Topics in Random Matrix Theory
Topics in Random Matrix Theory

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To Garth Gaudry, who set me on the road;
To my family, for their constant support;
And to the readers of my blog, for their feedback and contributions.
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In the winter of 2010, I taught a topics graduate course on random matrix theory, the lecture notes of which then formed the basis for this text. This course was inspired by recent developments in the subject, particularly with regard to the rigorous demonstration of universal laws for eigenvalue spacing distributions of Wigner matrices (see the recent survey [Gu2009b]). This course does not directly discuss these laws, but instead focuses on more foundational topics in random matrix theory upon which the most recent work has been based. For instance, the first part of the course is devoted to basic probabilistic tools such as concentration of measure and the central limit theorem, which are then used to establish basic results in random matrix theory, such as the Wigner semicircle law on the bulk distribution of eigenvalues of a Wigner random matrix, or the circular law on the distribution of eigenvalues of an iid matrix. Other fundamental methods, such as free probability, the theory of determinantal processes, and the method of resolvents, are also covered in the course.

This text begins in Chapter 1 with a review of the aspects of probability theory and linear algebra needed for the topics of discussion, but assumes some existing familiarity with both topics, as well as a first-year graduate-level understanding of measure theory (as covered for instance in my books [Ta2011, Ta2010]). If this text is used to give a graduate course, then Chapter 1 can largely be assigned as reading material (or reviewed as necessary), with the lectures then beginning with Section 2.1.

The core of the book is Chapter 2. While the focus of this chapter is ostensibly on random matrices, the first two sections of this chapter focus more on random scalar variables, in particular, discussing extensively the concentration of measure phenomenon and the central limit theorem in this
setting. These facts will be used repeatedly when we then turn our attention to random matrices, and also many of the proof techniques used in the scalar setting (such as the moment method) can be adapted to the matrix context. Several of the key results in this chapter are developed through the exercises, and the book is designed for a student who is willing to work through these exercises as an integral part of understanding the topics covered here.

The material in Chapter 3 is related to the main topics of this text, but is optional reading (although the material on Dyson Brownian motion from Section 3.1 is referenced several times in the main text).

This text is not intended as a comprehensive introduction to random matrix theory, which is by now a vast subject. For instance, only a small amount of attention is given to the important topic of invariant matrix ensembles, and we do not discuss connections between random matrix theory and number theory, or to physics. For these topics we refer the reader to other texts such as [AnGuZi2010], [DeGi2007], [De1999], [Fo2010], [Me2004]. We hope, however, that this text can serve as a foundation for the reader to then tackle these more advanced texts.

Acknowledgments

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terrytao.wordpress.com/category/teaching/254a-random-matrices

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The field of random matrix theory has seen an explosion of activity in recent years, with connections to many areas of mathematics and physics. However, this makes the current state of the field almost too large to survey in a single book. In this graduate text, we focus on one specific sector of the field, namely the spectral distribution of random Wigner matrix ensembles (such as the Gaussian Unitary Ensemble), as well as iid matrix ensembles. The text is largely self-contained and starts with a review of relevant aspects of probability theory and linear algebra. With over 200 exercises, the book is suitable as an introductory text for beginning graduate students seeking to enter the field.