoon lectures were for talks on recent developments. There were long breaks between lectures so that participants were able to engage in informal discussions. The lectures were roughly divided into six groups:

I. Mirror Principle (S.-T. Yau)

II. Toric Geometry and Mirror Symmetry (V. Batyrev, S. Hosono, T. Ito, B. Lian)

III. Periods, GKZ Hypergeometric Systems and Mirror Symmetry in Calabi–Yau Manifolds (C. Doran, S. Hosono, B. Lian, M. Saito)

IV. Moduli Theory of Calabi–Yau Manifolds (P. Berglund, I. Dolgachev, A. Todorov)

V. Regulators of Algebraic Cycles on Calabi–Yau Manifolds and Mirror Symmetry (A. Collino, P. L. del Angel, J. Lewis, S. Müller-Stach)

VI. Zeta-Functions and L-series of Calabi–Yau Varieties and Mirror Symmetry (P. Candelas, de la Ossa, Y. Goto, R. Schimmrigk, J. Stienstra, S. Tankeev, F. Villegas, H. Verrill, N. Yui)

About this Volume

This volume is the proceedings of the workshop, and it presents articles on the recent developments on Calabi–Yau Varieties and Mirror Symmetry. Some of the articles are written-up versions of the talks presented at the workshop, and others report on subsequent developments of the topics presented at the workshop. In addition, the editors have solicited articles from non-participants which are in total accordance with the workshop theme.

One of the goals of the Fields Communication Series is to publish articles that reach as wide audience as possible. Towards this goal, the editors encouraged each author to include an expository component to their paper. All papers included here have been refereed rigorously, and a number of articles went through extensive revisions to reach the standard imposed by the Fields Communication Series.

Here is a brief description of the articles in this volume.

The papers are divided roughly into two categories, namely Geometric Methods and Arithmetic Methods.

Geometric Methods

The articles of Batyrev and Materov, and of Chiang and Roan address a toric geometric approach to study Calabi–Yau manifolds. Batyrev and Materov, using the Cayley trick, define the notion of mixed toric residues and mixed Hessians associated with r Laurent polynomials $f_1, \cdots, f_r$, and conjecture that the values of mixed toric residues on the mixed Hessians are determined by mixed volumes of the Newton polytopes of $f_1, \cdots, f_r$. Further, using the mixed toric residues, they generalize their Toric Residue Mirror Conjecture to the case of Calabi–Yau complete intersections in Gorenstein toric Fano varieties obtained from nef-partitions of reflexive polytopes. Chiang and Roan determine the explict toric variety structure of $\mathrm{Hilb}^{A_1(n)}(\mathbf{C}^n)$ for $n = 4, 5$ where $A_1(n)$ is the special diagonal group of all order 2 elements, and further through the toric data of $\mathrm{Hilb}^{A_1(n)}(\mathbf{C}^n)$, they obtain certain toric crepant resolutions of $\mathbf{C}^n/A_1(n)$, and show that different crepant resolutions are connected by flops of n-folds for $n = 4, 5$.

The Picard–Fuchs differential equations and their interactions with K-theory via Abel–Jacobi maps are the main theme of the article of del Angel and Müller-Stach. The authors consider families of Calabi–Yau manifolds, and study the differential equations associated to the normal functions arising from algebraic cycles in the higher Chow groups. They describe the connection to the theory of analytically completely integrable Hamiltonian systems, and give several examples.

Lewis gives a survey on regulators of Chow cycles with a special emphasis on Calabi–Yau varieties. In this article he offers evidence that it is the Calabi–Yau varieties that provide the most interesting examples of regulator calculations. He also arrives at explicit formulae for the regulators under consideration

Hosono describes a method of counting BPS states using holomorphic anomaly equations. He studies Gromov–Witten invariants of a rational elliptic surface via holomorphic anomaly equations. After formulating invariants under the affine E_8 Weyl group symmetry, the author determines, from Gromov–Witten invariants, conjectured invariants and the number of BPS states.

Arithmetic Methods

The zeta-functions of a one-parameter family of quintic threefolds over finite fields, and of their mirror manifolds are studied in the article of Candelas, de la Ossa and Villegas. One of the remarkable properties of mirror symmetry is the existence of a mirror manifold $\mathcal{W}$ for which the roles of the complex structure parameters and the Kähler parameters are reversed with respect to those of the orginal Calabi–Yau manifold $\mathcal{M}$. The interpretation of the mirror symmetry phenomenon by means of the zeta-functions of a mirror pair of quintic Calabi–Yau threefolds is the main theme of their paper. A speculation is that there is a "quantum modification" of the zeta-functions that the zeta-functions of a mirror pair are the inverses of each other. In mirror symmetry an important role is played by the large complex structure limit, and the zeta-function seems to manifest an arithmetic analogue of the large complex structure limit which involves 5-adic expansion.

The modularity question of Calabi–Yau varieties are addressed in the articles of Yui, Verrill, and of Dieulefait and Manoharmayum. By the modularity of Calabi–Yau varieties defined over $\mathbf{Q}$, we mean that the associated L-series are conjecturally determined by some modular forms. Yui gives updates on the modularity question of Calabi–Yau varieties of dimensions $1, 2$ and 3 with a special emphasis on Calabi–Yau threefolds defined over $\mathbf{Q}$. Discussions on the modulaity of rigid Calabi–Yau threefolds are appended by the article of Verrill on the calculation of the L-series of certain rigid Calabi–Yau threefolds. Dieulefait and Manoharmayum give sufficient conditions for rigid Calabi–Yau threefolds over $\mathbf{Q}$ to be modular in terms of the 2–dimensional Galois representations. Yui further addresses the modularity question for non-rigid Calabi–Yau threefolds whose Galois representations are highly reducible.

Hypergeometric differential equations play essential role in the articles of Long, and of Villegas. Long gives an interpretation of a Shioda–Inose structure associated to a certain family of K3 surfaces in terms of Picard–Fuchs equations. In particular, the author constructs the j-invariant of a one-parameter family of elliptic curves which gives rise to a Shioda–Inose structure of a particular one-parameter family of M_n-polarized K3 surfaces. Villegas characterises all those Calabi–Yau manifolds which arise from hypergeometric functions. There are in total 14 such Calabi–Yau manifolds of dimension 3.

Various p-adic methods are employed in several articles. Ito, using p-adic Hodge theory and the Chebotarev density theorem, proves that birational smooth minimal models over $\mathbf{C}$ have equal Hodge numbers in all dimensions. This generalises a result of Batyrev on Betti numbers. Lian and Yau study the integrality property of certain n-th roots of the mirror maps of Calabi–Yau manifolds, using Dwork's theory of p-adic hypergeometric functions. Stienstra investigates, for families of smooth projective varieties over a localized polynomial ring $\mathbf{Z}[x_1, \cdots, x_r][D^{-1}]$, the conjugate filtration on De Rham cohomology $\otimes\mathbf{Z}/N\mathbf{Z}$. As $n \to \infty$ this leads to the concept of the *ordinary limit*, which seems to be the *non-archimedean analogue of the large complex structure limit.*

A number of articles discuss exclusively Calabi–Yau manifolds in positive characteristic. Goto studies K3 surfaces with symplectic group actions in characteristic $p > 0$. Among other things, he computes Picard numbers, the height of formal Brauer groups and the Artin invariant, with special attention to families of K3 surfaces in weighted projective spaces. In Stienstra's second paper *Ordinary*

Calabi–Yau-3 crystals, it is shown that crystals with the properties of crystalline cohomology of ordinary Calabi–Yau threefolds in characteristic $p > 0$ exhibit a remarkable similarity with the well known structure on the cohomology of complex Calabi–Yau threefolds near a boundary point of the moduli space with maximal unipotent local monodromy. In particular, there are canonical coordinates and an analogue of the prepotential of the Yukawa coupling in positive characteristic, as well as in p-adic settings.

The article of Schimmrigk *Aspects of conformal field theory from Calabi–Yau arithmetic* describes a link between arithmetic of Calabi–Yau manifolds via Hasse–Weil zeta-functions of the varieties, and the underlying conformal field theory. He shows that the algebraic number fields determined by the fusion rules of the conformal field theory are derived from the number-theoretic structure of the cohomological Hasse–Weil zeta-function of the variety.

In the second article, Schimmrigk and his collaborators address some mysterous observation made by Greg Moore that arithmetic structures of the so-called black hole attractor varieties admit complex multiplication. They describe further developments and generalizations of this observation, and argue how arithmetic structures are used in string theory.

This page intentionally left blank

The idea of mirror symmetry originated in physics, but in recent years, the field of mirror symmetry has exploded onto the mathematical scene. It has inspired many new developments in algebraic and arithmetic geometry, toric geometry, the theory of Riemann surfaces, and infinite-dimensional Lie algebras among others.

The developments in physics stimulated the interest of mathematicians in Calabi-Yau varieties. This led to the realization that the time is ripe for mathematicians, armed with many concrete examples and alerted by the mirror symmetry phenomenon, to focus on Calabi-Yau varieties and to test for these special varieties some of the great outstanding conjectures, e.g., the modularity conjecture for Calabi-Yau threefolds defined over the rationals, the Bloch-Beilinson conjectures, regulator maps of higher algebraic cycles, Picard-Fuchs differential equations, GKZ hypergeometric systems, and others.

The articles in this volume report on current developments. The papers are divided roughly into two categories: geometric methods and arithmetic methods. One of the significant outcomes of the workshop is that we are finally beginning to understand the mirror symmetry phenomenon from the arithmetic point of view, namely, in terms of zeta-functions and L-series of mirror pairs of Calabi-Yau threefolds.

The book is suitable for researchers interested in mirror symmetry and string theory.

ISBN 0-8218-3355-3

9 780821 833551

FIC/38