FIELDS INSTITUTE MONOGRAPHS

THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES

Lectures on Automorphic *L*-functions

James W. Cogdell
Henry H. Kim
M. Ram Murty

American Mathematical Society

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Providence, Rhode Island

The Fields Institute for Research in Mathematical Sciences

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Preface

"Mathematics goes to great pains to create expressions for relationships which pass empirical comprehension."

Carl Gustav Jung in "Memories, Dreams, Reflections"

This monograph is based on graduate courses which the authors gave at the Thematic Program on Automorphic Forms at the Fields Institute in the spring of 2003. The program was organized by J. Arthur, T. Haines, H. Kim, R. Murty, G. Pappas, and F. Shahidi. These courses were intended for postdocs and advanced graduate students in order to introduce them to the Langlands functoriality conjecture and its consequences in number theory and representation theory. In particular, we wanted to show them how automorphic L-functions play a crucial role in the theory. There have been some new developments in the theory, most notably, functoriality of the symmetric cube and symmetric fourth of cuspidal representations of $GL(2)$ and functoriality of classical groups. These developments make use of automorphic L-functions, namely, the combination of converse theorems of Cogdell and Piatetski-Shapiro and the Langlands-Shahidi method. The aim of the thematic program was to review these developments and encourage the discovery of as yet unknown implications of functoriality to number theory, and vice versa.

Besides the courses, there was a weekly seminar on automorphic forms given by members. There were two workshops: one on Shimura varieties and related topics, organized by T. Haines and G. Pappas, and the other on Automorphic L-functions, organized by H. Kim and R. Murty. In addition, S. Kudla gave the Coxeter Lectures on Arithmetic theta series and P. Sarnak gave the Distinguished Lectures on Automorphic L-functions and equidistribution.

This monograph is not at all a comprehensive account of automorphic forms. The most serious omissions are trace formulas and Shimura varieties, namely, the geometric point of view of automorphic forms. However, in order to compensate for this, there was a summer school on Harmonic Analysis, the Trace Formula and Shimura Varieties, sponsored by the Clay Mathematics Institute in the summer of 2003. It was organized by J. Arthur, D. Ellwood and R. Kottwitz. Their lecture notes will be published soon and we believe that it will complement our monograph very nicely.

Let us describe our monograph in detail. The Langlands functoriality conjecture can be roughly formulated as: if H and G are two reductive groups over a number field F, then to each homomorphism of L-groups $\phi : {}^LH \longrightarrow {}^LG$, there is associated a lift of automorphic representations of H to automorphic representations of G. One example would be: take H to be the group consisting of a single element and G to be $GL(2)$. Then ${}^LH$ is a Galois group and the problem is that of associating an automorphic form to a two-dimensional Galois representation.

A partial solution has been used by Andrew Wiles in his proof of Fermat's Last Theorem. It had been thought that only the trace formula developed by Arthur and others was promising for the functoriality conjecture. Indeed the trace formula method has been successful in some cases, most notably, cyclic base change of $GL(n)$ due to Arthur and Clozel.

At this moment, the converse theorems of Cogdell and Piatetski-Shapiro, if combined with the Langlands-Shahidi method, provide several instances of functoriality. These are L-function techniques. The converse theorems determine whether a certain global representation of $GL(n)$ which is just a product of all local ones, is an automorphic representation. In order to use the converse theorem, we need to study certain automorphic L-functions, namely, we need to prove that these automorphic L-functions are entire, satisfy functional equations, and are bounded in vertical strips. There are two ways of studying automorphic L-functions.

The first one is called the method of integral representations. It expresses certain automorphic L-functions as integrals of automorphic forms, often integrated against Eisenstein series. Sometimes, it is called the Rankin-Selberg method. This method has been investigated by a large number of mathematicians, going back to Hecke, Rankin and Selberg. However, one runs into serious difficulties upon studying the method at archimedean places. But the theory is complete for $GL(n)$. This is the subject of the first course by J. Cogdell. In Course One we present a comprehensive account of the theory of L-functions for $GL(n)$ via integral representations. We begin with the classical theory of Hecke. Then we turn to the modern theory of automorphic representations and their L-functions, both local and global, as developed by Jacquet, Piatetski-Shapiro and Shalika. We conclude with an exposition of the converse theorems for $GL(n)$ and their application to the question of global functoriality.

The second method of studying automorphic L-functions is called the Langlands-Shahidi method. This is the subject of the second course by H. Kim. Many automorphic L-functions appear as normalizing factors of intertwining operators in the constant terms of Eisenstein series. We can study them with the help of many properties of Eisenstein series. Even though the goal of this course was to explain recent striking results such as functoriality of the symmetric cube and symmetric fourth of cuspidal representations of $GL(2)$, we try to give a comprehensive account of the Langlands-Shahidi method, without assuming any previous knowledge of automorphic representations. Because of time constraints, many proofs had to be omitted. In those cases, we always provided the references for the proofs.

In the third course by R. Murty, we look at the applications of the Langlands functoriality conjecture to analytic number theory, especially to the Sato-Tate conjecture, the Ramanujan conjecture, the Selberg eigenvalue conjecture, Artin's holomorphy conjecture and the Langlands reciprocity conjecture. We emphasize how the Langlands program proposes to solve each of these conjectures. We then apply the recent work of Kim and Shahidi on symmetric power L-functions to these conjectures as well as related questions in analytic number theory.

James W. Cogdell
Henry H. Kim
M. Ram Murty
January 2004

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This book provides a comprehensive account of how automorphic L-functions play a crucial role in the Langlands program, especially, the Langlands functoriality conjecture, and in number theory. Recently there has been a major development in the Langlands functoriality conjecture by the use of automorphic L-functions, namely, by combining converse theorems of Cogdell and Piatetski-Shapiro with the Langlands-Shahidi method. This book introduces the reader to these developments step by step, and explains how the Langlands functoriality conjecture implies solutions to several outstanding conjectures in number theory, such as the Ramanujan conjecture, Sato-Tate conjecture, and Artin's conjecture. This book would be ideal for an introductory course in the Langlands program.

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