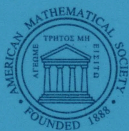


# Lectures on Quantum Groups

**Jens Carsten Jantzen**

**Graduate Studies  
in Mathematics**

**Volume 6**



**American Mathematical Society**

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## Editorial Board

James E. Humphreys  
Lance Small

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ABSTRACT. This book is an introduction to the theory of quantum groups. Its main objects are the quantized enveloping algebras introduced independently by Drinfeld and Jimbo. We study their finite dimensional representations, their centers, and their bases. In particular, we look at the crystal (or canonical) bases discovered independently by Lusztig and Kashiwara.

We first look at the quantum analogue of the Lie algebra  $\mathfrak{sl}_2$ , and then at the quantum analogue of arbitrary finite dimensional complex Lie algebras. The book is directed to anyone who wants to learn the subject and has been introduced to the theory of finite dimensional complex Lie algebras.

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10 9 8 7 6 5 4 3 14 13 12 11 10 09

# Contents

Introduction	1
Chapter 0. Gaussian Binomial Coefficients	5
Chapter 1. The Quantized Enveloping Algebra $U_q(\mathfrak{sl}_2)$	9
Chapter 2. Representations of $U_q(\mathfrak{sl}_2)$	17
Chapter 3. Tensor Products or: $U_q(\mathfrak{sl}_2)$ as a Hopf Algebra	31
Chapter 4. The Quantized Enveloping Algebra $U_q(\mathfrak{g})$	51
Chapter 5. Representations of $U_q(\mathfrak{g})$	69
Chapter 5A. Examples of Representations	87
Chapter 6. The Center and Bilinear Forms	105
Chapter 7. $R$ -matrices and $k_q[G]$	129
Chapter 8. Braid Group Actions and PBW Type Basis	141
Chapter 8A. Proof of Proposition 8.28	173
Chapter 9. Crystal Bases I	187
Chapter 10. Crystal Bases II	217
Chapter 11. Crystal Bases III	237
References	259
Notations	263
Index	265
Errata	267

## References

I refer to the books and papers by the names of the author(s) plus (where appropriate) a number. There are two exceptions: [H] = [Humphreys 1] and [L] = [Lusztig 7].

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## Notations

We use the standard notations  $\mathbf{Z}$  for the ring of integers,  $\mathbf{Q}$  for the field of rational numbers, and  $\mathbf{C}$  for the field of complex numbers without extra explanation

roman letters			
$C$	2.7	$T, T', {}^\omega T, {}^\omega T'$	8.2
$E$	1.1	$T_\alpha$	8.6, 8.14
$E^{(\tau)}$	8.2	$T'_\alpha, {}^\omega T_\alpha, {}^\omega T'_\alpha$	8.6
$E_\alpha$	4.3	$T_w$	8.18
$E_\alpha^{(\tau)}$	8.6	$U$	1.2, 4.6
$\tilde{E}_\alpha$	9.2, 9.4, 10.2	$U^+, U^-, U^0$	1.6, 4.6
$E_I$	4.12	$U_{ev}^0$	6.6
$F$	1.1	$U^\alpha$	4.8, 9.0
$F^{(\tau)}$	8.2	$\tilde{U}$	4.6
$F_\alpha$	4.3	$U^+[w], U^-[w]$	8.24
$F_\alpha^{(\tau)}$	8.6	$U_q(\mathfrak{g})$	4.3
$\tilde{F}_\alpha$	9.2, 9.4, 10.2	$U_q^+(\mathfrak{g}), U_q^-(\mathfrak{g}), U_q^0(\mathfrak{g})$	4.4
$F_I$	4.12	$\tilde{U}_q(\mathfrak{g})$	4.3
$G(b)$	11.10, 11.15	$U_q(\mathfrak{sl}_2)$	1.1
$G^\alpha(b)$	11.12	$U_{q_\alpha}(\mathfrak{sl}_2)$	4.4
$\text{Hom}_k(M, N)$	3.10	$U_{\mathbf{z}}, U_{\mathbf{z}}^+, U_{\mathbf{z}}^-, U_{\mathbf{z}}^0$	11.1
$HW(S)$	9.8	$W$	4.1
$K$	1.1	$Z$	2.19
$K_\alpha$	4.3	$Z(U)$	6.3
$K_\lambda$	4.4	$Z_b(\lambda)$	2.11
$L(n, +), L(n, -)$	2.6	$a_{\alpha\beta}$	4.3
$L(\lambda)$	5.5	$\text{ad}(a)$	4.18
$\tilde{L}(\lambda)$	5.9	$\tau \text{ad}(x)$	8.11
$L_{\mathbf{z}}(\lambda)$	11.4	$\text{ch}(M)$	5A.8
$M(\lambda)$	2.4, 5.5	$d_\alpha$	4.1
$\tilde{M}_{k'}(\mathbf{c})$	4.15	$e_\alpha(b)$	9.17
$P$	3.8, 7.1	$f_\alpha(b)$	9.17
$R$	3.18, 7.7	$\text{ht}(\mu)$	4.13
$R_{ij}$	7.6	$k$	1.1
$S$	3.6, 4.8	$k_q[G]$	7.11
$S^*$	7.11	$q$	1.1
$S'$	9.13	$q_\alpha$	4.2
		$r_\alpha, r'_\alpha$	6.14, 6.15, 8.26

$r'_{n\alpha}$	8A.4	$\gamma_\lambda$	6.4
$r'_{i\alpha,\beta}, r'_{\alpha,j\beta}$	8A.7	$\varepsilon$	3.4, 4.8
$s_\alpha$	4.1	$\varepsilon^*$	7.11
$\text{tr}_q$	3.10	$\varepsilon_\sigma$	5.3
type 1	3.19, 5.2	$\pi$	2.16, 6.2, 11.12
type $\sigma$	5.2	$\pi_\alpha$	8A.10
$u_{\alpha\beta}^+, u_{\alpha\beta}^-$	4.10	$\varpi_\beta$	4.1
$v_\lambda$	5.5, 9.5	$\rho$	4.9, 5A.8
$w_0$	8.24	$\tau$	1.2, 4.6
$\text{wt } I$	4.12	$\tau_1$	9.20
fraktur letters		$\chi(\lambda)$	5A.8
$\mathfrak{g}$	4.1	$\varphi_\lambda$	10.3
calligraphic letters		$\psi$	11.9
$\mathcal{B}(\lambda)$	9.5	$\omega$	1.2, 4.6
$\mathcal{B}(\infty)$	10.3	brackets, etc.	
$\mathcal{L}(\lambda)$	9.5	$[a]$	0.1
$\mathcal{L}(\infty)$	10.3	$[a]_\alpha$	4.2
$\mathcal{L}_Z(\infty), \mathcal{L}_Z(\lambda)$	11.6	$[a]_n$	0.1
$\mathcal{P}(\mu)$	5.19	$[a]_n^\alpha$	4.2
greek letters		$[n]!$	0.1
$\Delta$	3.1, 4.8	$[n]!_\alpha$	4.2
$\Delta^*$	7.11	$[K; a]$	1.3
$\Delta'$	9.13	$[K_\alpha; a]$	4.4
$\Theta = \Theta_{M, M'}$	3.12, 7.2	$[K_\alpha; a]_n$	11.1
$\Theta_n$	3.11	$(, \bar{\phantom{a}})$	4.1, 6.10
$\Theta_\mu$	7.1	$\langle \lambda, \alpha^\vee \rangle$	4.1
$\Theta^f$	3.13	$\langle u, v \rangle$	6.20
$\Theta_{ij}^f$	3.17, 7.5	exponents	
$\tilde{\Lambda}$	4.1	$M^U$	3.5
$\tilde{\Lambda}$	3.13	$M^*$	3.9, 5.3
$\Pi$	4.1	$M^\sigma$	5.2
$\Pi_s$	5A.2	$\varphi\Delta, \varphi S, \varphi\varepsilon$	3.8, 7.2
$\Phi$	4.1	indices	
$\Phi_s$	5A.2	$M_\lambda$	2.2, 5.4
$\alpha_0$	5A.2	$\mathcal{M}_\lambda$	9.3
$\gamma_i$	1.6	$M_{\lambda,\sigma}$	5.1

# Index

This index contains mainly references to definitions and main results. In some cases (such as “semisimplicity of  $U$ -modules”) you will be directed to subsections containing results on the topic, but where the phrase from the index does not occur explicitly in that form.

adjoint representation	4.18	hexagon identities	3.18/19, 7.8
admissible lattice	9.3	Hom space of	
antipode	3.6/7, 3.8, 9.13	modules	3.10, 5.3
augmentation	3.5	Hopf algebra	3.8, 4.8, 4.11, 7.11
braid group	7.6, 8.15	invariant form on $U$	6.20
braid relations	7.6, 8.15	isotypic component	9.10
canonical basis	11.16	Kostant’s partition	
center of $U$	2.17, 2.10, 6.25	function	5.19
character formula	5.15	largest short root	5A.2, 5A.9, 9.6, 9.21, 11.4
Clebsch-Gordan		locally nilpotent	5.7
formula	5A.8, 9.14	matrix coefficients	6.22, 7.10
coassociative	3.2	minuscule dominant	
cocommutative	3.8	weights	5A.1, 5A.9, 9.6, 9.21, 11.4
comultiplication	3.1/2, 3.8, 9.13	modules of type 1	3.19, 5.2
counit	3.4/5, 3.8	mod- $\tau_1$ -invariant	9.20
crystal base	9.4, 10.16	pairing of $U^{\leq 0}$ and	
crystal graph	9.28	$U^{\geq 0}$	6.12, 6.18, 8.28-30
divided powers	8.2, 8.6, 11.1	PBW type bases	1.5, 8.24
dominant short root	5A.2, 5A.9, 11.4	polarization	9.23, 10.16, 11.7
dual space of a		quantized enveloping	
module	3.9, 5.3, 5.16	algebra	1.1, 4.3/4
elementary move	8.18	quantum determinant	7.13
fixed points	3.5	quantum trace	3.10, 5.3
formal characters	5A.8	quantum Yang-Baxter	
fundamental weights	4.1	equation	3.17, 7.5/6
Gaussian binomial		reduced expression	8.18
coefficient	0.1, 0.4, 8.1, 10.5	$R$ -matrices	7.13, 8.30
grading of $U$	1.9, 4.7, 6.1		
Harish-Chandra			
homomorphism	6.4, 6.6, 6.25		
Hecke algebra	7.7		

- scalar product of roots 4.1
- semisimplicity of quad  $U$ -modules 2.9, 5.17
- simple  $U$ -modules 2.6, 2.13, 5.10
- tensor categories 3.19
- tensor products of  $U$ -modules 3.3, 3.14, 5.3, 5A.8/9, 7.3, 9.13
- trivial module 3.5, 5.3
- twisted comultiplication 3.8, 7.2
- twisted modules 2.13, 5.2, 5.11, 8.2, 11.9
- type of a  $U$ -module 5.2
- universal highest weight module 2.4, 5.5
- weight lattice 4.1
- weight space 2.2/3, 5.1
- weights of a module 2.2/3
- zero divisors 1.8, 8.25
- Z**-form 11.1

## Errata

p. 7, 2nd displayed equation: The numerator should be  $v^{a-b} - v^{-a-b} + v^{a+b} - v^{a-b}$

p. 7: Replace the last six lines by:

$$\begin{aligned} \sum_{i=0}^r (-1)^i v^{i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix} &= \sum_{i=0}^r (-1)^i v^{i(r-1)} \left( v^{-i} \begin{bmatrix} r-1 \\ i \end{bmatrix} + v^{r-i} \begin{bmatrix} r-1 \\ i-1 \end{bmatrix} \right) \\ &= \sum_{i=0}^{r-1} (-1)^i v^{i((r-1)-1)} \begin{bmatrix} r-1 \\ i \end{bmatrix} \\ &\quad - \sum_{j=0}^{r-1} (-1)^j v^{(j+1)(r-1)} v^{r-j-1} \begin{bmatrix} r-1 \\ j \end{bmatrix} \end{aligned}$$

where we interpret  $\begin{bmatrix} r-1 \\ -1 \end{bmatrix}$  as 0. The first sum is equal to 0 by induction, the second one is equal to

$$v^{2r-2} \sum_{j=0}^{r-1} (-1)^j v^{j((r-1)-1)} \begin{bmatrix} r-1 \\ j \end{bmatrix},$$

hence also 0 by induction.

p. 14: The first (three line) display in “1.5:” should end with  $Z^{n-1}X^r$  [not  $Z^{n+1}X^r$ ]

p. 24, l. –8: replace 2,12 by 2.12

p. 25, l. –7: replace 2.11 by 2.12

p. 26, lines 1, 2, and 4: replace  $p$  by  $l$

p. 26, line –4: replace  $U^0$  by  $U_0$

p. 27, Proof of 2.18, line 2: Lemma 2.17, not 2.18

p. 29, l. –1: In the summand for  $i = 0$  the product over  $j$  should run from  $j = 1$  to  $j = r - i - 1 = r - 1$ .

p. 30, first displayed line in 2.7:  $\cdots = -(q - q^{-1})^{-1}(K - K^{-1})E$ .

p. 30, last displayed line in 2.7:  $= \frac{(K - K^{-1})(q - q^{-1})}{(q - q^{-1})^2} E$ .

p. 32, l. 1: replace “ $E \otimes K^{-1} + 1 \otimes F$ ” by “ $E \otimes 1 + K \otimes E$ ”

- p. 42, l. -6: replace 3.2(4) by 3.1(4)
- p. 42, l. -2: replace 3.2(3) by 3.1(3)
- p. 47, l. 1: replace 0.1(2) by 0.2(1)
- p. 48, two lines before 3.10(7):  $\dots = q^{-r(r-1)}F(KF^rK^{-1})K^{r+1}$
- p. 58, lines -1 and -2: replace  $K_A^{-1}$  by  $K_{\text{wt}A}^{-1}$
- p. 60, in 4.14(6): replace  $\mu$  by  $\nu$
- p. 61, proof, line 7: formula
- p. 64, in 4.18(7):  $\dots = q_\alpha^{i(1-r)}E_\beta$ .
- p. 66, two lines before 4.21(4): the left hand sides (not “right”)
- p. 67, lines 6–8: The sentence beginning “Theorem 4.21.a and ...” should be reformulated as follows: “The description of the bases of  $\tilde{U}^{\geq 0}$  and  $\tilde{U}^{\leq 0}$  in 4.17 implies the multiplication induces surjective linear maps  $U^0 \otimes U^+ \rightarrow U^{\geq 0}$  and  $U^- \otimes U^0 \rightarrow U^{\leq 0}$ ; Theorem 4.21.a shows that these maps are bijective.”
- p. 68, l. 1: replace (4) by 4.4(4).
- p. 68, l. 10: replace 4.18(1) by 0.2(4).
- p. 70, Proof, l. 5: replace  $\pm q^a$  by  $\pm q_\alpha^a$ .
- p. 70, l. -3: this equation should be numbered (2).
- p. 71, l. -7: replace 3.10(6) by 3.5(2), replace 3.10(7) by 3.10(6).
- p. 71, l. -6: replace 3.10(8) by 3.10(9).
- p. 72, 5.4, l. -1: replace  $M(q^{-a-2})$  by  $M(q_\alpha^{-a-2})$ .
- p. 74, l. -7: replace  $I'$  by  $I$ .
- p. 75, l. 1 of proof of 5.9: replace 5.5(1) by 5.5(4).
- p. 77: the fourth line after (2) should begin “for all  $j$  such that ...”, (i.e, delete “all  $\alpha \in \Pi$  and”)
- p. 77, three lines before (3): the index of  $({}^\omega \tilde{L}(\lambda'))$  should be  $-\lambda' + \nu_0$ , not  $\lambda' - \nu_0$
- p. 79, one and three lines after 5.12(7): replace  $[a + r]$  by  $[a + s]$
- p. 79, 5.12(8): The factor after  $[K_\alpha; 0]$  should be  $F_{\beta_{i+1}}$
- p. 80, line after (5): replace 5.1(2) by 5.1(3).
- p. 81, 5.14(3):  $I = (\beta_1, \beta_2, \dots, \beta_r)$ .
- p. 81, l. -10/ -9 ... homomorphism of algebras  $\mathbf{Q}[v, v^{-1}] \rightarrow k$  that takes  $v \dots$
- p. 83, l. 9: a homomorphic image of  $M(\mu)$  [not of  $M(\lambda)$ ]



- p. 83, l. 10: replace 5.6.b by 5.4.b
- p. 84, l. -6: replace 5.19(1) by 5.18
- p. 84: Formula (3) in 5.19 should be numbered (2). And the direct sum should be over  $\nu \leq \lambda - \mu$ .
- p. 89: The alignment in (2) should be fixed.
- p. 93, l. 4: add a “)” at the end.
- p. 102, in 5A.8(6): The direct sum should start with  $i = 0$  [not:  $i = 1$ ]
- p. 102, in 5A.8(6): The direct sum should start with  $i = 0$  [not:  $i = 1$ ]
- p. 109, l. 7: replace 5.2(1) by 5.2(2).
- p. 111, 6.8, l. -2: replace (2) by (1).
- p. 112, l. 2 after (1): replace 5.1(2) by 5.1(1).
- p. 113, in 6.10(8): Replace  $F_J$  by  $E_J$  in the middle of the first line
- p. 116, l. 1: replace  $K_\alpha$  by  $K_{\mu+\mu'-\alpha}$
- p. 117, l. 5: replace 6.14(4) by 6.14(3)
- p. 117, l. -1: replace 6.10(3) by 6.13(1)
- p. 122, Proof, l. 2: replace 6.19(2) by 6.20(2)
- p. 131, l. 3 of 7.4: replace “by Lemma 4.14” by “by 4.13(1)”
- p. 134, two lines before (4): replace the second  $v_{i+1}$  by  $v_{i+2}$
- p. 138, in 7.12(2): Replace  $R_{hl}^{rs}$  (on the left hand side of the equation) by  $R_{lh}^{rs}$
- p. 138, l. 4 of proof of 7.12:  $\dots = \sum_{h,l,r,s} R_{lh}^{rs}(c'_{li}c_{hj})(u)e_r \otimes e'_s$
- p. 148, l. -6: The  $q_\alpha$ -factor should be  $q_\alpha^{(i+1)(r-2m)-j(i+1)}$
- p. 148, l. -5: The  $q_\alpha$ -factor should be  $q_\alpha^{(i+1)(r-2m)-(j-1)(i+1)}$
- p. 148, l. -1: The  $q_\alpha$ -factor should be  $q_\alpha^{(i+1)(r-2m)-ji}$
- p. 149, l. 7 and l. 9: replace  $a(m-j+1)$  by  $a(m-j-1)$
- p. 150, l. 3/4 after the first display: delete “Proposition”
- p. 152, l. 4:  $= (E_\alpha u' - K_\alpha u K_\alpha^{-1} E_\alpha) T_\alpha(v)$
- p. 161, l. 2 of proof of Lemma 8.21: insert  $\sum_{i=0}^r$  after the last “=”
- p. 161, l. -5 of 8.21:  $E_\alpha^i$  [not  $E_\alpha^{(i)}$ ]
- p. 162, l. 7 of 8.23:  $E_\beta^a E_\alpha^b$  resp. of all  $E_\alpha^a E_\beta^b \dots$  [no parentheses in the exponents]

- p. 164, l. 7 after (5):  $q^{-(\gamma_i, \gamma_j)}$  [replace  $\beta$  by  $\gamma$  and note the sign!]
- p. 167, l. 1: replace 8,26 by 8.26
- p. 191, l. 2 of the proof of b): description
- p. 191, in 9.6(2): It is more logical to write  $\tilde{F}_\alpha x_{\mu-\alpha} = F_\alpha^{(2)} x_\mu = 0$ .
- p. 193, in 9.7(2): replace  $F_\alpha^{(j)}$  by  $F_\alpha^{(j-1)}$
- p. 194, l. -6: replace  $\mathcal{B}$  by  $\mathcal{B}(\lambda)$ .
- p. 195, l. 10: replace  $v_s$  by  $v_r$
- p. 195, first display: both direct sums should be  $\bigoplus_{i=1}^r$
- p. 197, l. 13:  $\mathcal{N} = \mathcal{M} \cap N$  [not  $= \mathcal{N} \cap N$ ]
- p. 197, l. 16: ... with  $\varphi_i(v_\lambda) = v_i$  ... [not:  $\varphi(v_\lambda)$ ]
- p. 199, l. 1: replace  $(\mathcal{M}, \mathcal{L})$  by  $(\mathcal{M}, \mathcal{B})$
- p. 200, l. 2 of Lemma 9.14: replace  $U_\alpha$  by  $U^\alpha$
- p. 202, l. 1: for all integers  $a \geq 0$  and  $b > 0$
- p. 202, l. 7: this yields one inclusion in (6) [not in (10)]
- p. 202, paragraph beginning with "It is now easy to get (1)-(4)": All occurrences of  $y$ ,  $z_0$  and  $z_1$  should be overlined. (This was forgotten once for  $y$  and three times each for  $z_0$  and  $z_1$ .)
- p. 203, l. -9: replace  $r \geq 0$  by  $r > 0$
- p. 203, 9.16(3): Left hand side should be  $\tilde{E}_\alpha^r(x \otimes b) = 0$
- p. 203, 9.16(5): replace "if  $i > 0$ " by "if  $0 < i \leq n + 1$ "
- p. 204, six lines after (6): replace  $\tilde{E}_\alpha^2(x \otimes b) = 0$  by  $\tilde{E}_\alpha^2(x \otimes b) \neq 0$  [This is the line where we get  $v \otimes \tilde{E}_\alpha(x \otimes b)$ .]
- p. 205, l. 10: replace  $\tilde{E}_\alpha^j b_0$  by  $\tilde{F}_\alpha^j b_0$
- p. 205, l. -3: replace 9.16 by 9.15
- p. 206, l. -2 of 9.18:  $\tilde{E}_\alpha^{m+1} b' = 0$  [not  $\tilde{E}_\alpha^m b = 0$ ]
- p. 206, l. -1 of 9.18:  $= \langle \lambda, \alpha^\vee \rangle$  [not  $= \langle \lambda, \alpha^\vee \rangle + 1$ ]
- p. 208, lines -2 and -3 in paragraph beginning "c)":  $\dim M > 0$  [not  $\dim M < 0$ ]
- p. 208, line -2 in paragraph beginning "c)": "...if and only if ..." [second 'if' is missing]

- p. 208, l. -16:  $M^* = \bigoplus_{\nu} (M_{\nu})^*$  [not:  $M = \dots$ ]
- p. 208, l. -10:  $= \psi(v)(\tau_1(u)v') = [\text{not } = \psi(v)(v, \tau_1(u)v') =]$
- p. 212, l. -16: this implies  $\overline{q^r x} = 0$  [not  $\overline{q^r y} = 0$ ]
- p. 214, l. 7: and Lemma 9.18 [not Lemma 9.17]
- p. 221, first line of 10.5:  $\dots$  for all  $a \in \mathbf{Z}, a > 0$  [i.e., add  $a > 0$ ]
- p. 221, line after 10.5(2): and for all  $a, n \in \mathbf{Z}$  with  $0 \leq n \leq a$
- p. 222: Lemma 10.5 is false as stated. In order to correct it, take  $x \in L(\lambda)_{\mu}$ ; we should then assume that  $\langle \mu, \alpha^{\vee} \rangle \neq m - 2$  in order for the congruence  $\widetilde{E}_{\alpha} F_{\alpha}^{(m-1)} x \equiv F_{\alpha}^{(m-2)} x$  in (5) to hold. Similarly, for the second congruence in 10.5(9) on p. 223 to hold, we need that  $\langle \mu, \alpha^{\vee} \rangle \geq m - i$  or  $\langle \mu, \alpha^{\vee} \rangle < m - 2i$ . The mistake in the proof occurs two lines after 10.5(8) on p. 223: The equality  $\widetilde{E}_{\alpha} F_{\alpha}^{(m-1)} x_0 = F_{\alpha}^{(m-2)} x_0$  holds only if  $\langle \mu, \alpha^{\vee} \rangle \geq m - i$  or  $\langle \mu, \alpha^{\vee} \rangle < m - 2i$  (in the second case because both sides are 0). — The correction makes necessary a small change in the proof of Lemma 10.6. We apply there Lemma 10.5 with  $x = yv_{\lambda} \in L(\lambda)_{\lambda - \nu + n\alpha}$  and  $m = n + 1$ . So we need to know that  $\langle \lambda - \nu + n\alpha, \alpha^{\vee} \rangle \neq n - 1$ , i.e., that  $\langle \lambda, \alpha^{\vee} \rangle \neq \langle \nu, \alpha^{\vee} \rangle - (n + 1)$ . That is no problem since we later assume that  $\langle \lambda, \alpha^{\vee} \rangle \geq \langle \nu, \alpha^{\vee} \rangle + n$ . So we just have to state that assumption somewhat earlier on in the proof.
- p. 222, l. -1:  $\begin{bmatrix} m - i + j \\ j \end{bmatrix}_{\alpha}$  [ $j$ , not  $i$ , in second row]
- p. 224, two lines before (4):  $q_{\alpha}^{i(\langle \lambda - \nu + n\alpha, \alpha^{\vee} \rangle + i + 1)}$  in the numerator [additional  $n$ ]
- p. 224, one line before (4):  $\lambda - \nu + (i + n)\alpha$
- p. 224, 10.6(4):  $q_{\alpha}^{i(2\langle \lambda - \nu, \alpha^{\vee} \rangle + 2i + 3n)}$  in the numerator [ $3n$  instead of  $4 - n$ ]
- p. 224, last display in 10.6: first factor should be  $q_{\alpha}^{i(2\langle \lambda - \nu, \alpha^{\vee} \rangle + 2i + 3n)}$  [as in previous correction], the middle factor on the right hand side should be  $q_{\alpha}^{i(\langle \lambda, \alpha^{\vee} \rangle - \langle \nu, \alpha^{\vee} \rangle + 2i + 4n + 1)}$
- p. 224, l. -5:  $\dots$  made at the end  $\dots$  [delete ‘in’]
- p. 224, l. -3: replace Proposition by Theorem
- p. 228, l. 8 replace Lemma 10.9 by Proposition 10.9
- p. 229, first line of Remark:  $(\mathcal{L}(\infty), \mathcal{B}(\infty))$
- p. 230, Prop. 10.14: “ $\overline{\varphi}_{\lambda}$  induces” [NOT:  $\overline{\varphi}_{\lambda}(b)$ ]
- p. 231, 10.16(1): right hand side is  $U_{\nu}^{+} K_{\nu}$
- p. 232, l. -3: replace 6.16(6) by 6.15(6)
- p. 233, l. -3 of 10.17: replace  $r_{\alpha}(x)$  by  $r'_{\alpha}(x)$
- p. 238, 11.1(5): Add an index  $\alpha$  to the Gaussian binomial coefficient

- p. 238, 11.1(6): Replace both  $\mu$  by  $\nu$
- p. 239, Lemma 11.2.b: replace *with*  $r'_\alpha(u) = 0$  for all  $n$  by *with*  $r'_\alpha(u_n) = 0$  for all  $n$
- p. 245, first line after 11.8(2): Replace  $U_{\mathbf{Z}}^0$  by  $U_{\mathbf{Z}}^-$
- p. 245, Proposition 11.9: The word ‘Proposition’ is in a wrong font
- p. 247, l. –2 in 11.10: add a “)” after  $q\mathcal{L}(\infty)_{-\nu+n\alpha}$
- p. 249, three lines after 11.12(2): 11.10.c [not 11.10.b]
- p. 250, two lines before 11.13(4):  $e_\alpha(d) = 0$  [not  $> 0$ ]
- p. 251, l. 3:  $f_\alpha(\bar{\varphi}_\lambda(b))$  [not  $(d)$ ]
- p. 251, first line after 11.13(6):  $S = \{G^\alpha(b)v_\lambda \mid b \in \mathcal{B}(\infty)_{-\nu}, \bar{\varphi}_\lambda(b) \neq 0, e_\alpha(b) \geq n\}$
- p. 251, first line after 11.14(1):  $\varphi_\lambda(b) = \tilde{F}_{\alpha_1} \dots \tilde{F}_{\alpha_{r-1}} \tilde{F}_\beta \varphi_\lambda(1) = 0$
- p. 252, l. 12: ... yields 11.10.b.
- p. 252, two lines before 11.15(1):  $S = \{G(b)v_\lambda \mid b \in \mathcal{B}(\infty)_{-\nu}, \bar{\varphi}_\lambda(b) \neq 0\}$
- p. 254, l. –3:  $= F_\alpha^{(r-i)} F_{\alpha+\beta}^{(i)} F_\beta^{(s-i)} + \dots$  AND: add an index  $\alpha$  to the Gaussian binomial coefficient
- p. 255, l. 3:  $\equiv F_\alpha^{(r-i)} F_{\alpha+\beta}^{(i)} F_\beta^{(s-i)}$
- p. 255, l. 5:  $\equiv F_\alpha^{(r-i)} F_\beta^{(s)} F_\alpha^{(i)}$
- p. 255, l. 6: So we see that  $F_\alpha^{(r-i)} F_\beta^{(s)} F_\alpha^{(i)} \dots$
- p. 255, Proposition 11.18:  $\tau G(b) = G(\tau b)$
- p. 255, line 2 of the proof of 11.18:  $\tau G(b) = G(\tau b)$
- p. 255, line 7 of the proof of 11.18:  $r'_\alpha(u'_j) = 0$
- p. 255, l. –8: since  $e_\alpha(b') = 0$  [NOT:  $> 0$ ]
- p. 257, four lines before 11.19(3):  $(U_{\mathbf{Z}}^-)_{-\mu} \otimes \mathbf{C}$
- p. 260, ref. Jimbo 2: replace “ $q$ -difference analogue” by “ $q$ -analogue”
- p. 261, ref. Rudakov & Shafarevich: Delete the comma at the end

*The material is very well motivated ... Of the various monographs available on quantum groups, this one ... seems the most suitable for most mathematicians new to the subject ... will also be appreciated by a lot of those with considerably more experience.*

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Since its origin, the theory of quantum groups has become one of the most fascinating topics of modern mathematics, with numerous applications to several sometimes rather disparate areas, including low-dimensional topology and mathematical physics. This book is one of the first expositions that is specifically directed to students who have no previous knowledge of the subject. The only prerequisite, in addition to standard linear algebra, is some acquaintance with the classical theory of complex semisimple Lie algebras. Starting with the quantum analog of  $\mathfrak{sl}_2$ , the author carefully leads the reader through all the details necessary for full understanding of the subject, particularly emphasizing similarities and differences with the classical theory. The final chapters of the book describe the Kashiwara–Lusztig theory of so-called crystal (or canonical) bases in representations of complex semisimple Lie algebras. The choice of the topics and the style of exposition make Jantzen's book an excellent textbook for a one-semester course on quantum groups.

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