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Preface

The systematic study of matrices began late in the history of mathematics, but matrix theory is an active area of research now and it has applications in numerical analysis, control and systems theory, optimization, combinatorics, mathematical physics, differential equations, probability and statistics, economics, information theory, and engineering.

One attractive feature of matrix theory is that many matrix problems can be solved naturally by using tools or ideas from other branches of mathematics such as analysis, algebra, graph theory, geometry and topology. The reverse situation also occurs, as shown in the last chapter.

This book is intended for use as a text for graduate or advanced undergraduate level courses, or as a reference for research workers. It is based on lecture notes for graduate courses I have taught five times at East China Normal University and once at Peking University. My aim is to provide a concise treatment of matrix theory. I hope the book contains the basic knowledge and conveys the flavor of the subject.

When I chose material for this book, I had the following criteria in mind: 1) important; 2) elegant; 3) ingenious; 4) interesting. Of course, a very small percentage of mathematics meets all of these criteria, but I hope the results and proofs here meet at least one of them. As a reader I feel that for clarity, the logical steps of a mathematical proof cannot be omitted, though routine calculations may be or should be. Whenever possible, I try to have a conceptual understanding of a result. I always emphasize methods and ideas.

Most of the exercises are taken from research papers, and they have some depth. Thus if the reader has difficulty in solving the problems in these exercises, she or he should not feel frustrated.

Parts of this book appeared in a book in Chinese with the same title published by the Higher Education Press in 2008.

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Shanghai, December 2012 Xingzhi Zhan

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Notation

$\mathbb{C}$ the field of complex numbers

$\mathbb{R}$ the field of real numbers

Ω^n the set of n-tuples with components from Ω

M_n the set of $n \times n$ complex matrices

$M_{m,n}$ the set of $m \times n$ complex matrices

$M_n(\Omega)$ the set of $n \times n$ matrices with entries from Ω

$A(i, j)$ entry of the matrix A in the i-th row and j-th column

$A[\alpha|\beta]$ submatrix of A that lies in the rows indexed by α and columns indexed by β.

$A(\alpha|\beta)$ submatrix of A obtained by deleting the rows indexed by α and columns indexed by β.

A^T transpose of the matrix A

A^* conjugate transpose of the complex matrix A

$\mathrm{diag}(d_1, \ldots, d_n)$ the diagonal matrix with diagonal entries $d_1, \ldots, d_n$

$A_1 \oplus A_2 \oplus \cdots \oplus A_k$ the block diagonal matrix $\mathrm{diag}(A_1, A_2, \ldots, A_k)$

I the identity matrix whose order is clear from the context

I_n the identity matrix of order n

$\triangleq$ by definition equal to

$\forall$ for all

$\sigma(A)$ spectrum of the matrix A

$\rho(A)$ spectral radius of A

$\langle \cdot, \cdot \rangle$ the standard Euclidean inner product

$\|\cdot\|$ norm on a vector space

$\|A\|_\infty$ spectral norm of A

$\|A\|_F$ Frobenius norm of A

$\|A\|_p$ Schatten p-norm of A

$\|A\|_{(k)}$ Fan k-norm of A

$\|\cdot\|^D$ dual norm of $\|\cdot\|$

$W(A)$ numerical range of A

$w(A)$ numerical radius of A

$\langle f_1, \dots, f_k \rangle$ the ideal generated by $f_1, \dots, f_k$

$\operatorname{ran} A$ range of A

$\ker A$ kernel of A

$A \otimes B$ tensor product of A and B

$A \circ B$ Hadamard product of A and B

$\operatorname{sv}(A)$ the set of the singular values of A

$\det A$ determinant of A

$\operatorname{per} A$ permanent of A

$\operatorname{tr} A$ trace of A

$\binom{n}{k}$ binomial coefficient, $n!/[k!(n-k)!]$

$C_k(A)$ k-th compound matrix of A

S_n the set of permutations of $1, 2, \dots, n$

$x \prec y$ x is majorized by y

$x \prec_w y$ x is weakly majorized by y

$x \prec_{\log} y$ x is log-majorized by y

$x \prec_{\mathrm{wlog}} y$ x is weakly log-majorized by y

$\operatorname{diag}(x)$ diagonal matrix whose diagonal entries are the components of x

$A \leq B$, $B \geq A$ $B - A$ is positive semidefinite or entry-wise nonnegative, depending on context

$s_i(A)$ the i-th largest singular value of A

$s(A)$ the vector of the singular values of A

$|A|$ $(A^*A)^{1/2}$ or $(|a_{ij}|)$ if $A = (a_{ij})$, depending on context

$D(A)$ digraph of the matrix A

$F[x_1, \dots, x_k]$ the ring of polynomials in the indeterminates $x_1, \dots, x_k$ over the field F

Index

Matrix theory is a classical topic of algebra that had originated, in its current form, in the middle of the 19th century. It is remarkable that for more than 150 years it continues to be an active area of research full of new discoveries and new applications.

This book presents modern perspectives of matrix theory at the level accessible to graduate students. It differs from other books on the subject in several aspects. First, the book treats certain topics that are not found in the standard textbooks, such as completion of partial matrices, sign patterns, applications of matrices in combinatorics, number theory, algebra, geometry, and polynomials. There is an appendix of unsolved problems with their history and current state. Second, there is some new material within traditional topics such as Hopf's eigenvalue bound for positive matrices with a proof, a proof of Horn's theorem on the converse of Weyl's theorem, a proof of Camion-Hoffman's theorem on the converse of the diagonal dominance theorem, and Audenaert's elegant proof of a norm inequality for commutators. Third, by using powerful tools such as the compound matrix and Gröbner bases of an ideal, much more concise and illuminating proofs are given for some previously known results. This makes it easier for the reader to gain basic knowledge in matrix theory and to learn about recent developments.

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