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THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES

Partially Hyperbolic Dynamics, Laminations, and Teichmüller Flow

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Preface

This volume collects a set of contributions by participants of the Workshop "Partially hyperbolic dynamics, laminations, and Teichmüller flow" held in the Fields Institute in January 2006. This Workshop brought together several leading experts in two very active fields of contemporary dynamical systems theory: partially hyperbolic dynamics and Teichmüller flow. They are unified by ideas coming from the theory of laminations and foliations, dynamical hyperbolicity, and ergodic theory, which made interaction between experts from different fields natural and productive. These are the main themes of the current volume.

Roughly speaking, the main goal of dynamical systems theory is to describe the statistical behavior of trajectories of a typical system of some natural class. Tremendous progress in this direction was made in the 1960–1970's by Smale, Anosov, Sinai, Ruelle, and other people, who achieved this goal for uniformly hyperbolic systems. These are systems for which all recurrent orbits diverge (uniformly) exponentially rapidly, either in forward or backward time. These systems exhibit sensitive dependence on initial conditions, and generate horseshoes, strange attractors, and a variety of complex behaviours. Paradoxically, this extreme instability makes them both treatable and statistically robust.

A natural extension of this class relaxes the property of exponential divergence of all recurrent orbits leading to the class of *partially hyperbolic* systems. In the mid 1990's C. Pugh and M. Shub formulated a program of exploring the *Boltzman Ergodicity Conjecture* for this class. This led to an explosive development of the field in the past decade that was reflected at the Workshop.

The main conjecture by Pugh and Shub is that the generic volume preserving, partially hyperbolic system is *robustly ergodic*. The impressive progress that has been recently achieved in this direction appears in the article by Federico and Maria Rodriguez Hertz and Raul Ures. Their article provides an up-to-date survey of partially hyperbolic dynamics. It includes a discussion of the recent work of Burns and Wilkinson which removes some of the integrability assumptions in Pugh and Shub's earlier results, and which weakens the "exponential bunching" hypotheses. In a separate paper by the same authors, the Pugh and Shub Conjecture is proved in the case that the neutral direction of the partially hyperbolic splitting has dimension one.

Beyond hyperbolicity and partial hyperbolicity lies the concept of a *dominated splitting*: orbits group themselves into families whose relative behaviour is exponentially distinct: the hyperbolic dichotomy that stable vectors exponentially shrink while unstable vectors exponentially stretch is replaced by exponential decay of the ratio of stable-to-unstable. The concept has its origins in the works of Pliss, Mañé, and Liao. Its current state is exposed in a survey paper by Enrique Pujals, with various applications to surface diffeomorphisms, billiards, and geodesic flows.

Finally, robust dynamics beyond those with dominated splittings are explored in a survey paper by Lorenzo Diaz. The key ideas there are those of "blenders" and "heterodimensional cycles." Diaz pays particular attention to the last outstanding case of Smale's Stability Conjecture from the 60s: "The generic C^1 surface diffeomorphism satisfies Axiom A and the no-cycle condition (it is therefore structurally stable)". This is an open question even on the 2-sphere!

The *Teichmüller flow* is a conservative flow on the moduli space of quadratic or abelian differentials on Riemann surfaces which plays a crucial role in the study of several basic classes of low-dimensional dynamical systems, such as *interval exchange transformations (IET's)* and *billiards in polygons* with rational angles. It can be viewed as the renorm-group for these dynamical systems.

The foundations of the field of Teichmüller dynamics were laid down about twenty years ago in the pioneering work of H. Masur and W. Veech who proved by its means the unique ergodicity of the typical IET (*Keane Conjecture*). More recently, A. Zorich and M. Kontsevich formulated a series of conjectures on the Lyapunov spectrum of the flow and on the related deviations for the ergodic averages of IET's. The *Zorich-Kontsevich Conjectures* were proved by G. Forni, with the exception of the simplicity of the spectrum. This last step has been recently accomplished by A. Avila and M. Viana.

The fundamental idea underlying all applications of Teichmüller dynamics to the theory of IET's, measured foliations, translation surfaces or rational polygonal billiards can be summarized as follows. Any such system determines an abelian or quadratic differential in the moduli space, and the dynamical properties of the system can be read off from the properties of the closure of the corresponding orbit of the natural $SL(2,\mathbb{R})$-action on the moduli space. In this perspective the fundamental problems of the theory are related to the classification of all $SL(2,\mathbb{R})$ orbit closures and of all $SL(2,\mathbb{R})$ invariant measures. For example, Veech proved that if the $SL(2,\mathbb{R})$ orbit of a translation surface in the moduli space is closed, then every directional flow is either completely periodic or uniquely ergodic (*Veech Dichotomy*). The study of such surfaces, called *Veech surfaces*, is currently a very active area of research. In particular, K. Calta and C. McMullen have independently obtained a complete classification of all Veech surfaces in genus two.

In her lively talk at the Workshop, Kariane Calta addressed one of the crucial open problems in Teichmüller dynamics, that is, the classification of all measures invariant under the $SL(2,\mathbb{R})$ action. She presented partial results for the moduli space of abelian differentials on surfaces of genus two. Her joint paper with John Smillie is a survey of the "algebraic theory" of translation surfaces (or abelian differentials) which is behind many spectacular recent advances in the theory of Veech surfaces and of $SL(2,\mathbb{R})$ invariant measures in genus two. In higher genus the theory is not yet as effective but many ideas, such as the notion of "J-invariant", can be generalized and will presumably play a crucial role, for instance in the description of exceptional surfaces, characterized by the property that their $SL(2,\mathbb{R})$ orbit is not dense in an appropriate stratum of the moduli space.

Related themes are discussed by John Smillie and Barak Weiss, who survey their recent work aimed at the classification of $SL(2,\mathbb{R})$ closed invariant sets, in particular of closed Teichmüller disks and of Veech surfaces, by means of several new geometric and combinatorial invariants of translation surfaces. Together with the survey by Calta and Smillie, this survey provides an efficient introduction to the theory of Veech surfaces and of Veech groups. The survey is concluded by a list of open problems.

Important applications of Teichmüller dynamics to IET's (and related systems) are presented in this volume by Marcelo Viana, who expanded his talk at the Workshop into a survey of Zorich theory on the deviations of ergodic averages of IET's and of his joint work with Avila mentioned above. As a preparation, this paper provides an introduction to the dynamics of IET's from the renormalization point of view of the *Rauzy-Veech-Zorich inductions*, which are equivalent to first return maps of the Teichmüller flow to appropriate transverse sections.

Another important application of Teichmüller dynamics is given by Yitwah Cheung and Alex Eskin who revisit Masur's work on the Keane conjecture by proving a stronger criterion for the unique ergodicity of IET's. They present an outline of their recent proof that a minimal IET is uniquely ergodic provided that the corresponding Teichmüller orbit does not escape to infinity too fast. The Cheung-Eskin result is nearly optimal and illustrates in a refined way the basic principle that "Diophantine" properties of points of the moduli space can be read off from the rate of escape of their orbits under the Teichmüller geodesic flow.

The ergodic theory of the Teichmüller flow was discussed at the Workshop in the talk by A. Bufetov who presented his proof of the Central Limit Theorem for the flow based on Veech's construction of the "zippered rectangle flow" and on the direct analysis of the related Rauzy-Veech-Zorich algorithms. In this volume, Artur Avila and Alexander Bufetov present the proof of a related theorem on the exponential decay of correlations for the Zorich induction. This result stems from earlier work by Bufetov on the problem and from work of Avila, Gouëzel and Yoccoz, who recently proved the exponential decay of correlations for the Teichmüller flow.

At the Workshop A. Zorich gave an introduction to the field of Teichmüller dynamics in the first two lectures of his mini-course. In the third lecture, he went on to describe the connected components of the strata of the moduli space of abelian differentials in topological and combinatorial terms. This discussion was further elaborated upon in the talk by H. Masur who described his joint work with Zorich on the generic degenerations of quadratic differentials near the boundary of the moduli space. This work extends to the case of quadratic differentials an earlier work by A. Eskin, H. Masur and A. Zorich dealing with abelian differentials. A complete outline of all the main ideas and methods of this development is presented in the paper by Howard Masur and Anton Zorich that closes the second part of the volume.

The last part of the book contains several contributions on miscellaneous topics related to the main themes of the Workshop. It is opened with an introductory survey by Andrzej Bis on foliations and laminations that describes basic properties of these objects with a focus on the entropy invariant. It proceeds with a number of examples of foliations related to fractal sets that possess various interesting properties.

Ruelle-Sinai thermodynamical formalism is one the most powerful methods in the smooth ergodic theory. However, its applications usually require some kind of symbolic model of a system under consideration. The survey paper by Yakov Pesin and Ke Zhang extends this machinery to a large class of dynamical systems admitting a "tower representation" (a specific symbolic description with infinitely many states).

In late 1970s Katok constructed an example of an area-preserving two-dimensional surface diffeomorphism which is measure-theoretically isomorphic to the Bernoulli shift. The idea was to find a smooth realization to a Thurston's pseudo-Anosov map. The paper by Marlies Gerber in this volume gives a new, technically simpler, version of this construction. (In fact, as Pesin has recently proved and presented at the Workshop, any manifold of dimension at least three admits such a diffeomorphism.)

In mid 1990's Lyubich and Minsky constructed a hyperbolic three-dimensional lamination associcated to any rational endomorphism f of the Riemann sphere. This lamination can be refined to a natural horocycle lamination that encodes properties of the derivative cocycle of f. The paper by Alexey Glutsuk proves minimality of this horocycle lamination in the convex co-compact case. This result is analogous to the classical Hedlund's Theorem for horocycle flows on Riemann surfaces.

The limit set of any finitely generated Kleinian group carries a nice geometric measure called the Patterson-Sullivan conformal measure. The last paper in the volume, by Mark Pollicott, gives a construction of such a measure for a class of non-discrete groups of Möbius transformations.

June 2007
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