Random Operators
Disorder Effects
on Quantum Spectra
and Dynamics
Random Operators
Disorder Effects on Quantum Spectra and Dynamics

Michael Aizenman
Simone Warzel

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Dedicated to Marta by Michael

and to Erna and Horst by Simone
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Disorder effects on quantum spectra and dynamics have drawn the attention of both physicists and mathematicians. In this introduction to the subject we aim to present some of the relevant mathematics, paying heed also to the physics perspective.

The techniques presented here combine elements of analysis and probability, and the mathematical discussion is accompanied by comments with a relevant physics perspective. The seeds of the subject were initially planted by theoretical and experimental physicists. The mathematical analysis was, however, enabled not by filling the gaps in the theoretical physics arguments, but through paths which proceed on different tracks. As in other areas of mathematical physics, a mathematical formulation of the theory is expected both to be of intrinsic interest and to potentially also facilitate further propagation of insights which originated in physics.

The text is based on notes from courses that were presented at our respective institutions and attended by graduate students and postdoctoral researchers. Some of the lectures were delivered by course participants, and for that purpose we found the availability of organized material to be of great value.

The chapters in the book were originally intended to provide reading material for, roughly, a week each; but it is clear that for such a pace omissions should be made and some of the material left for discretionary reading. The book starts with some of the core topics of random operator theory, which are also covered in other texts (e.g., [105, 82, 324, 228, 230, 367]). From Chapter 5 on, the discussion also includes material which has so far been presented in research papers and not so much in monographs on the subject. The mark * next to a section number indicates material which the reader is
advised to skip at first reading but which may later be found useful. The selection presented in the book is not exhaustive, and for some topics and methods the reader is referred to other resources.

During the work on this book we have been encouraged by family and many colleagues. In particular we wish to thank Yosi Avron, Marek Biskup, Joseph Imry, Vojkan Jaksic, Werner Kirsch, Hajo Leschke, Elliott Lieb, Peter Müller, Barry Simon, Uzy Smilansky, Sasha Sodin, and Philippe Sosoe for constructive suggestions. Above all Michael would like to thank his wife, Marta, for her support, patience, and wise advice.

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Michael Aizenman, Princeton and Rehovot
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This book provides an introduction to the mathematical theory of disorder effects on quantum spectra and dynamics. Topics covered range from the basic theory of spectra and dynamics of self-adjoint operators through Anderson localization—presented here via the fractional moment method, up to recent results on resonant delocalization.

The subject’s multifaceted presentation is organized into seventeen chapters, each focused on either a specific mathematical topic or on a demonstration of the theory’s relevance to physics, e.g., its implications for the quantum Hall effect. The mathematical chapters include general relations of quantum spectra and dynamics, ergodicity and its implications, methods for establishing spectral and dynamical localization regimes, applications and properties of the Green function, its relation to the eigenfunction correlator, fractional moments of Herglotz-Pick functions, the phase diagram for tree graph operators, resonant delocalization, the spectral statistics conjecture, and related results.

The text incorporates notes from courses that were presented at the authors’ respective institutions and attended by graduate students and postdoctoral researchers.

It has been almost 25 years since the last major book on this subject. The authors masterfully update the subject but more importantly present their own probabilistic insights in clear fashion. This wonderful book is ideal for both researchers and advanced students.

—Barry Simon, California Institute of Technology