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Topics in Spectral Geometry

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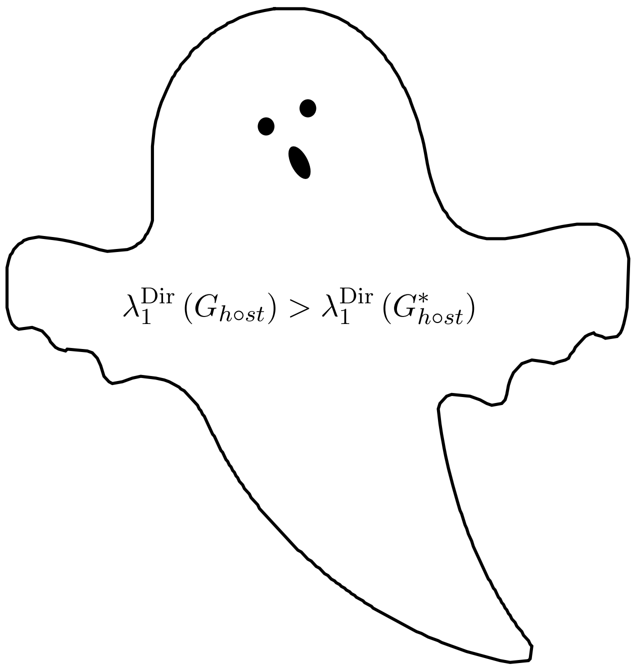

spectral *adj.* \ ˈspɛktrəl \

2. a. Having the character of a spectre or phantom; ghostly, unsubstantial, unreal.

5. a. Of or pertaining to, appearing or observed in, the spectrum. Also applied to a property or parameter which is being considered as a function of frequency or wave-length, or which pertains to a given frequency range or value within the spectrum.

From Oxford English Dictionary

Contents

Preface

Various distinct physical phenomena, such as wave propagation, heat diffusion, electron movement in quantum physics, oscillations of fluid in a container, can be modelled mathematically using the same differential operator — the Laplacian. Its spectral properties depend in a subtle way on the geometry of the underlying object, e.g., a Euclidean domain or a Riemannian manifold, on which the operator is defined. This dependence — or, rather, the interplay between the geometry and the spectrum — is the main subject of *spectral geometry*.

The roots of spectral geometry go back to the famous experiments of the physicist Ernst Chladni with vibrating plates in the late eighteenth century to early nineteenth century, as well as to the investigations of Lord Rayleigh on the theory of sound some decades later. The celebrated question of Mark Kac, "Can one hear the shape of a drum?", motivated a lot of research in the second half of the twentieth century and helped spectral geometry to emerge as a separate branch of geometric analysis.

Modern spectral geometry is a rapidly developing area of mathematics, with close connections to other fields, such as differential geometry, mathematical physics, number theory, dynamical systems, and numerical analysis. It is a vast subject, and by no means does this book pretend to be comprehensive. Our goal was to write a textbook that can be used for a graduate or an advanced undergraduate course, starting from the basics but at the same time covering some of the exciting recent developments in the area which can be explained without too many prerequisites. The authors have taught such courses over the past few years at different locations, in particular at the Université de Montréal and the Hebrew University of Jerusalem, and they have taught shorter courses at the Universities of Cardiff and Reading, as well as at several summer schools and instructional conferences; see, e.g., [**BouLev07**]. The present book is based in part on our lecture notes.

Acknowledgements. We gratefully acknowledge the influence of many earlier books on spectral geometry and related subjects, by Courant and Hilbert [**CouHil89**], Berger, Gauduchon, and Mazet [**BerGauMaz71**], Reed and Simon [**ReeSim75**], Bandle [**Ban80**], Chavel [**Cha84**], Bérard [**Bér86**], Davies [**Dav89**], [**Dav95**], Schoen and Yau [**SchYau94**], Rosenberg [**Ros97**], Henrot [**Hen06**], Helffer [**Hel13**], and Shubin [**Shu20**], to name just a few, as well as some of the more recent lecture notes by Laugesen [**Lau12**], Canzani [**Can13**], Buhovsky [**Buh16**], and Logunov and Malinnikova [**LogMal20**]. Of course, the standard disclaimer is that the choice of the topics in this book reflects the personal tastes and preferences of the authors.

Many people contributed to this book in different ways. It is a pleasure to thank our mentors Robert Brooks, Yakar Kannai, Victor Lidskii, Leonid Polterovich (to whom we are particularly thankful for encouraging this book project since its very early stages), and Dmitri Vassiliev, for introducing us to geometric spectral theory. Through the years, we have also been greatly influenced by collaborations and innumerable helpful discussions with Michiel van den Berg, E. Brian Davies, Lennie Friedlander, Nikolai Nadirashvili, Leonid Parnovski, and Mikhail Sodin, among others.

While preparing the manuscript we used the notes taken by Simon St-Amant at a course given by the third author, Iosif Polterovich, at the Université de Montréal. We are especially grateful to Matteo Capoferri, Philippe Charron, Stefano Decio, Emily Dryden, Alexandre Girouard, Asma Hassannezhad, Mikhail Karpukhin, Jean Lagacé, Antoine Métras, Nilima Nigam, and David Sher, who made a lot of insightful comments and suggestions on the preliminary draft of the book.

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We would also like to thank the anonymous referees for multiple helpful suggestions.

All the remaining errors, omissions, and inaccuracies are of course entirely ours.

We are grateful to the staff at the American Mathematical Society, in particular to Sergei Gelfand for his encouragement and helpful advice throughout the preparation of the book and to Brian Bartling for his assistance on typesetting.

At various stages of work on this book, the first author, Michael Levitin, has been visiting the Centre de recherches mathématiques in Montréal. Its hospitality is gratefully acknowledged.

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Last but not least, we would like to thank our families for their patience and support. Without them this project would have never been accomplished.

Introduction

Overview

The central theme of the book is spectral geometry of the Laplace operator on bounded Euclidean domains and compact Riemannian manifolds. Most of the time, we consider the classical Dirichlet or Neumann boundary conditions, except for the last chapter, where instead of the spectral parameter in the equation we look at the less explored Steklov problem with the spectral parameter in the boundary conditions.

The main topics discussed in the book can be summarised as follows:

- spectral theorems,
- eigenvalue inequalities,
- spectral asymptotics,
- nodal geometry,
- isospectrality and spectral invariants.

To cover these subjects we use a variety of techniques, such as variational principles, elliptic regularity, symmetrisation, conformal maps, harmonic analysis, and heat equation methods. Throughout the presentation we tried to keep a balance between the following principles:

- Focus on *phenomena*. For that reason, in many cases the proofs are given in the Euclidean setting, with indications on how the argument can be extended to the Riemannian case.
- Avoid *black boxes* as much as possible. While it is often unfeasible to present all the details, we at least tried to explain the main ideas behind the proofs.

- Keep *generality* reasonably wide to include most interesting examples. In particular, in the Euclidean setting we mostly consider Lipschitz boundaries, whilst on manifolds we deal with smooth Riemannian metrics.

The highlights of the book include:

- Spectral theorems and elliptic regularity. In particular, we discuss in detail both interior and boundary regularity of eigenfunctions.
- Weyl's law for the eigenvalue counting function.
- Friedlander–Filonov inequalities between Dirichlet and Neumann eigenvalues.
- Polya's conjecture for tiling domains and Berezin–Li–Yau–inequalities.
- Courant and Pleijel nodal domain theorems.
- Yau's conjecture on the size of the nodal sets.
- Isoperimetric inequalities for eigenvalues: Faber–Krahn, Cheeger, Szegő–Weinberger, and Hersch.
- Universal inequalities for eigenvalues.
- Heat trace asymptotics.
- Isospectrality and transplantation of eigenfunctions.
- Spectral geometry of the Steklov problem.

While many of these topics can be found in other books, having all these subjects under one cover makes this book quite different from the others. At times, our exposition of classical results contains some features which have not been emphasised previously. For example, we prove Courant's nodal domain theorem for Dirichlet eigenfunctions without any regularity assumptions on the boundary. Moreover, some of the material is based on recent research and therefore cannot be found in textbooks, such as the section on Yau's conjecture and essentially the entire chapter on the Steklov problem.

Plan of the book

The book is organised as follows.

In Chapter 1 we introduce our main hero, the Laplacian, and discuss several examples for which its eigenvalues and eigenfunctions can be calculated explicitly.

In Chapter 2 we lay the foundations for further material and explain the proofs of the weak and the strong spectral theorems for the Laplacian. This chapter includes mini-crash courses on the theory of selfadjoint unbounded linear operators, as well as on the Sobolev spaces and elliptic regularity. Our emphasis is on presenting the main tools and ideas, such as the Friedrichs extension, the a priori estimates, and Nirenberg's method of difference quotients, while referring the reader interested in full details to the existing literature.

Chapter 3 is concerned with the variational principles for eigenvalues and their applications. Apart from basic results such as domain monotonicity, Dirichlet–Neumann bracketing and Weyl's law, we prove the Friedlander–Filonov inequalities between Dirichlet and Neumann eigenvalues, the Berezin–Li–Yau inequalities, and Pólya's conjecture for tiling domains.

Chapter 4 focuses on the nodal geometry of eigenfunctions. We give a complete proof of Courant's nodal domain theorem, explaining some delicate issues arising for domains with nonsmooth boundary that have often been omitted in other sources. We also discuss Yau's conjecture on the volume of nodal sets, including recent breakthrough developments due to Logunov and Malinnikova. In particular, we give a sketch of the proof of a polynomial upper bound on the size of the nodal set. Some related topics, such as the density of the nodal set and the lower bound on the size of the nodal set in dimension two, are also presented. As an application of results on the local structure of the nodal set we prove multiplicity bounds for eigenvalues on surfaces.

In Chapter 5 we collect various geometric eigenvalue inequalities, such as the Faber–Krahn inequality, Cheeger's inequality, the Szegő–Weinberger inequality, as well as Hersch's inequality and other isoperimetric inequalities of surfaces. The latter is an actively developing subject and several recent advances are discussed in detail. This chapter also includes the universal inequalities, as well as related commutator identities.

The heat equation and results on heat kernel asymptotics are presented in Chapter 6. As an application, we prove Weyl's law on Riemannian manifolds. The spectral invariants arising from the heat asymptotics naturally lead us to the study of isospectrality. Some partial answers are given to the question "Can one hear the shape of a drum?", mentioned above. We present Milnor's example of flat isospectral tori which has fascinating connections to the theory of modular forms, and the celebrated Sunada construction of isospectral manifolds based on algebraic ideas. We also describe a rather elementary but ingenious transplantation technique that yields isospectral but not isometric planar domains. Some recent results on spectral rigidity are also discussed.

In the past decade, the study of the Steklov problem and of the Dirichlet-to-Neumann map became one of the most active directions in spectral geometry. This is the subject of Chapter 7. We define the Steklov spectrum and prove isoperimetric inequalities for Steklov eigenvalues. Using the connection between the Dirichlet-to-Neumann map and the boundary Laplacian, we obtain results on the asymptotics of the Steklov spectrum by means of the Hörmander–Pohozhaev identities and Weyl's law for the Laplacian on manifolds. We also provide a detailed exposition of recent results on the asymptotics of sloshing eigenvalues as well as Steklov eigenvalues on curvilinear polygons. Finally, we discuss the Dirichlet-to-Neumann map for the Helmholtz operator and use its properties to give another proof of the Friedlander–Filonov inequalities between Dirichlet and Neumann eigenvalues originally presented in Chapter 3.

Appendix A contains a short introduction to numerical spectral geometry, which provides the students with all the necessary tools for a quick numerical calculation of eigenvalues and eigenfunctions of planar domains.

In Appendix B we collect some standard background definitions and notation which we use throughout the book.

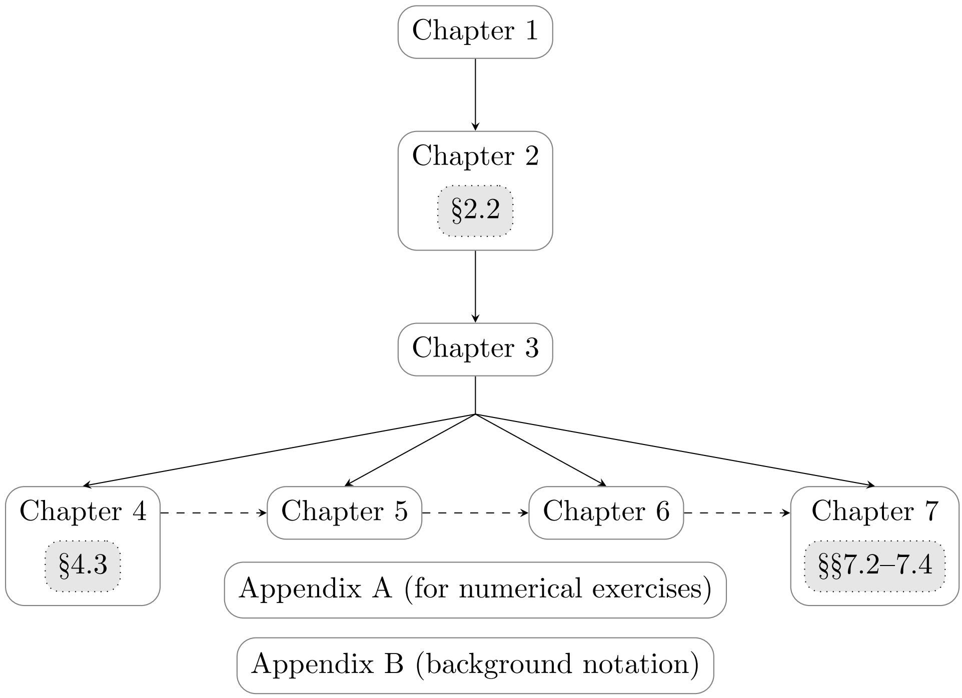
Possible courses based on this book

The book is to a large extent self-contained and is accessible to students and researchers with a basic knowledge of PDEs, functional analysis, and differential geometry. We do not *really* require prior knowledge of the theory of distributions and Sobolev spaces and explain the main notions we need. Throughout the book we often stay in the Euclidean setting and, where necessary, provide references for a reader unfamiliar with the fundamentals of Riemannian geometry. While graduate students in mathematics are the main targeted audience for the book, it could also be used, in parts, for teaching an advanced undergraduate course, as well as for both introductory and advanced mini-courses.

In our experience, essentially the whole book with the exception of the most advanced sections (§§2.2, 4.3, and 7.2–7.4) can be covered in a one-semester course. There are various ways to create shorter courses using the diagram of dependencies given on page xvii.

For example, one could teach the first three chapters only, or the first three chapters followed by one of the Chapters 4–7, with some minor additions and adjustments. Finally, the material of each of Chapters 1–3 can be taught as an introductory level mini-course, and each of the remaining chapters as a more advanced course.

Last, but not least, the book contains *many* exercises! The more difficult ones are provided with references and hints. A user-friendly tutorial



The diagram of chapter dependencies. The subsections in shaded boxes may be omitted for all but advanced courses. The dashed arrows indicate that while there is some dependency of material there, the corresponding chapters may be taught separately from each other.

on numerical spectral geometry presented in Appendix A could also help teachers who would like to introduce a computational component into their classes.

What is not in this book: Some further reading

As mentioned in the preface, the field of spectral geometry extends beyond the scope of this book. Below we discuss some interesting and important topics for further reading.

In order to keep the prerequisites to a minimum, we focused on results that can be presented without using pseudodifferential operators and microlocal analysis. As a consequence, apart from nodal geometry, we did not explore much the properties of eigenfunctions. We refer to [**Sog17**] and [**Zwo12**] for an exposition of results on asymptotic eigenfunction bounds, as well as questions arising in the fascinating area of *mathematical quantum chaos*, such as Shnirelman's quantum ergodicity theorem.

Throughout the book, we have almost exclusively dealt with the case of bounded domains and compact manifolds, for which the Laplace spectrum is discrete. A lot of interesting phenomena occur in other geometric setups. We refer to [**Bor16**] and [**DyaZwo19**] for recent developments of the spectral theory on infinite area hyperbolic spaces and the mathematical theory of resonances.

In this book, we focus on the Laplacian and the Dirichlet-to-Neumann map and do not touch other important operators. A modern exposition of the spectral theory of Schrödinger operators, with a particular focus on the celebrated Lieb–Thirring inequalities (closely linked to the Berezin–Li–Yau inequalities featured in Chapter 3), can be found in [**FraLapWei23**]. Many interesting geometric questions arise in the study of the spectrum of the Dirac operator, and we refer to [**BerGetVer04**, **Fri00**, **Gin09**] for further reading on this subject. Recent results on spectral geometry of potential operators, which are related to the Dirichlet-to-Neumann map, can be found in [**RuzSadSur20**]. A detailed introduction to the rich and actively developing theory of quantum graphs, which makes a cameo appearance in §7.3 of this book, can be found in [**BerKuc13**].

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