Convection-Diffusion Problems
An Introduction to Their Analysis and Numerical Solution

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Preface

Convection-diffusion problems attract much attention in the research literature. For numerical analysts working in this area, a standard reference is the text by Roos, Stynes, and Tobiska \cite{RST96,RST08}. This book contains a lot of useful information, but it is daunting for those beginners who have some familiarity with numerical methods and their analysis but who have not previously worked with convection-diffusion and other singularly perturbed differential equations. For many years I felt that an easier, more introductory book was needed to encourage new people to enter our fascinating research area. This belief was encouraged by the popularity of a survey article, “Steady-state convection-diffusion problems”, that I wrote for *Acta Numerica* in 2005 \cite{Sty05}. The present book is an extended and updated version of that 2005 article, and I have added exercises and other material to try to make it more attractive and more useful for the novice reader.

The feeling that a book of this type was desirable did not lead me to take any action until I was invited to present a course on this topic at the AARMS (Atlantic Association for Research in the Mathematical Sciences) Summer School at Dalhousie University in Halifax, Nova Scotia, Canada, during July 2015. The organisers encourage their lecturers to transform their lecture notes into books, and after much delay I have done this. I am very grateful to AARMS for their invitation to lecture and for the enjoyable month I spent in the delightful city of Halifax.

Here we list the prerequisites for the reader. In Chapters 1-3 some knowledge of two-point boundary value problems and their numerical solution by finite difference methods is enough for almost all of the material.
For Chapter 4 it is desirable to have some previous experience of partial differential equations. Chapter 5 uses only ideas from earlier chapters. Finite element methods (FEMs) appear for the first time in the long Chapter 6 and here I assume that the reader already has a general understanding of how FEMs are constructed and analysed. The Lebesgue spaces $L^p(\Omega)$ and the standard Sobolev spaces $H^k(\Omega)$ are used occasionally in the earlier chapters of the book and more heavily in Chapter 6 the reader should have some familiarity with these well-known concepts.

The book was written where I work, in the research paradise known as Beijing Computational Science Research Center. I owe a great debt to CSRC’s director Hai-Qing Lin for the positive environment he has created at CSRC through his friendly yet no-nonsense approach to productive research. My work was supported by the 1000 Talents (Foreign Experts) Program of the People’s Republic of China.

All comments on this book are welcome. No doubt it will (inevitably) contain some mistakes, so corrections are also welcome, though the fewer the better! My email address is m.stynes@csrc.ac.cn

Martin Stynes


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Many physical problems involve diffusive and convective (transport) processes. When diffusion dominates convection, standard numerical methods work satisfactorily. But when convection dominates diffusion, the standard methods become unstable, and special techniques are needed to compute accurate numerical approximations of the unknown solution. This convection-dominated regime is the focus of the book. After discussing at length the nature of solutions to convection-dominated convection-diffusion problems, the authors motivate and design numerical methods that are particularly suited to this class of problems.

At first they examine finite-difference methods for two-point boundary value problems, as their analysis requires little theoretical background. Upwinding, artificial diffusion, uniformly convergent methods, and Shishkin meshes are some of the topics presented. Throughout, the authors are concerned with the accuracy of solutions when the diffusion coefficient is close to zero. Later in the book they concentrate on finite element methods for problems posed in one and two dimensions.

This lucid yet thorough account of convection-dominated convection-diffusion problems and how to solve them numerically is meant for beginning graduate students, and it includes a large number of exercises. An up-to-date bibliography provides the reader with further reading.

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