

# Combinatorial Game Theory

**Aaron N. Siegel**

**Graduate Studies  
in Mathematics**

**Volume 146**



**American Mathematical Society**

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American Mathematical Society  
Providence, Rhode Island

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2010 *Mathematics Subject Classification*. Primary 91A46.

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### Library of Congress Cataloging-in-Publication Data

Siegel, Aaron N., 1977–

Combinatorial game theory / Aaron N. Siegel.

pages cm. — (Graduate studies in mathematics ; volume 146)

Includes bibliographical references and index.

ISBN 978-0-8218-5190-6 (alk. paper)

1. Game theory. 2. Combinatorial analysis. I. Title.

QA269.S5735 2013

519.3—dc23

2012043675

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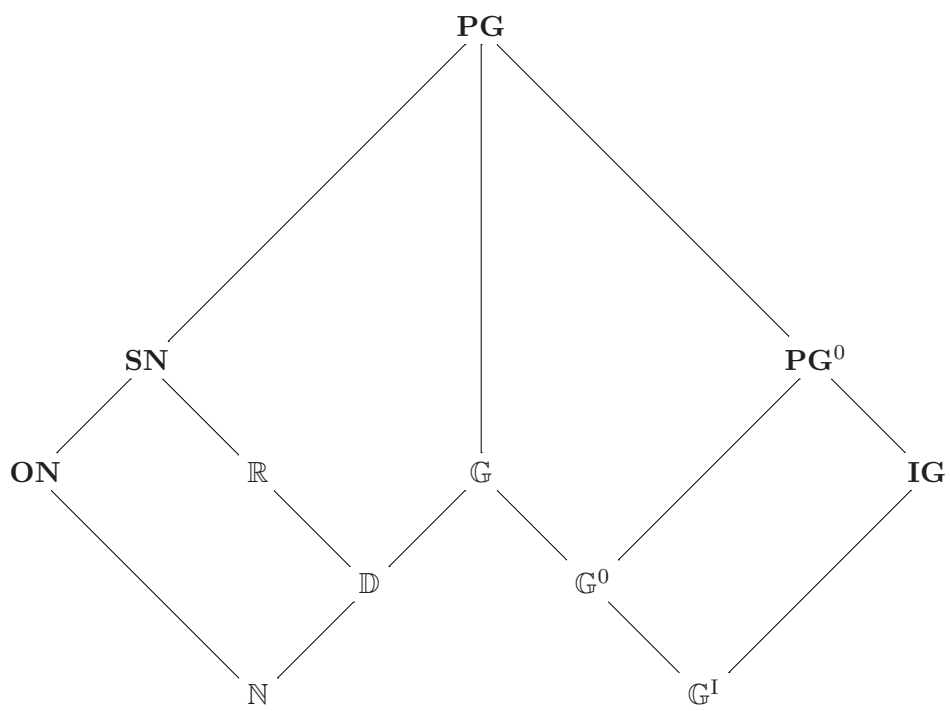
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To Elwyn Berlekamp





**Figure.** Various systems of numbers and games.





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# Preface

The disjunctive theory of combinatorial games can trace its roots to the work of Sprague and Grundy in the 1930s, but its modern form was born with the arrival of Conway's *On Numbers and Games* in 1976 and the classic *Winning Ways for Your Mathematical Plays* by Berlekamp, Conway, and Guy in 1982. In the ensuing three decades, combinatorial game theory has blossomed into a serious and active branch of combinatorics, with connections to coding theory, computational complexity, and commutative algebra.

This book is intended as a second course on combinatorial games, at the first- or second-year graduate level, and most readers will benefit from some prior exposure to the subject. *Winning Ways* is a fine introduction; in addition, an excellent new textbook by Albert, Nowakowski, and Wolfe, titled *Lessons in Play: An Introduction to Combinatorial Game Theory*, has recently appeared. Either (or both) of these references should serve as adequate preparation for this volume.

Nonetheless, this book is completely self-contained and traces the development of the theory from first principles and examples through many of its most recent advances. It should serve those who have read *Winning Ways* and crave a more rigorous development of the theory, as well as professionals seeking a cohesive reference for the many new ideas that have emerged in recent years. Among those advances appearing for the first time in textbook form (as far as I know) are Berlekamp's generalized temperature theory, Thane Plambeck's elegant theory of misère quotients, David Moews' results on the group structure of  $\mathbb{G}$ , and the construction of misère canonical forms for partizan games.

A great many people contributed insights, support, and suggestions. Elwyn Berlekamp has been a continued source of inspiration and encouragement throughout the past decade. Others whose input has influenced more than one page include Michael Albert, Dean Allemang, John Conway, Aviezri Fraenkel, J.P. Grossman, Richard Guy, Bob Hearn, Neil McKay, Richard Nowakowski, Thane Plambeck, Bill Spight, and David Wolfe. My wife, Olya, and daughter, Tali, have remained heroically supportive throughout endless years of writing and revising. Finally, I wish to thank my acquisitions editor, Ina Mette, and the AMS editorial committee for their seemingly infinite patience and unwavering support of this project.

Combinatorial game theory, in its modern form, is still a young field; many areas of the subject are still poorly understood and many questions remain unanswered. Open problems and conjectures are sprinkled throughout the text and are summarized in Appendix A. If this book provides a reservoir of tools and inspiration to attack these problems, then it will be a success.

Aaron Siegel  
San Francisco

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This bibliography includes material that is directly relevant to the subject matter covered in this book. Each reference includes a list of page numbers indicating where in the text it is cited; these appear in square brackets at the far right of the entry.

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# Glossary of Notation

This glossary is a list of various notation used throughout *Combinatorial Game Theory*, ordered by first appearance in the text. Each symbol is given together with a brief description, as well as the page (or pages) of the book on which its definition may be found.

Notation	Meaning or Description	Page(s)
$a \oplus b$	the nim-sum of $a$ and $b$	2, 438
$\mathcal{N}, \mathcal{P}, \mathcal{L}, \mathcal{R}$	the four (loopfree) outcome classes	6
$G^L, G^R$	a typical Left or Right option of $G$	8
$o(G)$	the outcome class of $G$	10, 55
$G + H$	the disjunctive sum of $G$ and $H$	11, 54
$G = H$	$G$ is equal to $H$	11, 55
$\mathbb{G}$	the group of short partizan game values	12, 55
$G \cong H$	$G$ is isomorphic to $H$	12
$\lfloor x \rfloor, \lceil x \rceil$	the floor and ceiling of $x$	24
$\text{lb}(n)$	the base-2 logarithm of $n$ , rounded down	24
$ \mathcal{S} $	the cardinality of the set $\mathcal{S}$	24
$\text{Pow}(\mathcal{S})$	the powerset of $\mathcal{S}$	24
$f[\mathcal{S}], f^{-1}[\mathcal{S}]$	the image and preimage of $\mathcal{S}$ under $f$	24
$\mathcal{A}^\omega$	the direct sum of countably many copies of $\mathcal{A}$	24
$\{G^L \mid G^R\}$	the game with typical Left option $G^L$ and typical right option $G^R$	53
$0$	zero, the empty game	54
$\tilde{\mathbb{G}}_n$	the set of games born by day $n$	54
$\tilde{\mathbb{G}}$	the set of short partizan games	54

Notation	Meaning or Description	Page(s)
$-G$	the negative of $G$	54
$G - H$	the disjunctive sum of $G$ and $-H$	54
$n \cdot G$	the disjunctive sum of $n$ copies of $G$	54
$\{G \parallel H \mid J\}$	the game $\{G \mid \{H \mid J\}\}$	
$G \geq H$	$G$ is greater than or equal to $H$	57
$G \triangleright H$	$G$ is greater than or confused with $H$	58
$G \not\asymp H$	$G$ is confused with $H$	58
$1, 2, -1, -2, \dots$	positive and negative integers	58
$*$	star	58
$\uparrow, \downarrow, \uparrow\uparrow, \downarrow\downarrow$	up, down, double-up, double-down	59
$\mathbb{G}^0$	the group of dicotic game values	60
$\mathbb{G}_n$	the set of game values born by day $n$	60
$b(G)$	the birthday of $G$	60, 400
$\mathbb{G}_n^0$	the set of dicotic game values born by day $n$	61
$\tilde{b}(G)$	the formal birthday of $G$	61
$\text{ch}(\mathcal{A})$	the set of children of games in $\mathcal{A}$	61
$\Delta^L(G), \Delta^R(G)$	a typical Left or Right incentive of $G$	62
$\mathbb{D}$	the group of dyadic rationals	69
$\frac{1}{2}, \frac{1}{4}, -\frac{1}{2}, -\frac{1}{4}, \dots$	dyadic rational numbers	69
$[a, b]$ or $]a, b[$	an open or closed interval with endpoints $a, b$	71, 71
$[a, b[$ or $]a, b]$	a half-open interval with endpoints $a, b$	71, 71
$\pm x$	the switch $\{x \mid -x\}$	73
$\mathcal{C}(G)$	the confusion interval of $G$	75
$L(G), R(G)$	the Left and Right stops of $G$	75
$m(G)$	the mean value of $G$	79
$G \ll H$	$G$ is infinitesimal relative to $H$	84
$*m$	star- $m$ , the number of order $m$	84
$\uparrow\uparrow, \downarrow\downarrow, \uparrow\uparrow\uparrow, \downarrow\downarrow\downarrow$	triple- and quadruple-up and down	85
$\uparrow n$	up- $n$ (shorthand for $n \cdot \uparrow$ )	85
$\uparrow n * m$	up- $n$ -star- $m$ (shorthand for $\uparrow n + *m$ )	85
$\dagger G, \dashv G$	tiny- $G$ and miny- $G$	88
$G : H$	the ordinal sum of $G$ and $H$	89
$\uparrow^n, \downarrow_n, \uparrow^{[n]}, \downarrow_{[n]}$	up- $n^{\text{th}}$ and relatives	94
$0.a_1a_2a_3\dots$	the uptimal with coefficients $a_1, a_2, a_3, \dots$	95
$\mathcal{UC}(G)$	the uptimal confusion interval of $G$	95

Notation	Meaning or Description	Page(s)
$\{0^k \mid G\}$	shorthand for $\{0 \parallel \{0^{k-1} \mid G\}\}$	96
$G^n, G^{[n]}$	generalized upimals	99
$G_t$	$G$ cooled by $t$	102
$t(G)$	the temperature of $G$	102
$L_t(G), R_t(G)$	the Left and Right scores of $G$ at $t$	103
$\tilde{L}_t(G), \tilde{R}_t(G)$	the Left and Right scaffolds of $G$ at $t$	108
$\int^T G$	$G$ heated by $T$	112
$\oint^T G$	$G$ overheated by $T$	114
$\oint_S^T G$	$G$ overheated from $S$ to $T$	115
$G \equiv H$	$G$ is infinitesimally close to $H$	124
$G \geq H$	$G$ is greater than or equal to $H$ modulo an infinitesimal difference	124
$\text{rcf}(G)$	the reduced canonical form of $G$	129
$G_*$	$G$ reduced by $*$	131
$\star$	a remote star	138
$G \sim H$	$G$ and $H$ are equivalent modulo $\star$	138
$G \gtrsim H$	$G$ is greater than or equal to $H$ modulo $\star$	138
$G \cdot \uparrow$	the Norton product of $G$ by $\uparrow$	141
$\text{aw}(G)$	the atomic weight of $G$	142
$\lfloor G \rfloor, \lceil G \rceil$	the floor and ceiling sets for $G$ in $\mathbb{G}_n$	159, 159
$G \vee H, G \wedge H$	the meet and join of $G$ and $H$ in $\mathbb{G}_n$	159, 159
$G^c$	the companion of $G$	163
$\mathcal{HT}_n$	the set of hereditarily transitive game values born by day $n$	167
$\langle \mathcal{S} \rangle$	the subgroup generated by $\mathcal{S}$	167
$\text{cl}(\mathcal{A})$	the closure of $\mathcal{A}$	168, 253
$G'$	a typical option of the impartial game $G$	179
$G' \in G$	$G'$ is an option of the impartial game $G$	180, 208
$\mathcal{G}(G)$	the nim value of $G$	180
$\text{mex}(\mathcal{S})$	the minimal excluded value of $\mathcal{S}$	180
$\mathbb{G}^\dagger$	the group of impartial game values	181
$\mathcal{G}(n)$	the nim value of a heap of size $n$	184
$\mathbf{d}_0.\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\dots$	the octal game with code digits $\mathbf{d}_0, \mathbf{d}_1, \mathbf{d}_2, \dots$	188
$\text{mex}_{\geq m}(\mathcal{S})$	the minimal excluded value of $\mathcal{S}$ among integers $\geq m$	203

Notation	Meaning or Description	Page(s)
$\mathcal{D}$	the outcome class of drawn positions	209
$\text{rank}(G)$	the rank of $G$ as a loopy impartial game	212
$a \otimes b$	the nim-product of $a$ and $b$	215, 439
$a \oslash b$	the nim-quotient of $a$ and $b$	219, 440
$a^{\overline{n}}$	the $n^{\text{th}}$ nim-power of $a$	219, 440
$o^+(G), o^-(G)$	the normal-play and misère-play outcomes of $G$	223
$\mathcal{G}^+(G), \mathcal{G}^-(G)$	the normal and misère nim values of $G$	229
$\mathcal{G}^\pm(G)$	the genus of $G$	231
$\delta(G)$	the discriminant of $G$	233
$*G_\#, *GHJ$	shorthand notation for misère impartial games	235
$\mathcal{G}^*(G)$	the extended genus of $G$	239
$G^-$	the mate of $G$	243
$G \bowtie H$	$G$ is linked to $H$	244
$\mathcal{Q}(\mathcal{A})$	the misère quotient of the set $\mathcal{A}$	250
$\mathcal{T}_n$	the tame misère quotient on $n$ generators	254
$G^\circ$	the adjoint of $G$	270
$G \ltimes H, G \bowtie H$	$G$ is downlinked or uplinked to $H$	274
$\overline{G}$	the negative of $G$ (synonymous with $-G$ )	282
$\hat{\mathcal{P}}, \check{\mathcal{P}}, \hat{\mathcal{N}}, \check{\mathcal{N}}$	the four partizan loopy outcome classes	282
<b>on, off, dud</b>	the loopy games: on, off, dud	284
<b>over, under</b>	the loopy games: over, under	291
$\Sigma_n^X(G)$	the $n^{\text{th}}$ sidling approximation to $G$ (from $X$ )	292
$G \sim_n H$	$G$ and $H$ are similar for $n$ moves	295
$G \simeq H$	$G$ and $H$ are similar	295
$\uparrow^{\text{on}}, \uparrow^{\text{on}}$	the loopy games: upon, upon <sup>th</sup>	296
<b>ace, pip<sub>n</sub></b>	the pip games	298
$\delta(G), \delta(G)$	the biased outcomes of $G$	309
$G \doteq H, G \doteq H$	$G$ is equal to $H$ (biased)	310
$G \gtrsim H, G \gtrsim H$	$G$ is greater than or equal to $H$ (biased)	310
$G = S \& T$	$G$ has onside $S$ and offside $T$	310
$G \uplus H, G \uplus H$	the upsum and downsum of $G$ and $H$	318
$G^\circ$	the degree (of loopiness) of $G$	322
$V(G)$	the variety of $G$	327
$\mathcal{E}_t^\delta$	the coupon stack of temperature $t$ and granularity $\delta$	336

Notation	Meaning or Description	Page(s)
$L_t^\delta(G), R_t^\delta(G)$	the enriched scores of $G$	336, 355
$L_{\text{full}}(G), R_{\text{full}}(G)$	the Left and Right fullstops of $G$	339, 355
$\mathcal{E}_t$	a thick coupon stack of temperature $t$	341
$t_b(G)$	the board temperature of $G$	344
$t_a(G)$	the ambient temperature of $G$	345
$s(G)$	the subposition count of $G$	372
$\theta_x$	the standard threat of size $x$	381
$\Theta_n$	the $n^{\text{th}}$ threat environment	381
$L_t^\sharp(G), R_t^\sharp(G)$	the Left komaster scores of $G$	381
$L_t^\flat(G), R_t^\flat(G)$	the Right komaster scores of $G$	381
$m^\sharp(G), m^\flat(G)$	the mast value with Left or Right as komaster	382
$t^\sharp(G), t^\flat(G)$	the temperature with Left or Right as komaster	382
$L_t^{\sharp\sharp}(G), R_t^{\sharp\sharp}(G)$	the Left komonster scores of $G$	392
$L_t^{\flat\flat}(G), R_t^{\flat\flat}(G)$	the Right komonster scores of $G$	392
$\omega$	omega, the least infinite ordinal	398
<b>PG</b>	the Group of long partizan game values	399
<b>IG</b>	the Group of long impartial game values	400
<b>PG<sup>0</sup></b>	the Group of long dicotic game values	400
<b>SN</b>	the Field of surreal numbers	401
$\pm\mathbb{N}$	the game $\pm(0, 1, 2, \dots)$	404
<b>PG<sup>∞</sup></b>	the Group of all-moderate game values	405
$\infty$	the game $\{\mathbb{N} \mid \pm\mathbb{N}\}$	406
$\infty^\alpha$	the largest moderate games	407
$x \times y$	the Conway product of long games $x$ and $y$	412
$x_\beta$	the $\beta^{\text{th}}$ approximation to the surreal number $x$	417
$\sigma \upharpoonright \beta$	the restriction of the sign sequence $\sigma$ to length $\beta$	418
$\text{supp}(x)$	the support of the surreal number $x$	428
<b>ON<sub>2</sub></b>	the simplest Field of characteristic 2	439
$\mathcal{P}_\gamma$	the set of predecessors of the ordinal $\gamma$	439





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ISBN 978-0-8218-5190-6



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