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To my parents,

who made it possible for me
to study mathematics

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Preface

These notes are based on a one-semester graduate course I gave at The Fields Institute in Fall, 1995 as part of the homotopy theory program which constituted the institute's major program that year. The purpose of the course was to bring graduate students who had completed a first course in algebraic topology to the point where they could understand research lectures in homotopy theory and to prepare them for the other, more specialized, graduate courses being held in conjunction with the program. The notes are divided into two parts: prerequisites and those which constituted the course proper.

Part I, the prerequisites, contains a review of material which might be taught in a first course in algebraic topology. Although it is probable that each of the topics discussed there has appeared in *some* such course, it seems unlikely that any individual first course would contain all of these topics. The students were expected to use this section to help fill in the gaps in their background knowledge. Chapter 4 (homological algebra) in particular contains some material which is likely to be new to most readers at this level. Not all of the material in Part I is applied in these notes; much is presented for completeness or to give a better overall picture of a topic. The reader may wish to skim or skip sections, referring back to them if needed. It is hoped that Part I will provide a useful summary for students and non-specialists who are interested in learning the basics of algebraic topology.

Experts in homotopy theory will recognize immediately upon seeing the chapter titles for Part II that it is not possible to cover all of those topics in detail in a one semester course; indeed many of them could constitute the material for such a course all by themselves. These notes, which were available in preliminary draft form at the time of the course, contain many details which were referred to during the lectures but not presented in detail.

An attempt has been made to make these notes self contained in an expositional sense. Even in the sections on prerequisites the development contains motivation for the major ideas and exact statements of definitions and theorems although the proofs are usually omitted in those sections. The percentage of statements for which a proof is given increases throughout the notes, starting at nearly 0% in Part I, increasing dramatically in Part II, nearing 100% by Chapter 11. Although time constraints prevented me from doing so in all cases, I was particularly concerned about providing a proof for those statements for which a proof is not readily available in the standard textbooks on the subject, since the original papers are often harder for students at this level to read.

The course consisted of eight major topics which appear as the eight chapters of Part II, numbered 7–14. The first of these, Chapter 7, is a lengthy chapter on "classical homotopy theory". It contains cellular approximation, the Hurewicz theorem, properties of H-spaces and co-H-spaces, Whitehead and Samelson products, and a large amount of space devoted to the manipulations of fibrations and cofibrations. It also introduces the James construction, Hopf-invariant maps, and

Lusternik-Schnirrelmann category. Some of this material appears in many places in the literature while other parts would fall into the category of "folk theorems". For example, many product and wedge decompositions which result from manipulating fibrations and cofibrations are given. These are well known and in everyday use but not easy to find in the literature.

Chapter 8 contains the basics of simplicial sets. It discusses Kan complexes, the singular complex, simplicial groups, simplicial abelian groups, and how the homotopy theory of simplicial sets parallels that of topological spaces. However some material, such as the simplicial classifying construction and simplicial loop construction, is omitted and readers who intend to make large use of simplicial sets will need to consult a book devoted explicitly to that subject for extra details. In these notes, simplicial sets are used primarily in discussing localization (Chapter 12). Although it is not explicitly a statement about simplicial sets. The fact that the quotient map from the singular chain complex of a space to that complex modulo its degenerate subcomplex is a chain homotopy equivalence appears in this chapter as a consequence of a theorem on simplicial abelian groups. This is used later in these notes in the section on Adams-Hilton models.

Chapter 9 is a brief introduction to fibre bundles. Basic definitions, classification theorems, and Milnor's construction are given. A student of algebraic topology would be well advised to read more on this subject, for example in Milnor's "Characteristic Classes" [MS].

The material of Chapter 10 on Hopf algebras is taken mostly from the classic paper by Milnor and Moore [MM] on this subject. The reader will have to refer to that source for proofs which (s)he cannot work out alone.

Chapter 11 is a long chapter on spectral sequences. The first half of the chapter develops the general theory of spectral sequences including a discussion of filtered complexes, exact couples, and the notion of convergence of the spectral sequence of a filtered complex. Then, with one exception, the major spectral sequences used in homotopy theory are discussed individually in the second half of the chapter. It is probably inevitable that parts of this chapter will be heavy going for readers who are seeing the material for the first time and rereading is recommended. The major spectral sequence in homotopy theory omitted from these notes is the Adams spectral sequence. In the context of the homotopy program at The Fields Institute, it would not have made sense to include that in this course since an entire course on the Adams spectral sequence was being given by Stan Kochman alongside this course. Readers are referred to the companion volume by Stan Kochman in this monograph series for a complete discussion of the Adams spectral sequence. The presentations of the two forms of the Eilenberg-Moore spectral sequence, that from base to fibre and that from fibre to base are given from opposite points of view. For the second, we take a geometric approach and obtain the spectral sequence by means of a suitable filtration on a space. This has some obvious advantages including giving us for free the commutativity of the spectral sequence with cohomology operations and ready generalization to generalized (co)homology theories. For the first Eilenberg-Moore spectral sequence we take a very algebraic approach closely related to Eilenberg and Moore's original proof. This allows the demonstration of homological techniques useful in other contexts and also demonstrates clearly how much information is lost when one passes from the singular chain complex on a space to its homology. The reader should be aware however that there are more geometric approaches to deriving this spectral sequence, although it would make sense to first develop the properties of a category of spectra to avoid having to express the statements in an awkward form. See Smith [Sm] for a derivation from this point of view. Also included in this chapter is a discussion of Adams-Hilton models. From a logical point of view, this material would fit better into Chapter 7, but the proof of the main theorem uses some material presented in Chapter 11. The section on Adams-Hilton models also contains a brief introduction to the homotopy theory of differential graded algebras.

Chapter 12 discusses localization and completion. Alternative methods of localization are mentioned, and details are given for the Bousfield-Kan construction with an outline of the proof that it does indeed construct a localization. There is also an introduction to cosimplicial sets which are used in the construction. The reader is referred to the book by Bousfield and Kan [BK] for more information about cosimplicial sets and the Bousfield-Kan construction.

The main purpose of Chapter 13 is to develop the connection between generalized cohomology theory and representing spaces, and in particular the Brown Representability Theorem. This provides the framework for the discussion of cohomology operations in the following chapter. Spectra and stable homotopy concepts are introduced, but there is much more that can be (and has been!) said on these subjects. More information can be found in Stan Kochman's monograph in this series. The reader is also referred to the books by Adams [A2] and Lewis, May, Steinberger [LMS] for detailed construction of categories of spectra and their properties. The reader interested in stable homotopy theory will want to read some works devoted entirely to that subject.

The last chapter applies material from many of the earlier chapters to the discussion of cohomology operations and in particular the construction and properties of the Steenrod algebra, and some sample applications. There is also a glimpse into the subject of secondary cohomology operations.

The topics covered in these notes are, for the most part, those referred to in Adams guide [A1]. An omission from the topics mentioned there is Postnikov systems, but the reader who has understood the main ideas of these notes should be able to find and understand a discussion of this without much difficulty. The main addition to the topics in Adams list is localization and completion (Chapter 13) whose growth into a major subject has taken place during the interval since [A1] appeared.

The reader of these notes should watch for occasional out-of-sequence numbers. In the interest of collecting together material on a given subject, occasionally the statement of a theorem whose proof requires later techniques will appear in an earlier section if it fits with the material at hand and the reader can be reasonably expected to understand the statement at that time. In these cases the fact that the theorem's number shows that it belongs to a later section will serve to alert the reader that the statement is being given only for reference and intuitional purposes, and that the statement will reappear in the appropriate place and will not be used in the interval in the proof of other theorems.

Another convention used in these notes is in connection with the symbol $\square$. This symbol is used to mean "end of discussion" and so in addition to appearing at the end of proofs may also appear at the end of examples or at the conclusion of the statement of a theorem whose proof has either been omitted or given previously.

I would like to thank those who attended the course both for their contributions to the course and for pointing out the mistakes and misprints which they observed in earlier drafts of these notes. In particular I mention Kasper Anderson, Wojcieck Chacholsky, Arleigh Crawford, Jasper Grodal, Sadok Kallel, Michael Mather, Jerome Scheerer, and Bjorn Schuster. Above all I want to thank Matvei Libine for his invaluable assistance in the preparation of these notes, ranging from correction of misprints and mathematical errors to stylistic suggestions and advice on which additional proofs or examples would be most useful to a reader seeing this material for the first time. Finally I would like to thank The Fields Institute for providing an atmosphere and facilities so conducive to the discussion of mathematics.

Paul Selick
October 1996

Summary of Global Notation

For each notation, we give a brief description and/or the nearest result or section number to the spot where the notation is first defined. Some notations appear more than once in the following list when the same symbol is used for different things in different contexts.

$\mathcal{A}$	14.2.1	the Steenrod algebra when the prime is understood
$\mathcal{A}_*$	14.5.1	the dual of the Steenrod algebra when the prime is understood
$\mathcal{A}_{(p)}$	14.2.1	the mod p Steenrod algebra
$\mathcal{A}(-)$	11.10.7	Adams-Hilton model of a space
$\mathcal{AB}$		the category of abelian groups
AW	5.5.2	Alexander-Whitney homomorphism
$B(-)$	9.2.2	classifying space of a topological group
$B_n(-)$	4.1.1	boundaries of a chain complex
$B_n(-)$	9.2.2	nth Milnor filtration on the classifying space of a topological group
$\mathbb{C}$		complex numbers
$C(-)$	7.1.16	reduced cone on a pointed space
$C_*(-)$	5.4.1	cellular chain complex of a space
C_f	7.1.16	reduced mapping cone of a function
CC	4.2.1	the category of chain complexes
CoTor	11.11.1	derived functor of $\square$
$\mathbb{C}P^n$		n-dimensional complex projective space
d		often used for the differential in a chain complex
$\hat{D}$	11.3.2	elements infinitely divisibly by i in the D-term of an exact couple
$\check{D}$	11.3.2	direct limit of D-term in an exact couple
d_i	8.1.1	boundary map of a simplicial set
d^r	11.3.2	differential in the E^r-term of a spectral sequence or the composition $j^r k^r$ in an exact couple
D^n		n-dimensional disk
D^r	11.3.2	term in an exact couple
$D(-)$	8.5.1	degenerate sub-chain-complex of a simplicial set
DGA	11.6.1	differential graded algebra
$E(-)$	9.2.2	total space of the universal bundle of a topological group
$E_n(-)$	9.2.2	nth Milnor filtration on the total space the universal bundle of a topological group
E^r	11.2.1	term in a spectral sequence or exact couple
$E(V)$		exterior algebra on a vector space or free module

$E[S]$		exterior algebra on the vector space or free module generated by a set
Ext	4.3.5	derived functor of Hom
EZ	5.5.2	Eilenberg-Zilber homomorphism
$\mathcal{F}_{(-)}$	11.1.1	a filtered object
$F_n(-)$	11.1.1	nth filtration of a filtered object
$\mathrm{FW}_n(-)$	7.9.1	nth fat wedge of a pointed space
$\mathrm{Gr}(-)$	11.1.1	associated graded object to a filtered object
h_X	7.4.3	Hurewicz homomorphism
$h_{(X,A)}$	7.4.3	relative Hurewicz homomorphism
$\mathrm{Hom}_{\underline{\underline{C}}}(-,-)$	1.1.1	set of morphisms in a category. Equivalent to $\underline{\underline{C}}(-,-)$
$H_*(-)$	4.1.1	homology of a chain complex or space
$\tilde{H}_*(-)$	4.1.1	reduced homology of a space
$H_*(-;G)$		homology of a chain complex or space with coefficients in a module
$H_*(-;{}^tL)$	11.9.1	homology of a space with local coefficients
$\tilde{H}_*(-;G)$	4.1.1	reduced homology of a space with coefficients in a module
$H^*(-)$	4.1.1	cohomology of a cochain complex or space
$\tilde{H}^*(-)$	4.1.1	reduced cohomology of a space
$H^*(-;G)$		cohomology of a cochain complex or space with coefficients in a module
$H^*(-;{}^tL)$	11.9.1	cohomology of a space with local coefficients
$\tilde{H}^*(-;G)$	4.1.1	reduced cohomology of a space with coefficients in a module
$H_*^{\mathrm{alg}}(-)$	4.3.8	algebraic group homology
$H^*_{\mathrm{alg}}(-)$	4.3.8	algebraic group cohomology
H_t	3.1.1	function obtained from a homotopy by fixing a value of t
I		the interval $[0,1]$ or its simplicial equivalent. Sometimes also used for the cellular chain complex of the interval
$I(-)$	10.1.2	augmentation ideal of a graded algebra
i_0		inclusion $x \mapsto (x,0)$ of a space into its product with $[0,1]$
i_1		inclusion $x \mapsto (x,1)$ of a space into its product with $[0,1]$
i^r	11.3.2	morphism between D^r terms in an exact couple
$\mathrm{Int}\,(-)$		interior of a set. Equivalent to $\mathring{(-)}$
$J(-)$	10.1.2	cokernel of the augmentation in a graded coalgebra
j^r	11.3.2	morphism from D^r to E^r in an exact couple
$J(-)$	7.9.1	James construction on a pointed space
$J_n(-)$	7.9.1	nth stage of James construction on a pointed space
k^r	11.3.2	morphism from E^r to D^r in an exact couple
$(-)_{\mathcal{K}}$	2.6.1	compactly generated topology of a space

$K(\pi, n)$	13.4.3	space with one nonvanishing homotopy group
$L(V)$		free Lie algebra on a vector space or free module
$L[S]$		free Lie algebra on the vector space or free module generated by a set
$L_n(-)$	4.3.4	left derived functor
$\varinjlim$	1.2	direct limit
$\varprojlim$	1.2	inverse limit
$\varprojlim^1$	4.7	derived functor of $\varprojlim$
M_f	7.1.16	reduced mapping cylinder of a function
$\mathrm{Map}\,(X, Y)$	2.3.8	space of continuous functions or function space of simplicial sets. Equivalent to Y^X
$\mathrm{Map}_*(X, Y)$	2.3.8	basepoint preserving functions between pointed spaces. Equivalent to Y^X when X and Y are pointed spaces
$(-)$-mods		category of modules over a ring
$\mathbb{N}$		natural numbers
N	4.3.8	sum within the group ring of all elements of a group
$N(-)$	8.5.1	normalized sub-chain-complex of a simplicial set
NDR	7.1.10	neighbourhood deformation retract
$P(-)$	10.4.1	primitive elements in an augmented coalgebra
$P(-)$	7.1.4	path space
$P'(-)$	7.9.1	Moore path space
P^f	7.1.16	mapping path space of a function
$\mathcal{P}$	14.3.9	Steenrod reduced power operation
$\mathcal{P}^{\Delta_k}$	14.5.3	Milnor primitive in the Steenrod algebra
PID		principal ideal domain
$\mathbb{Q}$		rationals
$Q(-)$	10.4.1	indecomposable quotient in an augmented algebra
$Q(-)$	13.3.1	infinite loop space formed by taking the direct limit of $\Omega^n S^n(-)$
Q^{Δ_k}	14.5.3	Milnor primitive in the Steenrod algebra
$\mathbb{R}$		real numbers
$\mathbb{R}P^n$		n-dimensional real projective space
$S(-)$		reduced suspension of a chain complex or a space
s		often used for a chain homotopy
s_i	8.1.1	degeneracy map of a simplicial set
S^n		n-dimensional sphere
S_n		symmetric group on n-symbols
$S_*(-)$	5.2	singular chain complex of a space
$S^*(-)$	5.6	singular cochain complex of a space
$S_*^{(k,A)}(-)$	7.4.1	Eilenberg subcomplex of a space
$S(V)$		free commutative algebra on a vector space or free module
$S[X]$		free commutative algebra on the vector space or free module generated by a set
sd	5.2.4	barycentric subdivision operator

$\mathcal{S}et_*$		the category of pointed sets
sh	7.5.14	shearing map
Sing $(-)$	8.4.1	singular complex of a simplicial set
$SpecS$		the category of spectral sequences
Sq	14.3.9	Steenrod square
SS		the category of simplicial sets
T	4.3.8	generator of a cyclic group
T	10.1.1	map which performs the graded interchange of the two factors of a tensor product
$T(-)$		tangent space of a manifold
$T(V)$		tensor algebra on a vector space or free module
$T[S]$		tensor algebra on the vector space or free module generated by a set
$\mathcal{T}op$		the category of topological spaces and continuous functions
$\mathcal{T}op_*$		the category of pointed topological spaces and basepoint preserving continuous functions
Tor	4.3.5	derived functor of $\otimes$
Tot$(-)$	11.7.1	total complex of a double complex. Also used for a simplicial set formed from a cosimplicial simplicial set
$\mathrm{Tot}^\pi(-)$	11.7.1	product total complex formed from a double complex
$U(-)$	10.5.1	universal enveloping algebra of a Lie algebra
$\mathbb{Z}$		integers
$\mathbb{Z}(-)$	8.5.4	simplicial set formed by taking the free abelian group in each degree
$\hat{\mathbb{Z}}_p$	12.2.3	p-adics
$Z_n(-)$	4.1.1	cycles of a chain complex
$Z^r(-)$	11.2	cycles in E^r-term of a spectral sequence
α	4.5	natural transformation from the identity arising from a pair of adjoint functors
β	4.5	natural transformation to the identity arising from a pair of adjoint functors
$\beta^{(r)}$	11.8.1	rth Bockstein homomorphism when the prime is understood
$\beta_p^{(r)}$	11.8.1	rth mod p Bockstein homomorphism
$\Gamma(V)$		divided polynomial algebra on a vector space or free module
$\Gamma[S]$	10.1.7	divided polynomial algebra on the vector space or free module generated by a set
$\gamma_j(-)$	10.1.7	a basis element in a divided polynomial algebra
∂	4.1.5	connecting homomorphism in a long exact homology sequence
∂	5.2.1	boundary map in the singular chain complex of a space
∂_j	5.2.1	homomorphism of singular homology groups defined by composition with inclusion of a face

δ	4.4.3	coboundary in a cochain complex or connecting homomorphism in a long exact cohomology sequence. In a cochain complex obtained by dualizing a chain complex, it is the Hom-dual of ∂, up to sign
δ^j	5.2	inclusion of a face into the standard n-simplex
Δ^*	8.1.6	category which can be used to define simplicial sets
Δ^n	5.2	the standard n-simplex
$\Delta[n]$	8.1.3	simplicial set whose realization is the standard n-simplex
ϵ		coaugmentation of a coalgebra or the "coaugmentation" map in a cotriple
η		augmentation of an algebra or the "augmentation" map in a triple
η	14.6.3	Hopf map from S^3 to S^2 or its suspension
η_n	14.6.3	map from S^{n+1} to S^n obtained by the suspending Hopf map
μ		multiplication of an H-space or Hopf algebra or the action of an H-space. Also used for the "multiplication" in a triple.
ι_n		generator of $H_n(S^n)$
θ_X	14.3.1	natural transformation used in constructing Steenrod operations
π'		projection onto the first factor of a product
π''		projection onto the second factor of a product
$\pi_n(-)$	3.1.3	homotopy group
$\pi_n^S(-)$	3.1.3	stable homotopy group
ψ		comultiplication of an co-H-space or Hopf algebra. Also used for the "comulitiplication" in a cotriple
σ_j	8.1.3	map from the standard $(n+1)$-simplex to the standard n-simplex which collapses a face
$\Omega(-)$	7.1.4	loop space
$\Omega'(-)$	7.9.1	Moore loop space
$\tau(-)$	11.9.4	transgression homomorphism
τ_k	14.5.1	generator of the dual of the Steenrod algebra
ξ_k	14.5.1	generator of the dual of the Steenrod algebra
χ		canonical conjugation of a Hopf algebra
$2^{(-)}$		set of subsets of a set
$\underline{\underline{C}}(-,-)$	1.1.1	set of morphisms in a category. Equivalent to $\mathrm{Hom}_{\underline{\underline{C}}}(-,-)$
$(-)^c$		complement of a set
$\overline{(-)}$		closure of a set
$\overset{\circ}{(-)}$		interior of a set. Equivalent to Int $(-)$
$\partial(-)$		boundary of a set

Y^X	2.3.8	space of continuous functions or of continuous basepoint preserving functions if the spaces are pointed. Equivalent respectively to either Map (X, Y) or $\text{Map}_*(X, Y)$. Can also mean function space of simplicial sets
$\cong$	1.1.2	isomorphism in a category, including homeomorphism between topological spaces
$\approx$	3.1.2	homotopy equivalence between pointed spaces
$\simeq$	3.1.1	homotopic for maps of spaces or chain complexes or simplicial sets
$\simeq\ (\text{rel}\,(-))$	3.1.1	homotopic relative to a subset
$\vee$	3.1.1	one-point union of pointed spaces
$\wedge$	3.1.1	reduced smash product of pointed spaces
$\coprod$	1.2	coproduct of objects in a category
$\prod$	1.2	product of objects in a category
$\perp$	1.2	map induced using the universal property of a coproduct
$\top$	1.2	map induced using the universal property of a product
$\otimes$	4.3.5	tensor product
$\square$	10.3.1	cotensor product
$(-)_\#$	3.1.1	map on homotopy induced by left composition
$\cup$		union
$\cup$	5.6.1	cup product or relative cup product
$\cap$		intersection
$\cap$	5.7.2	cap product or relative cap product
$X \cup_f Y$	1.7	attaching construction. The space obtained by gluing X to Y via f.
$\dashv$	4.5	adjoint functors
$\lvert - \rvert$		degree of an element in a graded set
$\lvert - \rvert$	8.1.3	realization of a simplicial set
$\alpha * \beta$	3.1.4	product of paths
$(-) * (-)$	7.7.1	reduced join of pointed spaces
$*$		basepoint of a pointed space or the map sending every element to the basepoint. Also used for the basepoint of a simplicial set
$G *_A H$	3.3.1	amalgamated free product
$[\alpha] \cdot [\beta]$	7.2.6	action of the fundamental groupoid
$(-)^\#$	3.1.1	map on homotopy induced by right composition
$(-)_*$	4.1.2	map induced on homology
$(-)^*$		map induced on cohomology
$(-)^*$		dual space
$(-)^\bullet$	12.2	cosimplicial object obtained from a triple
$(\hat{-})$		completion with respect to a filtration
$(-)_{(p)}$	12.2.2	localization of a space or module
$(-)^{(n)}$	2.7.1	n-skeleton of a CW-complex or simplicial set
$(-)^{(n)}$		n-fold reduced smash product of a pointed space

$\gamma_!$	7.1.3	
$f^!$	9.1.2	pullback of a bundle
$E^r \Rightarrow \mathcal{F}_{(-)}$	14.4.4	spectral sequence abuts to filtered object
$\langle -, - \rangle$	4.4.3	Kroncker product
$\langle -, - \rangle$	7.8.1	Samelson product
$[-, -]$	7.8.1	Whitehead product. Also used for the operation in a Lie algebra
$[-, -]$	3.1.3	pointed homotopy classes of pointed maps rel basepoint
$\{-, -\}$	3.1.3	stable homotopy classes of maps
$\{\gamma, \beta, \alpha\}$	14.7	Toda bracket
X/G	3.2.3	space of cosets
X/A	7.1.6	cofibre of an inclusion
$\underline{n}$		degree n self-map of a sphere
$\underline{R}$	4.4.3	chain complex concentrated in degree 0
$R(V)$		polynomial algebra on an R-module
$R[S]$		polynomial algebra on the R-module generated by a set
$R[G]$		group ring of the group G over the ring R
$R_\infty(-)$	12.2	Bousfield-Kan contruction of the ring R on a pointed simplical set

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This text is based on a one-semester graduate course taught by the author at The Fields Institute in fall 1995 as part of the homotopy theory program which constituted the Institute's major program that year. The intent of the course was to bring graduate students who had completed a first course in algebraic topology to the point where they could understand research lectures in homotopy theory and to prepare them for the other, more specialized graduate courses being held in conjunction with the program. The notes are divided into two parts: prerequisites and the course proper.

Part I, the prerequisites, contains a review of material often taught in a first course in algebraic topology. It should provide a useful summary for students and non-specialists who are interested in learning the basics of algebraic topology. Included are some basic category theory, point set topology, the fundamental group, homological algebra, singular and cellular homology, and Poincaré duality.

Part II covers fibrations and cofibrations, Hurewicz and cellular approximation theorems, topics in classical homotopy theory, simplicial sets, fiber bundles, Hopf algebras, spectral sequences, localization, generalized homology and cohomology operations.

This book collects in one place the material that a researcher in algebraic topology must know. The author has attempted to make this text a self-contained exposition. Precise statements and proofs are given of "folk" theorems which are difficult to find or do not exist in the literature.

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