GRADUATE STUDIES IN MATHEMATICS 205

Invitation to Partial Differential Equations

Mikhail Shubin

EDITED BY
Maxim Braverman
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AMS American Mathematical Society

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Foreword

Misha Shubin, our dear colleague at Northeastern University, worked for many years trying to finish this book on partial differential equations. Unfortunately, his failing health prevented him from completing this task. Sergei Gelfand of the American Mathematical Society approached us asking whether we could put the finishing touches on the manuscript so that it could be published. We agreed to do this hoping to preserve Misha's writing style which is at the same time rigorous yet accessible. Many chapters in the book required only minor changes; others required substantial revision and additions because they had not been completed before his health deteriorated. It has taken us quite a while to finish this project, but we now feel that the book is in sufficiently good condition that it can be shared with the mathematical community. We hope that it will be appreciated as a lasting tribute to the legacy of Professor Mikhail Shubin.

Maxim Braverman
Robert McOwen
Peter Topalov

Boston, November 2019

Preface

I shall be telling this with a sigh
Somewhere ages and ages hence:
Two roads diverged in a wood, and I –
I took the one less traveled by,
And that has made all the difference.

From *The Road Not Taken*
by Robert Frost

It will not be an overstatement to say that everything we see around us (as well as hear, touch, etc.) can be described by partial differential equations or, in a slightly more restricted way, by equations of mathematical physics. Among the processes and phenomena whose adequate models are based on such equations, we encounter, for example, string vibrations, waves on the surface of water, sound, electromagnetic waves (in particular, light), propagation of heat and diffusion, gravitational attraction of planets and galaxies, behavior of electrons in atoms and molecules, and also the Big Bang that led to the creation of our Universe.

It is no wonder, therefore, that today the theory of partial differential equations (PDE) forms a vast branch of mathematics and mathematical physics that uses methods of all the remaining parts of mathematics (from which PDE theory is inseparable). In turn, PDE influence numerous parts of mathematics, physics, chemistry, biology, and other sciences.

A disappointing conclusion is that no book, to say nothing of a textbook, can be sufficiently complete in describing this area. Only one possibility remains: to show the reader pictures at an exhibition—several typical methods and problems—chosen according to the author's taste. In doing so, the author has to severely restrict the choice of material.

This book originally contained a practically intact transcript of the PDE course which I taught twice to students in experimental groups at the Department of Mechanics and Mathematics of the Moscow State University and later, in a shortened version, in Northeastern University (Boston). Since then several other people taught this course in other universities (in particular, I. S. Louhivaara was the first to teach it outside Moscow State University—namely, at Freie Universität (Berlin). Now the transcript has been considerably extended, becoming practically a new book.

In Moscow, all problems presented in this book were used in recitations accompanying the lectures. There was one lecture per week during the whole academic year (two semesters); recitations were held once a week during the first semester and twice a week during the second one.

My intention was to create a course that is both concise and modern. These two requirements proved to be contradictory. To make the course concise, I was, first, forced to explain some ideas on simplest examples and, second, to omit many topics which should be included in a modern course but would have expanded the course beyond a reasonable length. With particular regret I have omitted the Schrödinger equation and related material. I hope the gap will be filled by a parallel course on quantum mechanics.

In any modern course on equations of mathematical physics, something should, perhaps, be said about nonlinear equations but I found it extremely difficult to select this "something".

The bibliography at the end of the book contains a number of textbooks where the reader can find information complementing this course. Naturally, the material of theses books overlaps, but it was difficult to do a more precise selection, so I leave this selection to the reader's taste.

The problems in the book are not just exercises. Some of them add essential information but require no new ideas to solve them.

Acknowledgments. I am very grateful to Professor S. P. Novikov, who organized the experimental groups and invited me to teach PDE to them. I am also greatly indebted to the professors of the chair of differential equations of the Moscow State University for many valuable discussions.

I owe a lot to Professor Louhivaara, who provided unexpectedly generous help in editing an early version of this text and encouraged me to publish this book. I also had very productive discussions about this course with Professor P. Albin, which led to extremely many useful improvements and corrections. Professors R. Mazzeo and I. Polterovich taught the course using the lecture notes, and I owe them thanks for many useful suggestions.

I am thankful to Professor O. Milatovic for his help in verifying problems.

I am also grateful to Professor D. A. Leites, who participated considerably in the translation of this book into English.

Selected notational conventions

Cross references inside the book are natural: Theorem 6.2 stands for Theorem 2 in Chapter 6, Problem 6.2 is Problem 2 from Chapter 6, etc.

$:=$ means "denotes by definition".

$\equiv$ means "identically equal to".

By $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ we will denote the set of all integers, nonnegative integers, positive integers, real numbers, and complex numbers, respectively.

I denotes the identity operator.

The *Heaviside function* is $\theta(z) = \begin{cases} 1 & \text{for } z \geq 0, \\ 0 & \text{for } z < 0. \end{cases}$

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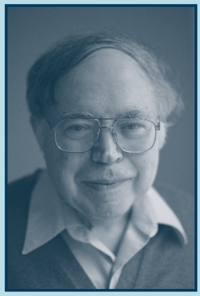
This book is based on notes from a beginning graduate course on partial differential equations. Prerequisites for using the book are a solid undergraduate course in real analysis. There are more than 100 exercises in the book. Some of them are just exercises, whereas others, even though they may require new ideas to solve them, provide additional important information about the subject.



It is a great pleasure to see this book—written by a great master of the subject—finally in print. This treatment of a core part of mathematics and its applications offers the student both a solid foundation in basic calculations techniques in the subject, as well as a basic introduction to the more general machinery, e.g., distributions, Sobolev spaces, etc., which are such a key part of any modern treatment. As such this book is ideal for more advanced undergraduates as well as mathematically inclined students from engineering or the natural sciences. Shubin has a lovely intuitive writing style which provides a gentle introduction to this beautiful subject. Many good exercises (and solutions) are provided!

—Rafe Mazzeo, Stanford University

This text provides an excellent semester's introduction to classical and modern topics in linear PDE, suitable for students with a background in advanced calculus and Lebesgue integration. The author intersperses treatments of the Laplace, heat, and wave equations with developments of various functional analytic tools, particularly distribution theory and spectral theory, introducing key concepts while deftly avoiding heavy technicalities.

—Michael Taylor, University of North Carolina, Chapel Hill

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