FIELDS INSTITUTE COMMUNICATIONS

THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES

Vertex Operator Algebras in Mathematics and Physics

Stephen Berman
Yuly Billig
Yi-Zhi Huang
James Lepowsky
Editors

American Mathematical Society

Vertex Operator Algebras in Mathematics and Physics

This page intentionally left blank

http://dx.doi.org/10.1090/fic/039

FIELDS INSTITUTE COMMUNICATIONS

THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES

Vertex Operator Algebras in Mathematics and Physics

Stephen Berman
Yuly Billig
Yi-Zhi Huang
James Lepowsky
Editors

American Mathematical Society
Providence, Rhode Island

The Fields Institute for Research in Mathematical Sciences

The Fields Institute is named in honour of the Canadian mathematician John Charles Fields (1863–1932). Fields was a visionary who received many honours for his scientific work, including election to the Royal Society of Canada in 1909 and to the Royal Society of London in 1913. Among other accomplishments in the service of the international mathematics community, Fields was responsible for establishing the world's most prestigious prize for mathematics research—the Fields Medal.

The Fields Institute for Research in Mathematical Sciences is supported by grants from the Ontario Ministry of Education and Training and the Natural Sciences and Engineering Research Council of Canada. The Institute is sponsored by McMaster University, the University of Toronto, the University of Waterloo, and York University, and has affiliated universities in Ontario and across Canada.

2000 *Mathematics Subject Classification.* Primary 17B69, 17B81, 17B67, 17B68, 17B65, 81T40, 81T30, 81R10, 57R56.

Library of Congress Cataloging-in-Publication Data

Vertex operator algebras in mathematics and physics / Stephen Berman . . . [et al.], editors.
p. cm. — (Fields Institute communications, ISSN 1069-5265 ; v. 39)
Includes bibliographical references.
ISBN 0-8218-2856-8 (acid-free paper)
1. Vertex operator algebras—Congresses. 2. Mathematical physics—Congresses. I. Berman, Stephen, 1944–. II. Series.

QA326.V47 2003
512′.55—dc22 2003062777

This publication was prepared by The Fields Institute.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 08 07 06 05 04 03

Contents

Preface

Vertex operator algebras are a class of algebras underlying a number of recent constructions, results and themes in mathematics. These structures arose naturally from vertex operator constructions of representations of affine Lie algebras and in the work of Frenkel-Lepowsky-Meurman and Borcherds on the "moonshine module" for the Monster finite simple group. The physical notion of "chiral algebra" in two-dimensional conformal quantum field theory is essentially equivalent to the mathematical notion of vertex operator algebra. These algebras can be understood as "string-theoretic analogues" of Lie algebras and also of commutative associative algebras, and they play fundamental roles in a range of areas of mathematics and physics.

In November 1983, before the notion of vertex (operator) algebra was formalized, a conference titled "Vertex Operators in Mathematics and Physics" was held at The Mathematical Sciences Research Institute in Berkeley, and one of us was a co-organizer of that conference. New and recent developments in mathematics and physics at that time were announced and discussed, many of them centering around the theme of certain operators called "vertex operators." These developments involved representations of affine Lie algebras and of the Virasoro algebra, combinatorial identities, the moonshine module and the Monster, integrable systems, conformal field theory and string theory. Two-dimensional conformal field theory was being formalized around that time, and string theory was destined to flourish in "the first string theory revolution" in 1984. There was already a sense that interesting things were likely to happen if mathematicians and physicists interacted at workshops such as these. The proceedings of that conference were published in the Mathematical Sciences Research Institute Publications #3, edited by J. Lepowsky, S. Mandelstam and I. M. Singer, Springer–Verlag, New York, 1985.

During the years after that conference, we have witnessed many deep and exciting developments in the areas of mathematics and physics related to vertex operators. The study of representations of affine Lie algebras and the moonshine module led to the notion of vertex (operator) algebra, and Borcherds completed the proof of the "monstrous moonshine" conjectures of Conway-Norton. The study of integrable systems stimulated the theory of quantum groups. Two-dimensional conformal field theory was greatly developed as a mathematical and physical theory linking a number of seemingly disparate areas. As a result of the "revolution" in 1984, string theory became the main candidate for a unified theory of all the fundamental forces. These subjects have become some of the most active research areas in mathematics and physics.

From the beginning of these developments, some of the most significant progress on both the physics and mathematics sides has continued to benefit from cross-pollination between the physical and mathematical points of view. To help promote such interaction, a workshop called "Vertex Operator Algebras in Mathematics and

Physics" was held at The Fields Institute for Research in Mathematical Sciences, Toronto, October 23–27, 2000. It was felt that such a workshop would be very appropriate at this time. In some sense, this conference, devoted to a variety of topics related to the theme of vertex operator algebras, can be viewed as a kind of sequel to the 1983 conference. We hope that it has promoted continuing and mutually beneficial interaction between mathematicians and physicists. The present volume consists of papers based on many of the talks at the conference.

Here is a brief outline of the papers in this volume. In the paper of T. Abe and K. Nagatomo, the authors establish the finiteness of the dimensions of conformal blocks over the projective line associated to a vertex operator algebra satisfying a certain finiteness condition. P. Bantay's paper reviews the theory of permutation orbifolds and applies it to symmetric product orbifolds and the conformal-field-theoretic "congruence subgroup problem." The paper by J. Fuchs and C. Schweigert studies algebra objects in tensor categories and applies the results to the description of boundary conditions in two-dimensional conformal field theory. R. Griess's paper is one in a series of papers exploring vertex operator algebras of a certain type, their automorphism groups, and the automorphism groups of their weight 2 parts.

In the paper by G. Höhn, the notion of genus of a quadratic form is generalized to vertex operator algebras, and extension problems for vertex operator algebras are related to the associated modular braided tensor categories. Y.-Z. Huang's paper discusses the relationship between Riemann surfaces with boundaries and the theory of vertex operator algebras, in connection with the definition of two-dimensional conformal field theory in the sense of Kontsevich and Segal and its generalization to open-closed conformal field theories. In the paper by H. Li, the author's earlier results on how to construct (generalized) vertex algebras from certain sets of vertex operators are summarized, with an emphasis on the viewpoint that vertex (operator) algebras are "algebras" of vertex operators, and related results are developed. A. Milas's paper studies Lie (super)algebras of differential operators on the circle from the vertex operator algebra point of view, and discusses relations with the zeta-function and modularity.

The paper of M. Primc extends results in the author's joint work with A. Meurman on defining relations for standard modules for $A_1^{(1)}$ and $A_2^{(1)}$, and relations among the relations, to general affine Lie algebras. A. Recknagel's paper reviews central ideas from string theory and conformal field theory and discusses "translation rules" from the standard description of strings and branes to notions from the world-sheet approach. In the paper by V. Schomerus, the author reviews and develops recent work on the relation between noncommutative geometry, open strings and D-branes in nontrivial backgrounds. The Schweigert–Fuchs paper investigates the mathematical structure of the world sheet in two-dimensional conformal field theories.

The present time seems particularly ripe for a resurgence of interaction between mathematicians and physicists studying a range of aspects of vertex operator algebra theory and related subjects. We hope that the papers in this volume will be helpful to both experts and beginners and will contribute to new developments in this exciting area of mathematics and physics.

We would like to thank The Fields Institute for inviting us to organize this conference within their program entitled "Infinite Dimensional Lie Theory and Its

Applications." Thanks are also due to their cheerful and expert staff for helping with the running of the conference and with the preparation of this volume.

S. Berman
Y. Billig
Y.-Z. Huang
J. Lepowsky

This page intentionally left blank

Schedule of Talks

Monday, October 23

Riemann surfaces with boundaries and vertex operator algebra theory
Yi-Zhi Huang (Rutgers University)

Automorphic and combinatorial properties of free field theories at higher genus
Geoffrey Mason (University of California, Santa Cruz)

D-branes in curved space
Michael Douglas (Rutgers University)

Regular representations and Zhu's $A(V)$-theory
Haisheng Li (Rutgers University)

Tuesday, October 24

Vertex operator algebras and dual pairs
Chongying Dong (University of California, Santa Cruz)

Solitonic representations and boundary categories
Jürgen Fuchs (Karlstad University)

The minimal model on the projective line
Kiyokazu Nagatomo (Osaka University)

Seiberg - Witten differential as a period of rational elliptic surface
Akihiro Tsuchiya (Nagoya University)

Wednesday, October 25

From branes to boundary conformal field theory
Andreas Recknagel (King's College, University of London)

Brane dynamics and non-commutative geometry
Volker Schomerus (Max Planck Institute)

Recent results on automorphism groups of VOAs
Robert Griess (University of Michigan)

Aspects of the construction of conformal field theories
James Lepowsky (Rutgers University)

Thursday, October 26

Integrable highest weight modules and vertex operator algebras
Arne Meurman (Lund University)

Basic representations for classical affine Lie algebras
Mirko Primc (University of Zagreb)

Sheaves of vertex algebras
Fyodor Malikov (University of Southern California)

Vertex operator algebras and ζ-functions
Antun Milas (Rutgers University)

Friday, October 25

Change of variables formulas for $N = 1$ Neveu-Schwarz vertex operator superalgebras and alternate notions of superconformality
Katrina Barron (University of California, Santa Cruz)

Genera of vertex operator algebras and 3-dimensional topological quantum field theories
Gerald Höhn (University of Freiburg)

Permutation orbifolds
Peter Bantay (Eötvös Loránd University)

$\widehat{\mathfrak{sl}}_2$-type vertex operator algebras and semi-infinite constructions of integrable representations
Boris Feigin (Landau Institute of Theoretical Physics)

Titles in This Series

39 **Stephen Berman, Yuly Billig, Yi-Zhi Huang, and James Lepowsky, Editors,** Vector operator algebras in mathematics and physics, 2003

38 **Noriko Yui and James D. Lewis, Editors,** Calabi-Yau varieties and mirror symmetry, 2003

37 **Panos Pardalos and Henry Wolkowicz, Editors,** Novel approaches to hard discrete optimization, 2003

36 **Shigui Ruan, Gail S. K. Wolkowicz, and Jianhong Wu, Editors,** Dynamical systems and their applications in biology, 2003

35 **Yakov Eliashberg, Boris Khesin, and François Lalonde, Editors,** Symplectic and contact topology: Interactions and perspectives, 2003

34 **T. J. Lyons and T. S. Salisbury, Editors,** Numerical methods and stochastics, 2002

33 **Franz-Viktor Kuhlmann, Salma Kuhlmann, and Murray Marshall, Editors,** Valuation theory and its applications, Volume II, 2003

32 **Franz-Viktor Kuhlmann, Salma Kuhlmann, and Murray Marshall, Editors,** Valuation theory and its applications, Volume I, 2002

31 **A. Galves, J. K. Hale, and C. Rocha, Editors,** Differential equations and dynamical systems, 2002

30 **Roberto Longo, Editor,** Mathematical physics in mathematics and physics: Quantum and operator algebraic aspects, 2001

29 **Teresa Faria and Pedro Freitas, Editors,** Topics in functional differential and difference equations, 2001

28 **David R. McDonald and Stephen R. E. Turner, Editors,** Analysis of communication networks: Call centres, traffic and performance, 2000

27 **Shui Feng and Anna T. Lawniczak, Editors,** Hydrodynamic limits and related topics, 2000

26 **Neal Madras, Editor,** Monte Carlo methods, 2000

25 **A. G. Ramm, P. N. Shivakumar, and A. V. Strauss, Editors,** Operator theory and its applications, 2000

24 **Edward Bierstone, Boris Khesin, Askold Khovanskii, and Jerrold E. Marsden, Editors,** The Arnoldfest: Proceedings of a conference in honour of V. I. Arnold for his sixtieth birthday, 1999

23 **Myrna H. Wooders, Editor,** Topics in mathematical economics and game theory. Essays in honor of Robert J. Aumann, 1999

22 **Graciela Chichilnisky, Editor,** Topology and markets, 1999

21 **Shigui Ruan, Gail S. K. Wolkowicz, and Jianhong Wu, Editors,** Differential equations with applications to biology, 1998

20 **Peter A. Fillmore and James A. Mingo, Editors,** Operator algebras and their applications II, 1998

19 **William G. Dwyer, Stephen Halperin, Richard Kane, Stanley O. Kochman, Mark E. Mahowald, and Paul S. Selick (Editor-in-Chief), Editors,** Stable and unstable homotopy, 1998

18 **Panos M. Pardalos and Henry Wolkowicz, Editors,** Topics in semidefinite and interior-point methods, 1998

17 **Joachim J. R. Cuntz and Masoud Khalkhali, Editors,** Cyclic cohomology and noncommutative geometry, 1997

16 **Victor P. Snaith, Editor,** Algebraic K-theory, 1997

15 **Stephen P. Braham, Jack D. Gegenberg, and Robert J. McKellar, Editors,** Sixth Canadian conference on general relativity and relativistic astrophysics, 1997

14 **Mourad E. H. Ismail, David R. Masson, and Mizan Rahman, Editors,** Special functions, q-series and related topics, 1997

TITLES IN THIS SERIES

13 **Peter A. Fillmore and James A. Mingo, Editors,** Operator algebras and their applications, 1997

12 **Dan-Virgil Voiculescu, Editor,** Free probability theory, 1997

11 **Colleen D. Cutler and Daniel T. Kaplan, Editors,** Nonlinear dynamics and time series: Building a bridge between the natural and statistical sciences, 1997

10 **Jerrold E. Marsden, George W. Patrick, and William F. Shadwick, Editors,** Integration algorithms and classical mechanics, 1996

9 **W. H. Kliemann, W. F. Langford, and N. S. Namachchivaya, Editors,** Nonlinear dynamics and stochastic mechanics, 1996

8 **Larry M. Bates and David L. Rod, Editors,** Conservative systems and quantum chaos, 1996

7 **William F. Shadwick, Perinkulam Sambamurthy Krishnaprasad, and Tudor Stefan Ratiu, Editors,** Mechanics day, 1996

6 **Anna T. Lawniczak and Raymond Kapral, Editors,** Pattern formation and lattice gas automata, 1996

5 **John Chadam, Martin Golubitsky, William Langford, and Brian Wetton, Editors,** Pattern formation: Symmetry methods and applications, 1996

4 **William F. Langford and Wayne Nagata, Editors,** Normal forms and homoclinic chaos, 1995

3 **Anthony Bloch, Editor,** Hamiltonian and gradient flows, algorithms and control, 1994

2 **K. A. Morris, Editor,** Control of flexible structures, 1993

1 **Michael J. Enos, Editor,** Dynamics and control of mechanical systems: The falling cat and related problems, 1993

This book presents the proceedings from the workshop, "Vertex Operator Algebras in Mathematics and Physics", held at The Fields Institute. It consists of papers based on many of the talks given at the conference by leading experts in the algebraic, geometric, and physical aspects of vertex operator algebra theory.

The book is suitable for graduate students and research mathematicians interested in the major themes and important developments on the frontier of research in vertex operator algebra theory and its applications in mathematics and physics.

ISBN 0-8218-2856-8

9 780821 828564

FIC/39