Groups and Topological Dynamics
Groups and Topological Dynamics

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To Oksana
Contents

Preface ix

Chapter 1. Dynamical systems 1
  §1.1. Introduction by examples 2
  §1.2. Subshifts 25
  §1.3. Minimal Cantor systems 40
  §1.4. Hyperbolic dynamics 79
  §1.5. Holomorphic dynamics 110
Exercises 122

Chapter 2. Group actions 129
  §2.1. Structure of orbits 130
  §2.2. Micro-supported actions and Rubin’s theorem 150
  §2.3. Automata 163
  §2.4. Groups acting on rooted trees 179
Exercises 212

Chapter 3. Groupoids 219
  §3.1. Basic definitions 219
  §3.2. Actions and correspondences 232
  §3.3. Fundamental groups 259
  §3.4. Complexes of groups and orbispaces 265
  §3.5. Compactly generated groupoids 286
  §3.6. Hyperbolic groupoids 297
### Contents

<table>
<thead>
<tr>
<th>Chapter 4. Iterated monodromy groups</th>
<th>311</th>
</tr>
</thead>
<tbody>
<tr>
<td>§4.1. Iterated monodromy groups of self-coverings</td>
<td>312</td>
</tr>
<tr>
<td>§4.2. Self-similar groups</td>
<td>322</td>
</tr>
<tr>
<td>§4.3. Expanding maps and contracting groups</td>
<td>333</td>
</tr>
<tr>
<td>§4.4. Iterated monodromy groups of correspondences</td>
<td>349</td>
</tr>
<tr>
<td>§4.5. Hyperbolicity</td>
<td>374</td>
</tr>
<tr>
<td>§4.6. Iterations of polynomials</td>
<td>414</td>
</tr>
<tr>
<td>§4.7. Dynamics on the sphere</td>
<td>438</td>
</tr>
<tr>
<td>§4.8. Other applications</td>
<td>469</td>
</tr>
<tr>
<td>Exercises</td>
<td>491</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 5. Groups from groupoids</th>
<th>501</th>
</tr>
</thead>
<tbody>
<tr>
<td>§5.1. Full groups</td>
<td>501</td>
</tr>
<tr>
<td>§5.2. AF groupoids and bounded type</td>
<td>518</td>
</tr>
<tr>
<td>§5.3. Torsion groups</td>
<td>536</td>
</tr>
<tr>
<td>§5.4. Homology of totally disconnected étale groupoids</td>
<td>563</td>
</tr>
<tr>
<td>§5.5. Almost finite groupoids</td>
<td>576</td>
</tr>
<tr>
<td>§5.6. Purely infinite groupoids</td>
<td>584</td>
</tr>
<tr>
<td>Exercises</td>
<td>591</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 6. Growth and amenability</th>
<th>597</th>
</tr>
</thead>
<tbody>
<tr>
<td>§6.1. Growth of groups</td>
<td>597</td>
</tr>
<tr>
<td>§6.2. Groups of intermediate growth</td>
<td>599</td>
</tr>
<tr>
<td>§6.3. Inverted orbits</td>
<td>617</td>
</tr>
<tr>
<td>§6.4. Linearly repetitive actions</td>
<td>624</td>
</tr>
<tr>
<td>§6.5. Families of groups and non-uniform exponential growth</td>
<td>630</td>
</tr>
<tr>
<td>§6.6. Amenability</td>
<td>637</td>
</tr>
<tr>
<td>Exercises</td>
<td>669</td>
</tr>
</tbody>
</table>

| Bibliography | 673 |
| Index | 689 |
The main subject of the book are group-theoretic aspects of topological dynamics with a focus on the study of asymptotic properties of groups of dynamical origin and on using group theory in symbolic dynamics.

One of the most basic ways to construct a group is to describe a set of symmetries of a structure either by taking all its symmetries, or by considering a group generated by a collection of symmetries. It is natural to study groups defined in this way by using the underlying symmetric structure. For example, groups of isometries of a metric space are effectively studied using the geometry of the space, fundamental groups of topological spaces are closely related the universal coverings, groups of matrices are studied via their action on the vector spaces. In a similar way, many interesting and important groups are defined and studied using dynamics of their action on topological spaces.

Traditionally, dynamical systems are defined as actions of subgroups or subsemigroups of the group of real numbers, representing time evolution of a system. We are mainly interested in more complicated groups. In fact, most of our groups are rather “exotic” from the point of view of more traditional group theory. They are either defined as groups generated by a given set of homeomorphisms of a topological space, or as groups naturally associated with classical dynamical systems (as their full groups or iterated monodromy groups, for example). Classical group-theoretic methods do not work well in these situations. For example, such groups usually can not be defined by a finite set of relations, and can not be built from simpler groups using classical group theoretic constructions. On the other hand, dynamical approaches are fruitful: one can study topological properties of the orbits.
of the action, the action on open sets, e.t.c., and derive properties of groups from the obtained information.

Very often properties of the orbits are easy to understand (in particular, when they come from well studied classical dynamical systems), so they are the main tool in the study of properties of the group. For example, while geometry of the Cayley graphs of the group may be very complicated, the geometry of the graphs of the action on the orbits may be very simple.

This dynamical approach to group theory is especially effective for understanding asymptotic properties of the groups. For instance, all currently known examples of groups of intermediate growth (i.e., such groups that the number of elements that can be written as a product of generators of length at most $n$ grows faster than any polynomial but slower than any exponential function of $n$) are built and studied using their action on topological spaces and dynamical properties of their orbits. Similarly, all known examples of non-elementary amenable groups are also defined and studied using topological dynamics. One of the main purposes of this book is to serve as an introduction to this approach to asymptotic group theory.

The interplay between group theory and topological dynamics also goes in the opposite direction. We will see how a locally expanding covering map $f : \mathcal{X} \to \mathcal{X}$ of a compact metric space can be completely encoded in an algebraic structure consisting of a group, called the \textit{iterated monodromy group}, and a “self-similarity” on it (which can be described in different equivalent ways: as a virtual endomorphism of the group, as a homomorphism from the group to the wreath product of the group with a symmetric group, or as a pair of commuting actions on a set). One can reconstruct the dynamical system $f \subset \mathcal{X}$ from the iterated monodromy group, and also use the algebraic data to study topological properties of the space and the map. For example, one can construct an approximation of the space and of the map by a piecewise linear map between polyhedra starting from the algebraic invariant. Conversely, the properties of the iterated monodromy groups can be deduced from the properties of the dynamical system. In most cases the iterated monodromy groups are also exotic groups from the point of view of classical group theory. For example, some of them are of intermediate growth, non-elementary amenable. Most of them are not finitely presented.

Iterated monodromy groups and self-similarity of groups was the subject of the previous book \cite{Nek05}. The current book contains some new applications of iterated monodromy groups (e.g., iterated monodromy groups of maps of several variables, using iterated monodromy groups to study semi-conjugacies of dynamical systems, some new algebraic facts about iterated monodromy groups, e.t.c.), it is less focused on the algebraic properties of self-similar groups, and puts the theory of iterated monodromy groups into
a wider framework of groups associated with dynamical systems. We have also included many examples of applications of iterated monodromy groups as exercises.

As it was mentioned above, one of the main objects of study in topological dynamics are topological properties of orbits. It became clear in recent decades that orbits by themselves, even when we “forget” the action, carry a lot of interesting information and have rich algebraic structure (even in the Borel setting, as in the theory of Borel equivalence relations \[\text{[KM04]}\]). One way of formalizing this approach is via the notion of a groupoid, i.e., a small category of isomorphisms. Here the objects of the category are points of the space and two points are isomorphic if they belong to one orbit. It is natural in the context of topological dynamics to enrich this groupoid by adding some topological information, for example by looking at the germs of the action. Due to their nature as a bridge between topological dynamics and algebra, topological groupoids play an essential role in the study of group theoretical aspects of topological dynamics.

A part of the material is based on two graduate courses taught by the author at Texas A&M University. Other parts where taught by the author in several mini-courses. It is aimed at graduate students and researchers interested in group theory and topological dynamics.

The first chapter is an introduction to classical examples and basic notions of topological dynamics. It serves as a “crash course” in topological dynamics for group theorists and also develops some tools and techniques that will be important in later chapters (for example ordered Bratteli diagrams, subshifts, substitutional systems). There is also a section containing basic facts in holomorphic dynamics that will be needed later for the theory of iterated monodromy groups.

The second chapter studies some general aspects of groups acting on topological spaces. In particular, we develop basic techniques of orbital graphs, give a proof of a simplified version of M. Rubin’s reconstruction theorem, and give basic definitions of theory of automata (transducers) used to define homeomorphisms of symbolic Cantor sets. A separate section is devoted to groups acting on rooted trees and related automata theory.

Chapter 3 is an introduction to the theory of topological groupoids. We use groupoids for two reasons: to encode the dynamical information about the action and to consider dynamical systems on orbispaces (generalizations of orbifolds). The former are used, for example, to define topological full groups and to study their properties. The latter are important for the theory of iterated monodromy groups (for example in the case of sub-hyperbolic complex rational functions). So far, existing textbooks on groupoid theory were focused on other aspects of groupoid theory: foliations, \(C^*\)-algebras,
or homotopy theory. We hope that our book will fill the gap and introduce topological groupoids to current and future specialists in groups and dynamics.

Iterated monodromy groups and their applications are studied in Chapter 4. The main focus of the chapter is algebraic theory and symbolic dynamics of expanding covering maps.

Chapter 5 is devoted to topological full groups. We describe a method of constructing simple finitely generated groups associated with dynamical systems and how properties of the groupoid of germs of the action are related with the properties of the group. One of its sections describes examples of finitely generated infinite torsion groups (groups of Burnside type) constructed from dynamical systems. All known amenable groups of Burnside type are constructed in this way.

The last chapter is in some sense the culmination of the book. We use a large part of the developed techniques to construct groups of intermediate growth and non-elementary amenable groups.

Exercises are listed at the end of each chapter. They are roughly divided into three categories marked by asterisks at their numbers: problems for which one just has to apply the definitions in a more or less straightforward way (without asterisk), problems for which one has to come up with a non-trivial idea (one asterisk), and mini-research projects (two asterisks).

The author wants to thank Laurent Bartholdi, Collin Bleak, Rostislav Grigorchuk, Nicolas Matte Bon, Constantine Medynets, Josiah Owens, Dylan Thurston, and the anonymous referees for suggestions improving the text of the book. The work of the author was partially supported by NSF grant DMS 1709480.
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Index

action, 1
  amenable, 640
  equicontinuous, 187
  expansive, 27
  extensively amenable, 651
  free (of groupoid), 235
  level-transitive, 182
  linearly repetitive, 626
  micro-supported, 151
  minimal, 3, 142
  of groupoids, 235
  on scwol, 266
  proper (of groupoid), 235
  residually finite, 185
  self-similar, 192
  standard, 313
  topologically transitive, 142

adding machine, 15, 182

adic
  homeomorphism, 51
  transformation, 50

A(6) alternating full group, 508

almost finitary, 205

anchor of an action, 233

angle doubling, 5

Anosov diffeomorphism, 94
Arnold’s cat map, 17

asymptotic morphism, 302

automaton, 163
  Aleshin, 192
  Basilica, 210
  dendroid, 418
  finite, 105

Grigorchuk, 210

groupoid, 362
Hanoi tower, 210

non-deterministic, 173

partial, 168

polynomial activity growth, 208

synchronous, 190

time-varying, 177

transformation defined by, 164

Bernoulli measure, 7

bi-Lipschitz equivalent log-scales, 89

biaction, 236
  composition of, 237
  inverse of, 237

bisection, 224

biset, 323
  associated with a map, 323

boundary
  hyperbolic space, 258
  rooted tree, 180

boundary connection, 528

branching index, 180

Bratteli diagram, 46
  equivalent, 47

of a sequence of Rokhlin-Kakutani
  partitions, 53

ordered, 49

properly ordered, 51

simple, 52

stationary, 59, 62

telescoping of, 47

topological, 179

689
Busemann quasi-cocycle, 300, 301
Cantor-Bendixson rank, 512
Cayley graph of groupoid, 288
Chabauty space, 146
complex of groups, 267
cover scwol, 268
component
of multisection, 502
composition of biactions, 237
conjugacy, 7
contraction coefficient, 342
correspondence
covering, 349
hyperbolic, 375
on moduli space, 452
topological, 176
cover
semi-Markovian, 352
critical exponent, 345
cycle
attracting, 113
irrationally indifferent, 113
parabolic, 113
repelling, 113
cylindrical set, 6, 25
DA-attractor, 96
dendroid
sequence of permutations, 416
set of automorphisms of a tree, 417
domain
of a multisection, 502
\(D_0\) iterated monodromy groups, 452
dynamical system, 1
adic, 50, 51
Anosov diffeomorphism, 91
DA-attractor, 96
Denjoy, 41
expanding, 80
finitely presented, 104
Kakutani equivalent, 56
limit, 380
minimal, 8
Ruelle-Smale, 92
Smale space, 92
topologically transitive, 8
Williams solenoid, 99
effective groupoid, 226
equivalence
asymptotic, 377
defined by a functor, 247
Kakutani, 56
of groupoids, 241
stable and unstable, 88
expansive cover, 292
expansivity entourage, 27
external ray, 119
faithful quotient, 327
Fatou component, 113
Fatou set, 112
Fölner condition, 638
free action of groupoid, 236
functor between groupoids, 221
fundamental group
of complex of groups, 268
of groupoid, 262
\(\mathcal{S}\)-path, 299
\(G\)-regular points, 138
\(G\)-singular, 138
generating pair, 286
geodesic quasi-flow, 301
geometric realization
of a scwol, 266
germ of an action, 138
graph, 129
amenable, 206
closure of, 141
linearly repetitive, 146
locally contained, 136
locally isomorphic, 136
long range, 134
of an action, 130
of germs, 135
orbital, 130
perfectly labeled, 130
repetitive, 135
Schreier, 130
topological, 137
well labeled, 291
graph shift, 292
graph subshift, 292
group
alternating full group \(A(\mathcal{S})\), 506
amenable, 640
Basilica, 210
branch, 194
Brieussel’s, 542
Burnside type, 657
contracting, 340
defined by a labeled graph, 130
dimension, 565
dimension group, 565
elementary amenable, 648
Fabrykowski-Gupta, 612
full, 501
Grigorchuk, 131, 210, 540, 600
growth of, 598
Gupta-Sidki, 540
Hanoi tower, 210
Heisenberg, 471
Higman-Thompson, 168, 585
Houghton, 132
iterated monodromy, 313
long range, 210
of action of a groupoid, 234
of rational homeomorphisms, 167
ordered abelian, 566
self-replicating, 329
self-similar, 323
symmetric full group $S(\mathfrak{g})$, 506
Thompson, 135
topological full group of a
homeomorphism, 171
tower of Hanoi, 207
weakly branch, 194
groupoid
eétale, 224
abstract, 219
AF, 520
AF-by-discrete, 526
almost finite, 576
approximately finite, 520
bounded type, 527
Cayley graph of, 288
compactly generated, 286
connected, 202
developable, 264
dual, 303
effective, 226
expansive, 202
fundamental group of, 262
generated by a map, 228
graded hyperbolic, 301
holonomy, 229
hyperbolic, 298
iterated monodromy, 338
localization of, 258
locally connected, 202
locally simply connected, 202
of an action, 224
of biaction, 234
of bisections, 282
of germs, 225
of stable equivalence relation, 231
of unstable equivalence relation, 231
path connected, 262
principal, 222
proper, 227
pull-back of, 222
purely infinite, 584
Ruelle, 232
stable equivalence relation, 223
tail, 519
topological, 223
trivial, 221
groupoid cover, 284
groupoids
equivalence of, 241
morphisms of, 239
growth
exponential, 599
intermediate, 599
polynomial, 599
sub-exponential, 599
uniform exponential, 633
growth type, 598
homoclinic, 88
horseshoe map, 8
hyperbolic
component, 118
virtual endomorphism, 375
index map, 582
inverse of biaction, 237
inverted orbit, 617
irrational rotation, 2
isotropy group, 222
iterated monodromy action, 313
iterated monodromy group, 313
itinerary
for Markov partition, 19
Julia set, 112
filled, 117
kernel of a semiconjugacy, 104
Kesten’s criterion, 644
$\mathfrak{R}(v)$, $\mathfrak{R}(u, v)$ iterated monodromy
groups, 428
length function, 597
level stabilizer, 182
Index

limit orbispace, 380
limit space, 384
local development, 275
local direct product, 94
localization, 253
log-scale, 83

Mandelbrot set, 118
mapping torus, 15
Markov partition, 18, 107
mating, 480
mean, 639
metric
  associated with a log-scale, 83
  visual, 298
minimal, 3
model of a self-similar group, 105
Moore diagram, 166
morphism of groupoids, 239
multiplier, 113
multisection, 502

natural extension, 14
neighborhood stabilizer $G_{(x)}$, 138
nucleus, 340
topological, 412

odometer
  binary, 15
  generalized, 41
orbifold, 281
Thurston, 439
orbispace, 281
orbit
  inverted, 617
  of a groupoid, 222

paradoxical decomposition, 637
partial self-covering, 312
path in groupoids, 259
$P_d$, 208
permutational wreath product, 619
Poincaré metric, 111
polynomial
  Basilica, 320
  Chebyshev, 320
portrait, 345
primitive substitution, 34
principal groupoid, 222
proper action of groupoid, 235
proper groupoid, 227
pseudo-orbit, 105
pseudogroup, 225
pseudogroup of bisections, 226
pull-back of groupoid, 232
quasi-cocycle, 300

$r$ range (for graphs), 129
$r$ range for groupoids, 220
rational function
  hyperbolic, 114
  post-critically finite, 116
  sub-hyperbolic, 115
rectangle, 89
regular point of an action, 138
Reiter’s condition, 641
rigid stabilizer, 151, 194
RK set, 520
Rokhlin-Kakutani
  partition, 44
  set, 520
tower, 44
rooted tree
  level-transitive, 181
  spherically homogeneous, 181
rotation, 2
$S_A$ space of perfectly labeled graphs, 139
scwol, 265
semiconjugacy, 7
sequence
  kneading, 428
  Thue-Morse, 33
$S(\mathcal{F})$ symmetric full group, 506
shadowing, 105
shift
  full, 25
  of finite type, 29
  one-sided, 24
  substitutional, 32
  topological Markov, 30
two-sided, 26
singular point of an action, 138
Smale quasi-flow, 305
Smale space, 92
Smale-Williams solenoid, 13
solenoid, 13
$s$ source (for graphs), 129
$s$ source for groupoids, 220
space
  of labeled rooted graphs, 139
  of marked groups, 150
  of paths in a Bratteli diagram, 47
Index

of units, 221
  underlying, 281
spider
  invariant, 419
strong injectivity constant, 97
subshift, 26
  complexity of, 36
  defined by a matrix, 30
  entropy of, 38
  linearly repetitive, 77
  of finity type, 29
  sofic, 105
  substitutional, 32
  substitutional Fibonacci, 33
  Thue-Morse, 33
  Toeplitz, 42
  topological Markov, 30
substitution, 32
  Chacon, 35
  Fibonacci, 33
  period doubling, 43
  primitive, 34
  Thue-Morse, 33

tile, 525
topological conjugacy, 7
transitions
  in a non-deterministic automaton, 173
transversal
  of a groupoid, 222
  of a virtual endomorphism, 374
topological, 286
tree of preimages, 312
tree shift, 292
uniformly recurrent subgroup, 148
unit of a groupoid, 220
URS, 148

Vershik map, 50
Vershik-Bratteli diagram, 49
Vietoris-Rips complex, 297
virtual endomorphism
  of a group, 329
  of a groupoid, 359
wreath recursion, 182 193 327
X* spherically homogeneous tree, 181
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