GRADUATE STUDIES IN MATHEMATICS 199

Applied Mathematics

Applied Stochastic Analysis

Weinan E
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AMERICAN MATHEMATICAL SOCIETY

10.1090/gsm/199

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American Mathematical Society
Providence, Rhode Island

2010 *Mathematics Subject Classification.* Primary 60-01, 60J22, 60H10, 60H35, 62P10, 62P35, 65C05, 65C10, 82-01.

For additional information and updates on this book, visit
www.ams.org/bookpages/gsm-199

Library of Congress Cataloging-in-Publication Data

Names: E, Weinan, 1963- author. | Li, Tiejun, 1974- author. | Vanden-Eijnden, Eric, 1968- author.
Title: Applied stochastic analysis / Weinan E, Tiejun Li, Eric Vanden-Eijnden.
Description: Providence, Rhode Island : American Mathematical Society, [2019] | Series: Graduate studies in mathematics ; volume 199 | Includes bibliographical references and index.
Identifiers: LCCN 2019000898 | ISBN 9781470449339 (alk. paper)
Subjects: LCSH: Stochastic analysis. | AMS: Probability theory and stochastic processes – Instructional exposition (textbooks, tutorial papers, etc.). msc | Probability theory and stochastic processes – Markov processes – Computational methods in Markov chains. msc | Probability theory and stochastic processes – Stochastic analysis – Stochastic ordinary differential equations. msc | Probability theory and stochastic processes – Stochastic analysis – Computational methods for stochastic equations. msc | Statistics – Applications – Applications to biology and medical sciences. msc | Statistics – Applications – Applications to physics. msc | Numerical analysis – Probabilistic methods, simulation and stochastic differential equations – Monte Carlo methods. msc | Numerical analysis – Probabilistic methods, simulation and stochastic differential equations – Random number generation. msc | Statistical mechanics, structure of matter – Instructional exposition (textbooks, tutorial papers, etc.). msc
Classification: LCC QA274.2 .E23 2019 | DDC 519.2–dc23
LC record available at https://lccn.loc.gov/2019000898

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10 9 8 7 6 5 4 3 2 1 24 23 22 21 20 19

To our families

Hongjun, Jane, and Ilene
Xueqing, Baitian, and Baile
Jasna, Colette, Pauline, and Anais et Lilia

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Part 2. Advanced Topics

Introduction to the Series

It is fairly well accepted that to learn pure mathematics, a student has to take analysis, algebra, geometry, and topology. Until now there has not been an equally well-accepted curriculum for applied mathematics. There are many reasons one can think of. For one thing, applied mathematics is truly very diverse. Traditional subjects such as numerical analysis, statistics, operational research, etc., can all be regarded as part of applied mathematics. Beyond that, a huge amount of applied mathematics is practiced in other scientific and engineering disciplines. For example, a large part of fluid dynamics research has become computational. The same can be said about theoretical chemistry, biology, material science, etc. The algorithmic issues that arise in these disciplines are at the core of applied mathematics. Machine learning, a subject that deals with mathematical models and algorithms for complex data, is, at its heart, a very mathematical discipline, although at the moment it is much more commonly practiced in computer science departments.

In 2002, David Cai, Shi Jin, Eric Vanden-Eijnden, Pingwen Zhang, and I started the applied mathematics summer school in Beijing, with the objective of creating a systematic and unified curriculum for students in applied mathematics. Since then, this summer school has been repeated every year at Peking University. The main theme of the summer school has also evolved. But in the early years, the main topics were applied stochastic analysis, differential equations, numerical algorithms, and a mathematical introduction to physics. In recent years, a mathematical introduction to machine learning has also been added to the list. In addition to the summer school, these courses have also been taught from time to time during regular terms at New York University, Peking University, and Princeton University. The lecture notes from these courses are the main origin of this textbook series. The early participants, including some of the younger people who were students or teaching assistants early on, have also become contributors to this book series.

After many years, this series of courses has finally taken shape. Obviously, this has not come easy and is the joint effort of many people. I would like to express my sincere gratitude to Tiejun Li, Pingbing Ming, Shi Jin, Eric Vanden-Eijnden, Pingwen Zhang, and other collaborators involved for their commitment and dedication to this effort. I am also very grateful to Mrs. Yanyun Liu, Baomei Li, Tian Tian, Yuan Tian for their tireless efforts to help run the summer school. Above all, I would like to pay tribute to someone who dedicated his life to this cause, David Cai. David had a passion for applied mathematics and its applications to science. He believed strongly that one has to find ways to inspire talented young people to go into applied mathematics and to teach them applied mathematics in the right way. For many years, he taught a course on introducing physics to applied mathematicians in the summer school. To create a platform for practicing the philosophy embodied in this project, he cofounded the Institute of Natural Sciences at Shanghai Jiaotong University, which has become one of the most active centers for applied mathematics in China. His passing away last year was an immense loss, not just for all of us involved in this project, but also for the applied mathematics community as a whole.

This book is the first in this series, covering probability theory and stochastic processes in a style that we believe is most suited for applied mathematicians. Other subjects covered in this series will include numerical algorithms, the calculus of variations and differential equations, a mathematical introduction to machine learning, and a mathematical introduction to physics and physical modeling. The stochastic analysis and differential equations courses summarize the most relevant aspects of pure mathematics, the algorithms course presents the most important technical tool of applied mathematics, and the learning and physical modeling courses provide a bridge to the real world and to other scientific disciplines. The selection of topics represents a possibly biased view about the true fundamentals of applied mathematics. In particular, we emphasize three themes throughout this textbook series: learning, modeling, and algorithms. Learning is about data and intelligent decision making. Modeling is concerned with physics-based models. The study of algorithms provides the practical tool for building and interrogating the models, whether machine learning–based or physics-based. We believe that these three themes should be the major pillars of applied mathematics, the applied math analog of algebra, analysis, and geometry.

While physical modeling and numerical algorithms have been the dominating driving force in applied mathematics for years, machine learning is a relatively new comer. However, there are very good reasons to believe that machine learning will not only change AI but also the way we do physical modeling. With this last missing component in place, applied mathematics will become the natural platform for integrating machine learning and physical modeling. This represents a new style for doing scientific research, a style in which the data-driven Keplerian paradigm and the first principle-driven Newtonian paradigm are integrated to give rise to unprecedented technical power. It is our hope that this series of textbooks will be of some help for making that a reality!

Weinan E, Princeton, 2018

Preface

This book is written for students and researchers in applied mathematics with an interest in science and engineering. Our main purpose is to provide a mathematically solid introduction to the basic ideas and tools in probability theory and stochastic analysis. Starting from the basics of random variables and probability theory, we go on to discuss limit theorems, Markov chains, diffusion processes, and random fields. Since the kind of readers we have in mind typically have some background in differential equations, we put more weight on the differential equation approach. In comparison, we have neglected entirely martingale theory even though it is a very important part of stochastic analysis. The diffusion process occupies a central role in this book. We have presented three different ways of looking at the diffusion process: the approach of using stochastic differential equations, the Fokker-Planck equation approach, and the path integral approach. The first allows us to introduce stochastic calculus. The second approach provides a link between differential equations and stochastic analysis. The path integral approach is very much preferred by physicists and is also suited for performing asymptotic analysis. In addition, it can be extended to random fields.

In choosing the style of the presentation, we have tried to strike a balance between rigor and the heuristic approach. We have tried to give the reader an idea about the kind of mathematical construction or mathematical argument that goes into the subject matter, but at the same time, we often stop short of proving all the theorems we state or we prove the theorems under stronger assumptions. Whenever possible, we have tried to give the intuitive picture behind the mathematical constructions.

Another emphasis is on numerical algorithms, including Monte Carlo methods, numerical schemes for solving stochastic differential equations, and the stochastic simulation algorithm. The book ends with a discussion on two application areas, statistical mechanics and chemical kinetics, and a discussion on rare events, which is perhaps the most important manifestation of the effect of noise.

The material contained in this book has been taught in various forms at Peking University, Princeton University, and New York University since 2001. It is now a required course for the special applied mathematics program at Peking University.

Weinan E
Tiejun Li
Eric Vanden-Eijnden

December 2018

Notation

(1) Geometry notation:
   (a) $\mathbb{N}$: Natural numbers, $\mathbb{N} = \{0, 1, \ldots\}$.
   (b) $\mathbb{Z}$: Integers.
   (c) $\mathbb{Q}$: Rational numbers.
   (d) $\mathbb{R}$: Real numbers.
   (e) $\bar{\mathbb{R}}$: Extended real numbers, $\bar{\mathbb{R}} = [-\infty, \infty]$.
   (f) $\mathbb{R}_+$: Nonnegative real numbers.
   (g) $\bar{\mathbb{R}}_+$: Extended nonnegative real numbers, $\bar{\mathbb{R}}_+ = [0, \infty]$.
   (h) $\mathbb{S}^{n-1}$: $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$.
   (i) $\mathbb{R}^{\mathbf{T}}$: Collection of all real functions on time domain $\mathbf{T}$.

(2) Probability notation:
   (a) $\mathbb{P}$: Probability measure.
   (b) $\mathbb{E}$: Mathematical expectation.
   (c) $\mathbb{P}^i$, $\mathbb{P}^x$, or $\mathbb{P}^\mu$: Probability distribution conditioned on $X_0 = i$, $X_0 = x$, or $X_0 \sim \mu$.
   (d) $\mathbb{E}^i$, $\mathbb{E}^x$, $\mathbb{E}^\mu$: Mathematical expectation with respect to $\mathbb{P}^i$, $\mathbb{P}^x$, or $\mathbb{P}^\mu$.
   (e) $\mathbb{E}^{x,t}$: Mathematical expectation conditioned on $X_t = x$.
   (f) Ω: Sample space.
   (g) $\mathcal{F}$: σ-algebra in probability space.
   (h) $\mathcal{R}, \mathcal{R}^d$: The Borel σ-algebra on $\mathbb{R}$ or $\mathbb{R}^d$.
   (i) $\sigma(\mathcal{B})$: The smallest σ-algebra generated by sets in $\mathcal{B}$.
   (j) $\mathcal{R}^{\mathbf{T}}$: The product Borel σ-algebra on $\mathbb{R}^{\mathbf{T}}$.
   (k) $\mathcal{U}(A)$: Uniform distribution on set A.

(l) $\mathcal{P}(\lambda)$: Poisson distribution with mean λ.
(m) $\mathcal{E}(\lambda)$: Exponential distribution with mean λ^{-1}.
(n) $[X, X]_t$: Quadratic variation process of X.
(o) $X \sim \mathcal{P}(\lambda)$: Distribution of X. The right-hand side can be distributions like $\mathcal{P}(\lambda)$ or $N(\mu, \sigma^2)$, etc.

(3) Function spaces:
(a) $C_c^\infty(\mathbb{R}^d)$ or $C_c^k(\mathbb{R}^d)$: Smooth or C^k-functions with compact support in $\mathbb{R}^d$.
(b) $C_0(\mathbb{R}^d)$: Continuous functions in $\mathbb{R}^d$ that vanish at infinity.
(c) $C_b(\mathbb{R}^d)$: Bounded continuous functions in $\mathbb{R}^d$.
(d) L_t^p or $L^p([0, T])$: L^p-functions as a function of t.
(e) L_ω^p or $L^p(\Omega)$: L^p-functions as a function of ω.
(f) $\mathscr{B}, \mathscr{H}$: Banach or Hilbert spaces.
(g) $\mathscr{B}^*$: Dual space of $\mathscr{B}$.

(4) Operators: $\mathcal{I}, \mathcal{P}, \mathcal{Q}, \mathcal{K}$, etc.

(5) Functions:
(a) $\lceil \cdot \rceil$: The ceil function. $\lceil x \rceil = m+1$ if $x \in [m, m+1)$ for $m \in \mathbb{Z}$.
(b) $\lfloor \cdot \rfloor$: The floor function. $\lfloor x \rfloor = m$ if $x \in [m, m+1)$ for $m \in \mathbb{Z}$.
(c) $|\boldsymbol{x}|$: ℓ^2-modulus of a vector $\boldsymbol{x} \in \mathbb{R}^d$: $|\boldsymbol{x}| = (\sum_{i=1}^d x_i^2)^{\frac{1}{2}}$.
(d) $\|f\|$: Norm of function f in some function space.
(e) $a \vee b$: Maximum of a and b: $a \vee b = \max(a, b)$.
(f) $a \wedge b$: Minimum of a and b: $a \wedge b = \min(a, b)$.
(g) $\langle f \rangle$: The average of f with respect to a measure μ: $\langle f \rangle = \int f(x)\mu(dx)$.
(h) $(\boldsymbol{x}, \boldsymbol{y})$: Inner product for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$: $(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T \boldsymbol{y}$.
(i) (f, g): Inner product for L^2-functions f, g: $(f, g) = \int f(x)g(x)dx$.
(j) $\langle f, g \rangle$: Dual product for $f \in \mathscr{B}^*$ and $g \in \mathscr{B}$.
(k) $A : B$: Twice contraction for second-order tensors, i.e., $A : B = \sum_{ij} a_{ij} b_{ji}$.
(l) $|S|$: The cardinality of set S.
(m) $|\Delta|$: The subdivision size when Δ is a subdivision of a domain.
(n) $\chi_A(x)$: Indicator function, i.e., $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise.
(o) $\delta(x - a)$: Dirac's delta-function at $x = a$.
(p) Range($\mathcal{P}$), Null($\mathcal{P}$): The range and null space of operator $\mathcal{P}$, i.e., Range($\mathcal{P}$) $= \{y | y = \mathcal{P}x\}$, Null($\mathcal{P}$) $= \{x | \mathcal{P}x = 0\}$.
(q) ${}^\perp\mathscr{C}$: Perpendicular subspace, i.e., ${}^\perp\mathscr{C} = \{x | x \in \mathscr{B}^*, \langle x, y \rangle = 0, \forall y \in \mathscr{C}\}$ where $\mathscr{C} \subset \mathscr{B}$.

(6) Symbols:
   (a) $c_\varepsilon \asymp d_\varepsilon$: Logrithmic equivalence, i.e., $\lim_{\varepsilon\to 0} \log c_\varepsilon / \log d_\varepsilon = 1$.
   (b) $c_\varepsilon \sim d_\varepsilon$ or $c_n \sim d_n$: Equivalence, i.e., $\lim_{\varepsilon\to 0} c_\varepsilon / d_\varepsilon = 1$ or $\lim_{n\to\infty} c_n/d_n = 1$.
   (c) $\mathcal{D}x$: Formal infinitesimal element in path space, $\mathcal{D}x = \prod_{0\le t\le T} dx_t$.

Appendix

A. Laplace Asymptotics and Varadhan's Lemma

Consider the Laplace integral

$$F(t) = \int_{\mathbb{R}} e^{th(x)} dx, \qquad t \gg 1,$$

where $h(x) \in C^2(\mathbb{R})$, $h(0)$ is the only global maximum, and $h''(0) \neq 0$. We further assume that for any $c > 0$, there exists $b > 0$ such that

$$h(x) - h(0) \leq -b \qquad \text{if } |x| \geq c.$$

Assume also that $h(x) \to -\infty$ fast enough as $x \to \infty$ to ensure the convergence of F for $t = 1$.

Lemma A.1 (Laplace method)**.** *We have the asymptotics*

$$F(t) \sim \sqrt{2\pi}(-th''(0))^{-\frac{1}{2}} \exp(th(0)) \qquad \text{as } t \to \infty, \tag{A.1}$$

where the equivalence $f(t) \sim g(t)$ *means that* $\lim_{t\to\infty} f(t)/g(t) = 1$.

The above asymptotic results can be stated as

$$\lim_{t\to\infty} \frac{1}{t} \log F(t) = \sup_{x\in\mathbb{R}} h(x).$$

This formulation is what we will use in the large deviation theory. Its abstract form in the infinite-dimensional setting is embodied in the so-called Varadhan's lemma to be discussed later [**DZ98**, **DS84**, **Var84**].

Proof. Without loss of generality, we can assume $h(0) = 0$ by shifting $h(x)$ correspondingly. With this condition, if $h(x) = h''(0)x^2/2$, $h''(0) < 0$, we have

$$\int_{\mathbb{R}} e^{th(x)}dx = \sqrt{2\pi}(-th''(0))^{-\frac{1}{2}}.$$

In general, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $|x| \le \delta$,

$$\left| h(x) - \frac{h''(0)}{2}x^2 \right| \le \epsilon x^2.$$

It follows that

$$\begin{aligned}\int_{[-\delta,\delta]} \exp\Big(\frac{tx^2}{2}(h''(0)-2\epsilon)\Big)dx &\le \int_{[-\delta,\delta]} \exp\Big(th(x)\Big)dx \\ &\le \int_{[-\delta,\delta]} \exp\Big(\frac{tx^2}{2}(h''(0)+2\epsilon)\Big)dx.\end{aligned}$$

By assumption, for this $\delta > 0$, there exists $\eta > 0$ such that $h(x) \le -\eta$ if $|x| \ge \delta$. Thus

$$\int_{|x|\ge\delta} \exp\Big(th(x)\Big)dx \le e^{-(t-1)\eta}\int_{\mathbb{R}} e^{h(x)}dx \sim \mathcal{O}(e^{-\alpha t}),\ \alpha > 0, \quad \text{for} \quad t > 1.$$

We first prove the upper bound:

$$\begin{aligned}\int_{\mathbb{R}} \exp\Big(th(x)\Big)dx \le &\int_{\mathbb{R}} \exp\Big(\frac{tx^2}{2}(h''(0)+2\epsilon)\Big)dx \\ &- \int_{|x|\ge\delta} \exp\Big(\frac{tx^2}{2}(h''(0)+2\epsilon)\Big)dx + \mathcal{O}(e^{-\alpha t}) \\ =&\sqrt{2\pi}\Big[t(-h''(0)-2\epsilon)\Big]^{-\frac{1}{2}} + \mathcal{O}(e^{-\beta t})\end{aligned}$$

where $\beta > 0$. In fact, we ask for $\epsilon < -h''(0)/2$ here.

The proof of the lower bound is similar. By the arbitrary smallness of ϵ, we have

$$\lim_{t\to\infty} F(t)/\sqrt{2\pi}(-th''(0))^{-\frac{1}{2}} = 1,$$

which completes the proof. □

Definition A.2 (Large deviation principle)**.** Let $\mathscr{X}$ be a complete separable metric space and let $\{\mathbb{P}^\varepsilon\}_{\varepsilon\ge 0}$ be a family of probability measures on the Borel subsets of $\mathscr{X}$. We say that $\mathbb{P}^\varepsilon$ satisfies the large deviation principle if there exists a rate functional $I : \mathscr{X} \to [0,\infty]$ such that:

(i) For any $\ell < \infty$,

$$\{x : I(x) \le \ell\} \text{ is compact.}$$

(ii) Upper bound. For each closed set $F \subset \mathscr{X}$,

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \ln \mathbb{P}^\varepsilon(F) \le -\inf_{x\in F} I(x).$$

(iii) Lower bound. For each open set $G \subset \mathscr{X}$,

$$\varliminf_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}^\varepsilon(G) \geq -\inf_{x \in G} I(x).$$

Theorem A.3 (Varadhan's lemma)**.** *Suppose that $\mathbb{P}^\varepsilon$ satisfies the large deviation principle with rate functional $I(\cdot)$ and $F \in C_b(\mathscr{X})$. Then*

$$\lim_{\varepsilon \to 0} \varepsilon \ln \int_{\mathscr{X}} \exp\left(\frac{1}{\varepsilon} F(x)\right) \mathbb{P}^\varepsilon(dx) = \sup_{x \in \mathscr{X}} (F(x) - I(x)). \tag{A.2}$$

The proof of Varadhan's lemma can be found in [**DZ98**, **Var84**].

B. Gronwall's Inequality

Theorem B.1 (Gronwall's inequality)**.** *Assume that the function $f : [0, \infty) \to \mathbb{R}^+$ satisfies the inequality*

$$f(t) \leq a(t) + \int_0^t b(s) f(s) ds,$$

where $a(t), b(t) \geq 0$. Then we have

$$f(t) \leq a(t) + \int_0^t a(s) b(s) \exp\left(\int_s^t b(u) du\right) ds. \tag{B.1}$$

Proof. Let $g(t) = \int_0^t b(s) f(s) ds$. We have

$$g'(t) \leq a(t) b(t) + b(t) g(t).$$

Define $h(t) = g(t) \exp(-\int_0^t b(s) ds)$. We obtain

$$h'(t) \leq a(t) b(t) \exp\left(-\int_0^t b(s) ds\right).$$

Integrating both sides from 0 to t, we get

$$g(t) \leq \int_0^t a(s) b(s) \exp\left(\int_s^t b(u) du\right) ds,$$

which yields the desired estimate. □

In the case when $a(t) \equiv a$ and $b(t) \equiv b$, we have

$$f(t) \leq a \exp(bt).$$

Theorem B.2 (Discrete Gronwall's inequality)**.** *Assume that F_n satisfies*

$$F_{n+1} \leq (1 + b_n \delta t) F_n + a_n, \quad F_0 \geq 0, \tag{B.2}$$

where $\delta t, a_n, b_n \geq 0$. Then we have

$$F_n \leq \exp\left(\left(\sum_{k=0}^{n-1} b_k\right) \delta t\right) F_0 + \sum_{k=0}^{n-1} \left(a_k \exp\left(\left(\sum_{l=k+1}^{n-1} b_l\right) \delta t\right)\right). \tag{B.3}$$

Proof. From (B.2), we have

$$F_n \le \prod_{k=0}^{n-1}(1+b_k\delta t)F_0 + \sum_{k=0}^{n-1}\left(a_k \sum_{l=k+1}^{n-1}(1+b_l\delta t)\right).$$

The estimate (B.2) follows by a straightforward application of the inequality $1+x \le e^x$. □

When $F_0 \le C\delta t^p$, $b_n \equiv b$, $a_n = K\delta t^{p+1}$, and $n\delta t \le T$, we have

$$F_n \le Ce^{bT}\delta t^p + Ke^{bT}\frac{\delta t^{p+1}}{1-e^{-b\delta t}}.$$

This is the commonly used pth-order error estimate in numerical analysis.

C. Measure and Integration

Let $(\Omega, \mathcal{F})$ be a measurable space.

Definition C.1 (Measure)**.** The measure $\mu : \mathcal{F} \to \bar{\mathbb{R}}_+ = [0, \infty]$ is a set function defined on $\mathcal{F}$ that satisfies

(a) $\mu(\emptyset) = 0$;

(b) the countable additivity; i.e., for pairwise disjoint sets $A_n \in \mathcal{F}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty}\mu(A_n),$$

where we assume the arithmetic rules (2.7) on the extended reals $\bar{\mathbb{R}}_+$.

When the set function μ takes values in $\bar{\mathbb{R}} = [-\infty, \infty]$ and only the countable additivity condition is assumed, μ is called a *signed measure*.

Definition C.2 (Algebra)**.** An algebra (or field) $\mathcal{F}_0$ is a collection of subsets of Ω that satisfies the following conditions:

(i) $\Omega \in \mathcal{F}_0$;

(ii) if $A \in \mathcal{F}_0$, then $A^c \in \mathcal{F}_0$;

(iii) if $A, B \in \mathcal{F}_0$, then $A \cup B \in \mathcal{F}_0$.

Theorem C.3 (Measure extension)**.** *A finite measure μ on an algebra $\mathcal{F}_0 \subset \mathcal{F}$, i.e., $\mu(\Omega) < \infty$, can be uniquely extended to a measure on $\sigma(\mathcal{F}_0)$.*

Definition C.4 (Measurable function)**.** A function $f : \Omega \to \mathbb{R}$ is called *measurable* or *$\mathcal{F}$-measurable* if $f^{-1}(A) \in \mathcal{F}$ for any $A \in \mathcal{R}$.

Definition C.5 (Simple function)**.** A function $f : \Omega \to \mathbb{R}$ is called a simple function if it has the representation

$$f(\omega) = \sum_{i=1}^{n} a_i \chi_{A_i}(\omega),$$

where $a_i \in \mathbb{R}$ and $A_i \in \mathcal{F}$ for $i = 1, \dots, n$.

Theorem C.6. *Any nonnegative measurable function f on space $(\Omega, \mathcal{F})$ can be approximated by a sequence of monotonically increasing nonnegative functions $\{f_n\}$; that is, $0 \le f_n(\omega) \le f_{n+1}(\omega)$ for any n and*

$$\lim_{n\to\infty} f_n(\omega) = f(\omega).$$

We will denote such a monotone approximation as $f_n \uparrow f$ for short.

Definition C.7 (Integration of a simple function)**.** The integral of the simple function $f(\omega) = \sum_{i=1}^{n} a_i \chi_{A_i}(\omega)$ is defined as

$$\mu(f) = \int_\Omega f \mu(d\omega) = \sum_{i=1}^{n} a_i \mu(A_i).$$

Theorem C.8 (Properties of the integral of simple functions)**.** *Suppose that f_n, g_n, f, and g are nonnegative simple functions. Then we have:*

(a) $\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g)$ *for any* $\alpha, \beta \in \mathbb{R}_+$.

(b) *If* $f \le g$, *then* $\mu(f) \le \mu(g)$.

(c) *If* $f_n \uparrow f$, *then* $\lim_{n\to\infty} \mu(f_n) = \mu(f)$.

(d) *If* f_n *and* g_n *are monotonically increasing and* $\lim_{n\to\infty} f_n \le \lim_{n\to\infty} g_n$, *then* $\lim_{n\to\infty} \mu(f_n) \le \lim_{n\to\infty} \mu(g_n)$.

Definition C.9. Let f be a nonnegative measurable function. The integral of f is defined as

$$\mu(f) = \int_\Omega f(\omega)\mu(d\omega) = \lim_{n\to\infty} \mu(f_n),$$

where $f_n \uparrow f$ are nonnegative functions.

It is easy to see that the integral is well-defined, say using Theorem C.8(d).

Definition C.10. Let f be a measurable function. The integral of f is defined as

$$\mu(f) = \int_\Omega f(\omega)\mu(d\omega) = \mu(f^+) - \mu(f^-),$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ are both nonnegative measurable functions. If both $\mu(f^+)$ and $\mu(f^-)$ are finite, f is called an integrable function.

Theorem C.11 (Monotone convergence theorem). *Suppose that* $\{f_n\}$ *are nonnegative integrable functions and* $f_n \uparrow f$ *almost everywhere. Then*

$$\lim_{n\to\infty} \mu(f_n) = \mu(f).$$

Theorem C.12 (Fatou lemma). *Let* $\{f_n\}$ *be nonnegative integrable functions. We have*

$$\mu(\liminf_{n\to\infty} f_n) \leq \liminf_{n\to\infty} \mu(f_n).$$

Theorem C.13 (Dominated convergence theorem). *Suppose that* $\{f_n\}$ *are integrable functions and* $f_n \to f$ *almost everywhere. If* $|f_n| \leq g$ *for any* n *and* $\mu(g) < \infty$, *then*

$$\lim_{n\to\infty} \mu(f_n) = \mu(f).$$

Definition C.14 (σ-finite measure). A measure μ on $(\Omega, \mathcal{F})$ is σ-finite if there exists a countable partition of Ω; i.e., $\Omega = \bigcup_{n=1}^{\infty} A_n$ where $\{A_n\}$ are pairwise disjoint, such that $\mu(A_n) < \infty$ for any n.

Definition C.15 (Absolute continuity). Let μ, η be a σ-finite measure and a signed measure, respectively. Here η is called absolutely continuous with respect to μ if $\mu(A) = 0$ implies $\eta(A) = 0$ for any $A \in \mathcal{F}$. It is also denoted as $\eta \ll \mu$ for short.

Theorem C.16 (Radon-Nikodym theorem). *Let* μ *be a* σ*-finite measure on* $(\Omega, \mathcal{F})$ *and let* η *be a signed measure which is absolutely continuous with respect to* μ. *Then there exists a measurable function* f *such that*

$$\eta(A) = \int_A f(\omega)\mu(d\omega)$$

for any $A \in \mathcal{F}$. *Here* f *is unique in the* μ*-equivalent sense; i.e.,* $f \overset{\mu}{\sim} g$ *if* $\mu(f = g) = 1$. *It is also called the Radon-Nikodym derivative of* η *with respect to* μ, *abbreviated as* $f = d\eta/d\mu$.

The readers may refer to [**Bil79**, **Cin11**, **Hal50**] for more details.

D. Martingales

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ or $\{\mathcal{F}_t\}_{t\geq 0}$.

Definition D.1 (Martingale). A continuous time stochastic process X_t is called an $\mathcal{F}_t$-martingale if $X_t \in L^1_\omega$ for any t, X_t is $\mathcal{F}_t$-adapted, and

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s \qquad \text{for all } s \leq t. \tag{D.1}$$

X_t is called a submartingale or supermartigale if the above equality is replaced by $\geq$ or $\leq$. These concepts can be defined for discrete time stochastic processes if we replace (D.1) by $\mathbb{E}(X_n|\mathcal{F}_m) = X_m$ for any $m \leq n$. The discrete submartingale or supermartingale can be defined similarly.

The following theorem is a straightforward application of the conditional Jensen inequality

Theorem D.2. *Assume ϕ is a convex function such that $\phi(X_t) \in L^1_\omega$ for any t. Then $\phi(X_t)$ is a submartingale.*

The simplest choices of ϕ include $\phi(x) = |x|$, $x^+ := x \vee 0$, or x^2 if X_t is a square-integrable martingale.

Theorem D.3 (Martingale inequalities)**.** *Let X_t be a submartingale with continuous paths. Then for any $[s,t] \subset [0,\infty)$ and $\lambda > 0$, we have:*

(i) *Doob's inequality:*

$$\mathbb{P}\left(\sup_{u\in[s,t]} X_u \geq \lambda\right) \leq \frac{\mathbb{E}X_t^+}{\lambda}.$$

(ii) *Doob's L^p-maximal inequality:*

$$\mathbb{E}\left(\sup_{u\in[s,t]} X_u\right)^p \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}X_t^p, \qquad p > 1.$$

Useful results follow immediately if we take $X_t = |Y_t|$ where Y_t is a martingale. Similar inequalities also hold for discrete time martingales. See [**Chu01**, **Dur10**, **KS91**] for more details.

E. Strong Markov Property

Consider a finite Markov chain $\{X_n\}_{n\in\mathbb{N}}$ on S with initial distribution μ and transition probability matrix $\boldsymbol{P}$.

Theorem E.1 (Markov property)**.** *Conditional on $X_m = i$ ($m \in \mathbb{N}$), $\{X_{m+n}\}_{n\in\mathbb{N}}$ is Markovian with initial distribution δ_i and transition probability matrix $\boldsymbol{P}$, and it is independent of $(X_0, X_1, \ldots, X_m)$.*

Theorem E.2 (Strong Markov property)**.** *Let N be a stopping time. Conditional on $\{N < \infty\}$ and $X_N = i$, $\{X_{N+n}\}_{n\in\mathbb{N}}$ is Markovian with initial distribution δ_i and transition probability matrix $\boldsymbol{P}$.*

The above results also hold for the Q-process $\{X_t\}_{t\geq 0}$ with generator $\boldsymbol{Q}$.

Theorem E.3 (Markov property)**.** *Conditional on $X_t = i$ ($t \geq 0$), $\{X_{t+s}\}_{s\geq 0}$ is Markovian with initial distribution δ_i and generator $\boldsymbol{Q}$, and it is independent of $\{X_r, r \leq t\}$.*

Theorem E.4 (Strong Markov property)**.** *Let T be a stopping time. Conditional on $\{T < \infty\}$ and $X_T = i$, $\{X_{T+t}\}_{t\geq 0}$ is Markovian with initial distribution δ_i and generator $\boldsymbol{Q}$.*

See [**Dur10**, **Nor97**] for more details.

F. Semigroup of Operators

Let $\mathscr{B}$ be a Banach space equipped with the norm $\|\cdot\|$.

Definition F.1 (Operator semigroup)**.** A family of bounded linear operators $\{S(t)\}_{t\geq 0} : \mathscr{B} \to \mathscr{B}$ forms a strongly continuous semigroup if for any $f \in \mathscr{B}$:

(i) $S(0)f = f$; i.e., $S(0) = I$.

(ii) $S(t)S(s)f = S(s)S(t)f = S(t+s)f$ for any $s, t \geq 0$.

(iii) $\|S(t)f - f\| \to 0$ for any $f \in \mathscr{B}$ as $t \to 0+$.

We will call $\{S(\cdot)\}$ a contraction semigroup if $\|S(t)\| \leq 1$ for any t, where $\|S(t)\|$ is the operator norm induced by the metric $\|\cdot\|$.

The simplest examples of semigroups include the solution operator for the system

$$\frac{d\boldsymbol{u}(t)}{dt} = \boldsymbol{A}\boldsymbol{u}(t),$$

where $\boldsymbol{u} \in \mathbb{R}^n, \boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\mathscr{B} = \mathbb{R}^n$, and $S(t)\boldsymbol{f} := \boldsymbol{u}(t)$ by solving the ODE with initial condition $\boldsymbol{u}(t)|_{t=0} = \boldsymbol{f}$. Similarly, consider the PDE

$$\partial_t u = \Delta u \quad \text{in } U, \qquad u = 0 \ \text{ on } \partial U,$$

where U is a bounded open set with smooth boundary. We can take $\mathscr{B} = L^2(U)$ and $S(t)f := u(t)$, where $u(t)$ is the solution of the above PDE with initial condition $u(x, t = 0) = f$.

Definition F.2 (Infinitesimal generator)**.** Denote

$$D(\mathcal{A}) = \left\{ f \in \mathscr{B} : \lim_{t\to 0+} \frac{S(t)f - f}{t} \text{ exits in } \mathscr{B} \right\}$$

and

$$\mathcal{A}f := \lim_{t\to 0+} \frac{1}{t}(S(t)f - f), \qquad f \in D(\mathcal{A}).$$

The operator $\mathcal{A}$ is called the infinitesimal generator of the semigroup $S(t)$; $D(\mathcal{A})$ is the domain of the operator $\mathcal{A}$.

It can be shown that the infinitesimal generators of the two examples above are $\mathcal{A} = \boldsymbol{A}$ and $\mathcal{A} = \Delta$ and their domains are $\mathbb{R}^n$ and the Sobolev space $H_0^1(U) \cup H^2(U)$, respectively.

Theorem F.3 (Basic properties)**.** *Let $f \in D(\mathcal{A})$. We have:*

(i) $S(t)f \in D(\mathcal{A})$ *and* $S(t)\mathcal{A}f = \mathcal{A}S(t)f$ *for any* $t \geq 0$.

(ii) $f(t) := S(t)f$ *is differentiable on* $(0, \infty)$ *and*

$$\frac{df(t)}{dt} = \mathcal{A}f(t), \qquad t > 0.$$

Theorem F.4. *The generator $\mathcal{A}$ is a closed operator and $D(\mathcal{A})$ is dense in $\mathscr{B}$.*

In general, the generator $\mathcal{A}$ is unbounded, e.g., $\mathcal{A} = \Delta$ in the previous PDE example, so we only have $D(\mathcal{A}) \subsetneq \mathscr{B}$.

Definition F.5 (Resolvent set and operator)**.** Let $\mathcal{A}$ be a closed linear operator with domain $D(\mathcal{A})$. The resolvent set of $\mathcal{A}$ is defined by

$$\rho(\mathcal{A}) = \{\lambda : \lambda \in \mathbb{R} \text{ and } \lambda I - \mathcal{A} \text{ is bijective from } D(\mathcal{A}) \text{ to } \mathscr{B}\}.$$

If $\lambda \in \rho(\mathcal{A})$, the resolvent operator $R_\lambda : \mathscr{B} \to \mathscr{B}$ is defined by

$$R_\lambda f := (\lambda I - \mathcal{A})^{-1} f.$$

The closed graph theorem ensures that $(\lambda I - \mathcal{A})^{-1}$ is a bounded linear operator if $\lambda \in \rho(\mathcal{A})$.

Theorem F.6 (Hille-Yosida theorem)**.** *Let $\mathcal{A}$ be a closed, densely defined linear operator on $\mathscr{B}$. Then $\mathcal{A}$ generates a contraction semigroup $\{S(t)\}_{t\geq 0}$ if and only if*

$$\lambda \in \rho(\mathcal{A}) \qquad \textit{and} \qquad \|R_\lambda\| \leq \frac{1}{\lambda} \qquad \textit{for all } \lambda > 0.$$

Interested readers are referred to [**Eva10**, **Paz83**, **Yos95**] for more details on semigroup theory.

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