

GRADUATE STUDIES
IN MATHEMATICS **210**

Combinatorics: The Art of Counting

Bruce E. Sagan



AMERICAN
MATHEMATICAL
SOCIETY

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To Sally, for her love and support

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Preface

Enumerative combinatorics has seen an explosive growth over the last 50 years. The purpose of this text is to give a gentle introduction to this exciting area of research. So, rather than trying to cover many different topics, I have chosen to give a more leisurely treatment of some of the highlights of the field. My goal has been to write the exposition so it could be read by a student at the advanced undergraduate or beginning graduate level, either as part of a course or for independent study. The reader will find it similar in tone to my book on the symmetric group. I have tried to keep the prerequisites to a minimum, assuming only basic courses in linear and abstract algebra as background. Certain recurring themes are emphasized, for example, the existence of sum and product rules first for sets, then for ordinary generating functions, and finally in the case of exponential generating functions. I have also included some recent material from the research literature which, to my knowledge, has not appeared in book form previously, such as the theory of quotient posets and the connection between pattern avoidance and quasisymmetric functions.

Most of the exercises should be doable with a reasonable amount of effort. A few unsolved conjectures have been included among the problems in the hope that an interested student might wish to tackle one of them. They are, of course, marked as such.

A few words about the title are in order. It is in part meant to be a tip of the hat to Donald Knuth's influential series of books *The art of computer programming*, Volumes 1–3 [51–53], which, among many other things, helped give birth to the study of pattern avoidance through its connection with stack sorting; see Exercise 36 in Chapter 1. I hope that the title also conveys some of the beauty found in this area of mathematics, for example, the elegance of the Hook Formula (equation (7.10)) for the number of standard Young tableaux. In addition I should mention that, due to my own preferences, this book concentrates on the enumerative side of combinatorics and mostly ignores the important extremal and existential parts of the field. The reader interested in these areas can consult the books of Flajolet and Sedgewick [25] and of van Lint [95].

This book grew out of the lecture notes which I have compiled over years of teaching the graduate combinatorics course at Michigan State University. I would like to thank the students in these classes for all the feedback they have given me about the various topics and their presentation. I am also indebted to the following colleagues, some of whom taught from a preliminary version of this book, who provided me with suggestions as well as catching numerous typographical errors: Matthias Beck, Moussa Benoumhani, Andreas Blass, Seth Chaiken, Sylvie Corteel, Georges Grekos, Richard Hensch, Nadia Lafrenière, Duncan Levear, and Tom Zaslavsky. Darij Grinberg deserves special mention for providing copious comments and corrections as well as providing a number of interesting exercises. I also received valuable feedback from four anonymous referees. Finally, I wish to express my appreciation of Ina Mette, my editor at the American Mathematical Society. Without her gentle support and persistence, this text would never have seen the light of day. Because I typeset this document myself, all errors can be blamed on my computer.

East Lansing, Michigan, 2020

List of Notation

Symbol	Definition	Page
$A(D)$	arc set of digraph D	21
$A(G)$	adjacency matrix of graph G	60
$\mathcal{A}(G)$	set of acyclic orientations of G	103
$a(G)$	number of acyclic orientations of G	103
$A([n], k)$	set of permutations π in \mathfrak{S}_n having k descents	121
$A(n, k)$	Eulerian number, cardinality of $A([n], k)$	121
$A_n(q)$	Eulerian polynomial	122
$\mathcal{A}(P)$	atom set of poset P	169
$\text{Asc } c$	ascent set of a proper coloring c	279
$\text{asc } c$	ascent number of a proper coloring c	279
$\text{Asc } \pi$	ascent set of permutation π	76
$\text{asc } \pi$	ascent number of permutation π	76
$\text{Av}_n(\pi)$	the set of permutations in \mathfrak{S}_n avoiding π	29
α^r	reversal of composition α	32
$\tilde{\alpha}$	expansion of composition α	274
$\alpha(C)$	rank composition of chain C	275
$B(G)$	incidence matrix of graph G	61
$B(T)$	set of partitions of the set T	10
B_n	Boolean algebra on $[n]$	140
B_∞	poset of subsets of \mathbb{P}	178
$B(n)$	n th Bell number	10

Symbol	Definition	Page
\mathbb{C}	complex numbers	1
$c_i(g)$	number of cycles of length i in group element g	197
CL_n	claw poset with n atoms	169
$\text{co } T$	content of tableau T	225
C_n	cycle with n vertices	19
C_n	chain poset of length n	139
$c_x(P)$	column insertion of element x into tableau P	245
C_∞	chain poset on \mathbb{N}	178
$C(n)$	Catalan number	26
$c([n], k)$	set of permutations in \mathfrak{S}_n with k cycles	12
$c(n, k)$	signless Stirling number of the first kind	12
$c_o(L, k)$	ordered k cycle decompositions of permutations of L	127
$\mathbb{C}X$	vector space generated by set X over \mathbb{C}	248
$\mathbb{C}[x]$	polynomial algebra in x over \mathbb{C}	71
$\mathbb{C}[[x]]$	formal power series algebra in x over \mathbb{C}	81
$\mathcal{C}(\pi)$	set of functions compatible with π	236
$\mathcal{C}_m(\pi)$	set of functions compatible with π bounded by m	236
$\text{Des } P$	descent set of tableau P	271
$\text{Des } \pi$	descent set of permutation π	75
$\text{des } \pi$	descent number of permutation π	76
D_n	lattice of divisors of n	140
D_∞	divisibility poset on \mathbb{P}	181
$D(n)$	derangement number	43
$\mathcal{D}(n)$	set of Dyck paths of semilength n	26
$\mathcal{D}(V)$	set of all digraphs on vertex set V	21
$\mathcal{D}(V, k)$	set of all digraphs on vertex set V with k edges	21
$\text{deg } m$	degree of a monomial	219
$\text{deg } v$	degree of vertex v in a graph	20
$\Delta f(n)$	forward difference operator of $f(n)$	162
$\delta_{x,y}$	Kronecker delta	7
$\delta(x, z)$	delta function of poset incidence algebra	159
$E(G)$	edge set of graph G	18
$E(L)$	set structure on label set L	125
$\overline{E}(L)$	nonempty set structure on label set L	125
E_n	Euler number	120
e_n	n th elementary symmetric function	221
$E(t)$	generating function for elementary symmetric functions	221
$\text{Exc } \pi$	set of excedances of permutation π	122
$\text{exc } \pi$	number of excedances of permutation π	122

Symbol	Definition	Page
Fix f	fix point set of a function f	44
f_n	Fibonacci number	3
F_n	Fibonacci number	2
\mathbb{F}_q	Galois field with q elements	79
$f(x)$	ordinary generating function	81
$f_S(x)$	weight-generating function for weighted set S	86
$F(n)$	binomial poset n -interval factorial function	178
$F(x)$	exponential generating function	117
$F_S(x)$	exponential generating function for structure S	125
F_S	fundamental quasisymmetric for set S	269
F_α	fundamental quasisymmetric for composition α	269
f^λ	number of standard Young tableaux of shape λ	225
Φ	fundamental map on permutations	122
ϕ	bijection between subsets and compositions	16
$G \setminus e$	graph G with edge e deleted	100
G/e	graph G with edge e contracted	101
$GL(V)$	general linear group over vector space V	287
$\mathcal{G}(V)$	set of all graphs on vertex set V	20
$\mathcal{G}(V, k)$	set of all graphs on vertex set V with k edges	20
G_x	stabilizer of element x under the action of group G	191
$H_c = H_{i,j}$	hook of cell $c = (i, j)$	230
$h_c = h_{i,j}$	hooklength of cell $c = (i, j)$	230
\mathcal{H}_n	set of hook diagrams with n cells	278
h_n	n th complete homogeneous symmetric function	221
$H(t)$	complete homogeneous generating function	221
ideg v	in-degree of vertex v in a digraph	21
Inv π	inversion set of permutation π	74
inv π	inversion number of permutation π	74
$\mathcal{I}(P)$	incidence algebra of poset P	158
$I(S)$	lower-order ideal generated by S in a poset	143
ISF($G; t$)	increasing spanning forest generating function of G	105
ISF $_m(G)$	set of m -edge increasing spanning forests of G	105
isf $_m(G)$	number of m -edge increasing spanning forests of G	105
$i_\lambda(G)$	number of independent type λ partitions in graph G	254
$\mathcal{J}(P)$	distributive lattice of lower-order ideals of poset P	151
K_n	complete graph with n vertices	19
K_n	lattice of compositions of n	140
$K_{\lambda,\mu}$	number of tableaux of shape λ and content μ	225

Symbol	Definition	Page
$L(G)$	Laplacian of graph G	62
$\mathcal{L}(G)$	bond lattice of graph G	167
$\mathcal{L}(P)$	set of linear extensions of P	238
$\ell(C)$	length of chain C in a poset	147
$\ell(\lambda)$	length of an integer partition λ	15
$\ell(\pi)$	length of a permutation π	4
$\lim_{k \rightarrow \infty} f_k(x)$	limit of a sequence of formal power series	84
$\text{lds } \pi$	length of a longest decreasing subsequence of π	245
$\text{lis } \pi$	length of a longest increasing subsequence of π	244
$L_n(q)$	lattice of subspaces of \mathbb{F}_q^n	140
$L_\infty(q)$	poset of subspaces of vector space V_∞ over \mathbb{F}_q	178
$L(V)$	lattice of subspaces of V	140
$\lambda(F)$	type of partition induced by edge set F	255
$\lambda!$	multiplicity factorial of partition λ	254
$\text{maj } \pi$	major index of permutation π	76
$M(n)$	Mertens function	183
$M(P)$	monomial quasisymmetric function for poset P	275
M_α	monomial quasisymmetric function	268
m_λ	monomial symmetric function	220
$\mu(P)$	Möbius function value on a poset P	154
$\mu(x)$	one-variable Möbius function evaluated at x	154
$\mu(x, z)$	two-variable Möbius function on the interval $[x, z]$	157
\mathbb{N}	nonnegative integers	1
$\text{NBC}_k(G)$	set of no broken circuit sets of k edges of G	102
$\text{nbc}_k(G)$	number of no broken circuit sets of k edges of G	102
$\mathcal{NE}(m, n)$	set of N - E lattice paths from $(0, 0)$ to (m, n)	26
$\text{odeg } v$	out-degree of vertex v in a digraph	21
\mathcal{O}_x	orbit of an element x under action of a group	190
$O(g)$	big oh notation applied to function g	182
$o(g)$	order of a group element g	210
\mathbb{P}	positive integers	1
P^*	dual of poset P	142
$\mathcal{PC}(G)$	set of proper colorings of G with the positive integers	279
$P(G; t)$	chromatic polynomial of graph G	100
$\text{Par } P$	set of P -partitions	238
$\text{Par}_m P$	set of P -partitions bounded by m	238
P_n	path with n vertices	19
$P(n)$	set of partitions of the integer n	13
$p(n)$	number of partitions of the integer n	13
p_n	n th power sum symmetric function	221
$P(t)$	power sum symmetric generating function	221

Symbol	Definition	Page
$\mathcal{P}(n, k)$	set of partitions of n into at most k parts	15
$p(n, k)$	number of partitions of n into at most k parts	15
$P(S)$	permutations of a set S	4
$P(S, k)$	permutations of length k of a set S	4
$P((S, k))$	words of length k over a set S	5
$P(\pi)$	insertion tableau of π	242
$\mathcal{P}(u; v)$	set of directed paths from u to v in a digraph	56
Π_n	partition lattice on $[n]$	140
$\Pi(\mathcal{S})$	partition structure on structure \mathcal{S}	131
$\Pi_e(\mathcal{S})$	even partition structure on structure \mathcal{S}	133
$\Pi_o(\mathcal{S})$	odd partition structure on structure \mathcal{S}	133
\mathbb{Q}	rational numbers	1
$Q(n)$	set of compositions of the integer n	16
$q(n)$	number of compositions of the integer n	16
$Q(n, k)$	set of compositions of n into k parts	16
$q(n, k)$	number of partitions of n into k parts	16
QSym	algebra of quasisymmetric functions	268
QSym $_n$	quasisymmetric functions of degree n	268
$Q(\pi)$	recording tableau of π	242
$Q_n(\Pi)$	quasisymmetric function for patterns Π	277
\mathbb{R}	real numbers	1
$\mathcal{RC}(\pi)$	set of functions reverse compatible with π	270
rk P	rank of a ranked poset P	147
Rk $_k P$	k th rank set of a ranked poset P	147
rk x	rank of an element x in a ranked poset	147
$\mathcal{R}(k, l)$	set of partitions contained in a $k \times l$ rectangle	79
RPar P	set of reverse P -partitions	271
$\mathcal{R}(P)$	reduced incidence algebra of a binomial poset	179
rpp $_n(\lambda)$	number of shape λ reverse plane partitions of n	233
rpar($P; \mathbf{x}$)	generating function for reverse P -partitions	271
$r_x(P)$	row insertion of element x into tableau P	241
$\rho(F)$	vertex partition induced by edge set F	255
$\rho : G \rightarrow \text{GL}(V)$	representation of group G	287
$\mathcal{S}(L)$	labeled structure on label set L	124
\mathfrak{S}	pattern poset	140
\mathfrak{S}_n	symmetric group on $[n]$	11
$Sf(n)$	summation operator applied to function $f(n)$	162
sgn	sign function on a signed set	44
sh T	shape of tableau T	225
$s(n, k)$	signed Stirling number of the first kind	13

Symbol	Definition	Page
$S(T, k)$	set of partitions of the set T into k blocks	10
$S(n, k)$	Stirling number of the second kind	10
$S_o(L, k)$	set of ordered partitions of the set L into k blocks	127
$ST(G)$	set of spanning trees of graph G	59
st	statistic on a set	74
std σ	standardization of the permutation σ	28
Supp x	support set of x in a product of claws	173
supp x	size of support set of x in a product of claws	173
Sym	algebra of symmetric functions	220
Sym_n	symmetric functions of degree n	220
$\text{SYT}(\lambda)$	set of standard Young tableaux of shape λ	224
$\text{SSYT}(\lambda)$	set of semistandard Young tableaux of shape λ	225
s_λ	Schur function	225
$T_{i,j}$	element in cell (i, j) of tableau T	225
\mathcal{T}_n	set of monomino-domino tilings of a row of n squares	3
$U(S)$	upper-order ideal generated by S in a poset	143
$V(D)$	vertex set of digraph D	21
$V(G)$	vertex set of graph G	18
V_∞	vector space with a countably infinite basis over \mathbb{F}_q	178
$w_k(P)$	Whitney number of the first kind for a poset P	156
$W_k(P)$	Whitney number of the second kind for a poset P	156
W_n	walk with n vertices	19
wt	weight function on a set	86
\mathbf{x}	a countably infinite set of variables	219
\mathbf{x}^c	monomial for a coloring c of a graph	253
\mathbf{x}^f	monomial for a function f	270
\mathbf{x}^T	monomial for a tableau T	225
X^g	fixed points of group element g acting on set X	192
$X(G; \mathbf{x})$	chromatic symmetric function of graph G	253
$X(G; \mathbf{x}, q)$	chromatic quasisymmetric function of graph G	280
Y	Young's lattice	140
\mathbb{Z}	set of integers	1
$\zeta(x, z)$	zeta function in the incidence algebra of a poset	159
$\zeta(s)$	Riemann zeta function	182
$z(g)$	cycle index of group element g	197
$Z(G)$	cycle index of group G	197
$\#S$	cardinality of the set S	1
$ f $	size (sum of values) of a function	236
$ S $	cardinality of the set S	1
$ T $	sum of entries of tableau T	233
$S \uplus T$	disjoint union of sets S and T	1

Symbol	Definition	Page
$ \lambda $	sum of the parts of partition λ	13
$\lambda \vdash n$	λ is a partition of n	13
$S \times T$	(Cartesian) product of sets S and T	1
$P \uplus Q$	disjoint union of posets P and Q	145
$P \oplus Q$	ordinal sum of posets P and Q	146
$P \times Q$	(Cartesian) product of posets P and Q	146
$[g]$	linear transformation for group element g	287
$[g]_B$	matrix in basis B for group element g	287
$[n]$	set of integers $\{1, 2, \dots, n\}$	4
$[n]_q$	q -analogue of nonnegative integer n	75
$[n]_q!$	q -analogue of $n!$	75
$[x^n]f(x)$	coefficient of x^n in $f(x)$	83
$n \downarrow_k$	n falling factorial with k factors	4
2^S	set of subsets of S	5
$\binom{S}{k}$	set of k -element subsets of S	6
$\binom{n}{k}$	binomial coefficient	7
$\begin{bmatrix} n \\ k \end{bmatrix}_q$	q -binomial coefficient	77
$\begin{bmatrix} V \\ k \end{bmatrix}$	k -dimensional subspaces of vector space V	79
$\{\{a, a, \dots\}\}$	multiset individual element notation	8
$\{\{a^2, \dots\}\}$	multiset multiplicity notation	8
$\left(\binom{S}{k}\right)$	set of k -element multisubsets of S	9
$\chi(G)$	chromatic number of G	99
$\chi(g)$	character of group element g	291
$x \leq y$	x is covered by y in a poset	140
$y \succ x$	y covers x in a poset	140
$\hat{0}$	the minimum element of a poset	142
$\hat{1}$	the maximum element of a poset	142
$[x, y]$	closed interval from x to y in a poset	143
$x \wedge y$	meet of x and y in a poset	148
$\bigwedge X$	meet of the subset X in a poset	149
$x \vee y$	join of x and y in a poset	149
$U + V$	sum of subspaces U and V	149
$f * g$	convolution of f and g in the incidence algebra	158
$\chi(P; t)$	characteristic polynomial of a ranked poset P	164
P / \sim	quotient of poset P by equivalence relation \sim	169
ω_n	primitive n th root of unity	210
$\pi \xrightarrow{\text{RS}} (P, Q)$	Robinson–Schensted map	242
$M \xrightarrow{\text{RSK}} (T, U)$	Robinson–Schensted–Knuth map	244

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