

GRADUATE STUDIES
IN MATHEMATICS **207**

**Organized Collapse:
An Introduction to
Discrete Morse Theory**

Dmitry N. Kozlov



AMERICAN
MATHEMATICAL
SOCIETY

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Providence, Rhode Island

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Dedicated to David, Esther, Judith, and Eva-Maria

蜘蛛殺すあとの淋しき夜寒哉

正岡 子規 (1867-1902)

After killing
a spider, how lonely I feel
in the cold of night!

Masaoka Shiki (1867-1902)

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Preamble

Applied topology is a modern mathematical subject which appeared in recent years as a concentration point for many methods, topological in nature, which were used in a wide variety of applications, both within and outside of classical mathematics. While displaying an amazing breadth, these methods have at least one thing in common—the emphasis on the computational aspect.

Clearly, whenever computations are involved, there arises the question of their complexity. No application will be truly useful if it results in a computationally intractable problem. Accordingly, the simplification of the required computations takes center stage. Whether within a topological, combinatorial, or algebraic context, we need to be able both to define single elementary simplifications, as well as to manage their sequences, telling us for instance which sequence is permissible, and also what the final result will be of applying a certain sequence of simplifications.

In topological context, the tool of choice for handling these considerations came to be popularly known as *discrete Morse theory*, and this book grew out of the author's thoughts about the role discrete Morse theory should play within the wider context of applied topology.

Although some of the viewpoints and even some of the results presented here are new, this is not a research manuscript. When writing this book, we tried to produce a text which would be useful for a beginning graduate, as well as advanced undergraduate students. Due to the large differences in the master studies curricula in various educational systems, we have tried to keep the presentation self-contained.

As a result, we have included a brief introduction to the basics of algebraic topology. So, while our target audience consists of graduate students in applied topology, the book can also be used by anyone looking for a first textbook treatment of homology. Furthermore, we intended the book to be of interest to students and specialists in computer science and engineering who would like to learn about applications of topology to their fields, as well as for research mathematicians interested in learning about the subject and applying it in their context.

During the work on this book the author was supported by the University of Bremen, Germany, as well as by the Okinawa Institute of Science and Technology, Japan. He expresses his deepest gratitude to both of these wonderful institutions.

Herbert Edelsbrunner has been both an initiator and a strong supporter of this undertaking. Without him this book would probably never have been started.

Once it was, writing it was a long and excruciating process. During this time, the author had the great benefit of experiencing constant and friendly encouragement of the AMS Editor Ina Mette. This support is recognized and very much appreciated.

After a text reaches a certain size, it starts having a life of its own. It is only with the help of careful proofreaders that the author may proceed beyond a certain point. Here, the author would especially like to mention his graduate student Leonard Wienke, whose notational and typographic suggestions have been most helpful.

Last, but not least, I would like to express my gratitude to my family, to whom this book is dedicated. Without them the book would either have been finished much sooner, or not finished at all.

Preface

The idea of homology

The main idea of algebraic topology is to try to use *algebraic structures* to say something qualitative about *topological spaces*. Over time one developed many such algebraic invariants. Perhaps the simplest one to define is the so-called *fundamental group* of a space. Most probably the reader has seen its definition in one form or another. Roughly speaking, one chooses a point in space and then defines some calculus using all possible loops anchored at this point. A clear weakness of such an invariant is that it does not tell us anything beyond the first few dimensions: taking any space and then attaching balls of dimension 3 and higher to that space will not be detected by the fundamental group at all. That in itself can be fixed by introducing higher-dimensional homotopy groups. What is much worse, from the point of view of applied topology, is that not only are these invariants hard to compute, but the famous result of P.S. Novikov, [No55], actually tells us that it is not decidable whether or not the fundamental group is trivial.

All these problems are solved if one passes to the so-called *homology groups*. These are defined in all dimensions and, if the framework is right, can be computed using linear algebra.

Before talking about invariants, though, let us contemplate for a moment how topological spaces can be described. Many classic examples, including curves, surfaces, or more generally manifolds, are given by their defining equations in Euclidean spaces. While useful in many other fields of mathematics, such as differential geometry, as well as in physics, this will give us

only a limited supply of spaces. Furthermore, the description using equations is indirect, making a constructive computation of algebraic invariants rather daunting.

An alternative approach is combinatorial in nature. Instead of using geometry as our guidance, we take some elementary building blocks, the so-called *simplices*, and then glue them together to produce a topological space, which we then call an abstract simplicial complex. This combinatorial gluing scheme then takes the role of the space description, and the actual topological space can be recovered from it, uniquely up to homeomorphism, using the so-called geometric realization construction.

In the opposite direction, the bridge from the continuous to the discrete is provided by the concept of triangulation. Although there is a number of underwater stones here, in essence well-behaved spaces can be triangulated so that an abstract simplicial complex can be produced. The main problem is that a triangulation, as opposed to the geometric realization, is in no way unique, and it is not feasible to combinatorially define a sensible equivalence relation on the set of simplicial complexes such that any two triangulations of the same topological space are equivalent. This conundrum can eventually be resolved, though not without using the technical tool of *simplicial approximation*.

One way to bypass these difficulties altogether is to take the idea of a combinatorial gluing scheme one step further, into the realm of abstract algebra. The sets of simplices are replaced by vector spaces or, more generally, by free abelian groups, and the gluing information gets baked into a linear map, the so-called *boundary operator*, leading to the concept of a *chain complex*. The resulting inter-relations between topology, combinatorics, and algebra are shown in Figure 0.1.

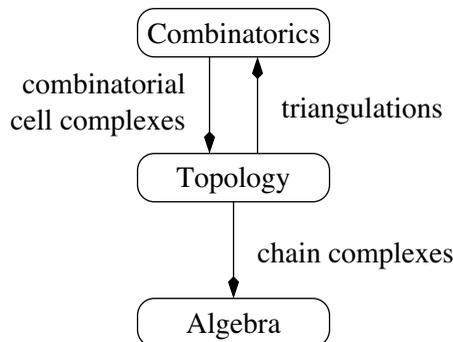


Figure 0.1. Inter-relations between topology, combinatorics, and algebra.

The idea of discrete Morse theory

In general, given two topological spaces, it is hard to tell whether they are homotopy equivalent. The intuitive picture of allowing stretchings of the space results in the formal concept of a *strong deformation retraction*, and it is possible to take it as a basis for an equivalence relation. Unfortunately, this is rather impractical, and completely useless from the computational point of view.

One is therefore sorely tempted to introduce such deformations in the combinatorial context of simplicial complexes. This leads to the so-called *simplicial collapses*, and their theory, known as *simple homotopy theory*, which was ingeniously developed by J.H.C. Whitehead in the 30s, [Co73, Wh50]. Unfortunately (or perhaps fortunately), it turned out that even when two topological spaces can be deformed to each other, in the desired way, the corresponding combinatorial deformation may not exist. In fact, there is a possible obstruction to the existence of such a deformation, the so-called *Whitehead torsion*, which can be found in the so-called *Whitehead group* of the appropriate fundamental group. On one hand, this leads to an interesting and subtle theory of combinatorial torsions. On the other, these “difficulties” led to the fact that the combinatorial track of development in topology was abandoned, to the advantage of the purely algebraic one.

In recent years, there arose many situations where the revival of combinatorial thinking could be of use. Many topological spaces are constructed not by starting from geometry, but rather directly as abstract simplicial complexes. Oftentimes, these are artificial objects which are associated to various situations in combinatorics, or other fields, constructed for the sole purpose of using the power of topology, where there was no topology to start with. One particular field, which has lately received a rather substantial development boost, is *applied topology*. Here simplicial complexes play an important role in many applied contexts, from data analysis to robotics. Just as in the combinatorial context, there is a renewed and natural desire to return to the theory of combinatorial deformations.

This is where *discrete Morse theory* comes in. This occurs on two levels. First, we would like to go beyond the usual simplicial collapses and to allow the sort of deformations which could be thought of as *internal collapses*. These will still yield homotopy equivalences, but will no longer preserve simplicial structure, producing more complicated gluing maps. Second, once we resign ourselves to perform sequences of such deformations, it is imperative to give conditions under which sequences are allowed, the so-called *acyclicity condition*, as well as to learn to do good book-keeping for collapsing sequences.

Once these objectives are achieved, we will have on our hands a theory which is very effective from the computational point of view. It can then be combined in a mutually profitable way with other tools of applied topology and combinatorics, see Table 0.1.

Field	Deformation
Topology	strong deformation retraction
Combinatorics	simplicial collapse
Algebra	change of basis

Table 0.1. The many faces of discrete Morse theory.

A sample application

To give an illustration of how discrete Morse theory works, let us consider the simplicial complex in Figure 0.2.

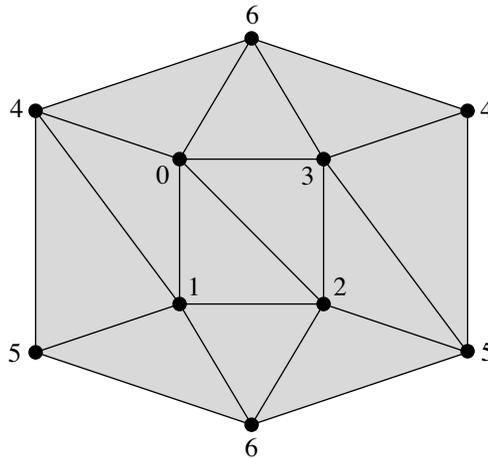


Figure 0.2. A triangulation of a pinched torus.

In this figure each set of vertices may span at most one simplex, so simplices with the same sets of vertices are to be identified. It is easy to see that we have a triangulation of a sphere with north and south poles identified, producing vertex 6. Alternatively, this can be viewed as the so-called *pinched torus*, obtained from a usual torus by shrinking an essential circle to a point.

In discrete Morse theory, we want to consider a sort of discrete flow, illustrated in Figure 0.3 by adding a number of arrows. This family of arrows

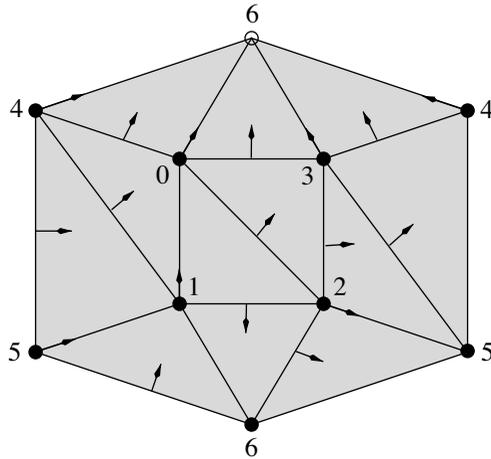


Figure 0.3. Acyclic simplex matching.

must satisfy an additional property—the acyclicity condition, which we will define later.

Later in the book, we will see that a fruitful way to work with such a discrete flow is to view it as a matching on the set of simplices, and to define various notions in terms of emerging combinatorics.

For example, a careful reader will see that the only simplices which are not matched in Figure 0.3 are the vertex 6, the edge 16, and the triangle 012. Discrete Morse theory (or, more specifically, Theorem 11.2) will allow us to immediately conclude that (the geometric realization of) our simplicial complex is homotopy equivalent to another space which is obtained by gluing a 2-disc onto a circle along some function on the boundary of that 2-disc.

This is of course, by itself, a great simplification of the initial space, and this is precisely how discrete Morse theory is often used. Since there are several functions along which the gluing may occur, there is more work needed for the complete understanding of the space. Still, this can be done by using a few more advanced topics of our theory, for example, by studying the associated Morse chain complex, and coupling this with the fact that the maps between circles are classified, up to homotopy, by their winding numbers.

In the end one can conclude that the topological space associated to our simplicial complex is homotopy equivalent to a circle and a 2-sphere, glued together at a single point; the standard “name” of this space is $S^2 \vee S^1$.

How to use this book

When writing this book, we have envisioned a number of various uses. To start with, it can certainly be adopted as a first text in algebraic topology, specifically as an introduction to homology. That being said, the author assumes the knowledge of fundamental groups, as would be the case at the University of Bremen, where fundamental groups are included in the undergraduate topology course. In a different system this has to be adjusted by either adding a crash course on the fundamental group, or skipping the topics which involve it. Parts 1 and 2 of this book may be taken to form a backbone of such an introductory course in homology theory.

Another target audience for this book consists of those readers who are interested in combinatorial topology, and specifically in the role played by discrete Morse theory. Such a reader should concentrate on Part 3, where the majority of examples are drawn from the vast pool of combinatorially defined simplicial, but not only simplicial, complexes. Many results of combinatorial topology are recovered there from the point of view of discrete Morse theory. The previous book by this author, [Ko08], may also be of benefit as a reference material here.

Finally, the author would be honoured if this text was found of interest by the dynamic and fast-growing community in applied topology. Certainly, a number of shortcuts can be done by an applied topologist interested only in simplicial or cellular models. A different, more computational, approach would essentially allow skipping Part 2, and also some of the examples. On the other hand, the algorithmic procedure from Chapter 12 for producing explicit homology generators from acyclic matchings would be of great interest, as would be Part 4, including the short Chapter 18 describing a relation to persistent homology.

Here is the summary of the contents of the four parts.

Part 1: Introduction to homology. We adopt the simplicial approach and gear the introduction towards advanced undergraduate or graduate students in mathematics, looking for their first exposure to homology theory, or computer science and engineering graduate students interested in applications of topology to their fields.

Part 2: More advanced topics in homology theory, including chain homotopy, long exact sequences, singular homology, and cellular homology. While invaluable for anyone learning algebraic topology, it may be skipped if applied topology is the primary focus.

Part 3: Basic discrete Morse theory. Graduate level introduction to the subject of discrete Morse theory. Formulation and proof of all the basic results which have so far been most useful. The emphasis here is on the

approach via simple homotopy theory, using collapses and acyclic matchings. In the end a connection to the historical framework using discrete Morse functions and discrete vector fields is made.

Part 4: Collection of topics from advanced discrete Morse theory. Algebraic Morse theory and the change of bases. Discrete Morse theory via poset maps with small fibers. Connections to persistent homology.

Prerequisites

As prerequisites, we expect that the reader is familiar with linear algebra, group theory, and point-set topology. In particular, we expect the knowledge of the quotient group and the classification theorem for finitely generated abelian groups. We do not require detailed knowledge of finite fields, however we expect familiarity with the field with 2 elements, which we denote \mathbb{Z}_2 , and vector spaces over that field.

Beyond group theory, we assume that the reader is familiar with parts of abstract algebra, including rings, modules, and tensor products, and that he has basic knowledge of the concepts from category theory.

Finally, the knowledge of the fundamental group would come in handy as the guiding principle for some of the intuition; that being said, it is not a strict prerequisite for most of the book.

Guide to the literature

Discrete Morse theory was introduced by Robin Forman in his seminal paper [Fo98]. Since then the subject has been treated in the textbook form by Knudson, [Kn15], and more recently by Scoville, [Sc19]. There have also been chapter-long treatments by Forman, [Fo02a], and by the author, [Ko08, Chapter 11]. Our approach here is closest to the last reference.

Furthermore, a large number of research articles related to this subject have appeared. This book consists of four parts, and we have decided to provide the reader with references and suggestions for further reading at the end of Parts 2, 3, and 4.

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