

Poncelet's Theorem



Leopold Flatto

Poncelet's Theorem

Leopold Flatto

The graphic of the Poncelet grid used on the cover is taken from
The Poncelet Grid by R. Schwartz,
Advances in Geometry (2007), 157–175. Used with permission.

2000 *Mathematics Subject Classification*. Primary 51–01, 51M04, 51N15.

For additional information and updates on this book, visit
www.ams.org/bookpages/mbk-56

Library of Congress Cataloging-in-Publication Data

Flatto, Leopold, 1929–

Poncelet's theorem / Leopold Flatto.

p. cm.

Includes bibliographical references and index.

ISBN 978-0-8218-4375-8 (alk. paper)

1. Poncelet's theorem. 2. Geometry, Projective. I. Title.

QA471.F53 2008

516'.5—dc22

2008025623

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294, USA. Requests can also be made by e-mail to reprint-permission@ams.org.

© 2009 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights
except those granted to the United States Government.

Copyright of Chapter 15 reverts to the public domain
28 years after publication.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 14 13 12 11 10 09

In Loving Memory

To Zehava, whose encouragement and spirit made this book possible

Contents

Preface	xi
List of Commonly Used Symbols	xv
Chapter 1. Introduction	1
1.1. The Theorems of Poncelet and Cayley	1
1.2. The Poncelet and Steiner Theorems— A Misleading Analogy	6
1.3. The Real Case of Poncelet's Theorem	9
1.4. Related Topics	10
Part I. Projective Geometry	
Chapter 2. Basic Notions of Projective Geometry	15
2.1. Projective Plane	15
2.2. Projectivities	19
2.3. Projective Line	22
2.4. Algebraic Curves	24

Chapter 3. Conics	31
3.1. Conics	31
3.2. Intersection of Line and Conic	34
3.3. Reduced Form	36
3.4. Projective Structure on a Smooth Conic	38
3.5. Parametric Equations of Smooth Conics	39
Chapter 4. Intersection of Two Conics	43
4.1. Intersection Numbers	43
4.2. Bezout's Theorem for Conics	51
4.3. Conic Pencils	53
4.4. Degenerate Conics in a Conic Pencil	55
Part II. Complex Analysis	
Chapter 5. Riemann Surfaces	61
5.1. Definition of Riemann Surface	61
5.2. Examples of Riemann Surfaces	65
5.3. More Examples of Riemann Surfaces. Algebraic Curves	68
5.4. Examples of Conformal Maps	74
5.5. Covering Surfaces	76
5.6. Isomorphisms of Tori	79
Chapter 6. Elliptic Functions	83
6.1. Elliptic Functions	83
6.2. The Weierstrass \wp -Function	86
6.3. The Functions ζ and σ	89
6.4. Differential Equation for \wp	92
6.5. The Elliptic Function $w = sn(z)$	94
Chapter 7. The Modular Function	97
7.1. The Functions g_2, g_3	97
7.2. The Modular Function J	98
7.3. Fundamental Region for Γ	100

7.4. Fourier Expansion of J	102
7.5. Values of J	104
7.6. Solution to the Inversion Problem	108
Chapter 8. Elliptic Curves	111
8.1. Elliptic Curves	111
8.2. Algebraic Models	113
8.3. Division Points of C/Λ	115
8.4. Division Points of \mathcal{S}	117
Part III. Poncelet and Cayley Theorems	
Chapter 9. Poncelet's Theorem	123
9.1. Poncelet Correspondence	123
9.2. Algebraic Equation for \mathcal{M}	125
9.3. Complex Structure on \mathcal{M}	128
9.4. \mathcal{M} is an Elliptic Curve	130
9.5. The Automorphisms σ, τ , and η	131
9.6. Proof of Poncelet's Theorem	132
Chapter 10. Cayley's Theorem	135
10.1. Origin of \mathcal{M}	135
10.2. Algebraic Equation for \mathcal{M}	136
10.3. Proof of Cayley's Theorem	138
Chapter 11. Non-generic Cases	141
11.1. Fixed Points of η	141
11.2. Equations for C, D , and \mathcal{M}	142
11.3. The Riemann Surface \mathcal{M}_0	144
11.4. Formulas for η	147
11.5. Poncelet's Theorem	148
11.6. Existence of Circuminscribed n -Gons	150

Chapter 12. The Real Case of Poncelet's Theorem	153
12.1. Poncelet's Theorem for Two Circles	153
12.2. Poncelet's Theorem for Two Ellipses	155
12.3. Topological Conjugacy	157

Part IV. Related Topics

Chapter 13. Billiards in an Ellipse	165
13.1. Billiards in an Ellipse. Caustics	165
13.2. The Map η_R	167
13.3. Description of \mathcal{M}_R	168
13.4. Invariant Measure. Rotation Number	170
13.5. Billiard Trajectories with the Same Caustic	172
13.6. Derivation of Invariant Measure	173
13.7. Proofs of Theorems 13.3 and 13.4	177
Chapter 14. Double Queues	179
14.1. The Two-Demands Model	180
14.2. Formulas	182
14.3. Riemann Surface	183
14.4. Automorphy Conditions	184
14.5. The Regions \mathcal{D}_z and \mathcal{D}_w	184
14.6. Analytic Continuation	186

Supplement

Chapter 15. Billiards and the Poncelet Theorem	
S. TABACHNIKOV	191
15.1. Mathematical Billiards	191
15.2. Integrable Case	195
15.3. Poncelet Grid	198
15.4. Poncelet Theorem on Quadratic Surfaces	204
15.5. Outer Billiards in the Hyperbolic Plane	207
References	210

Appendices

Appendix A. Factorization of Homogeneous Polynomials	215
Appendix B. Degenerate Conics of a Conic Pencil. Proof of Theorem 4.9	219
Appendix C. Lifting Theorems	223
C.1. Homotopy	223
C.2. Lifting Theorems	224
Appendix D. Proof of Theorem 11.5	229
Appendix E. Billiards in an Ellipse. Proof of Theorem 13.1	233
References	237
Index	239

Preface

One of the most important and beautiful theorems in projective geometry is that of Poncelet, concerning closed polygons which are inscribed in one conic and circumscribed about another (the exact statement is given in §1.1). The theorem is also of great depth in that it relates to a large body of mathematics. The aim of this book is to explore these relations, which provide much insight into several important mathematical topics.

The topics in question are Poncelet's theorem, billiards in an ellipse, and double queues. At first sight these topics seem unrelated, belonging to three distinct mathematical fields: geometry, dynamical systems, and probability. But there is a hidden thread tying these topics together: the existence of an underlying structure (we name it the Poncelet correspondence \mathcal{M} (see §1.1)) which turns out to be an elliptic curve. As is well known, elliptic curves can be endowed with a group structure, and the exploitation of this structure sheds much light on the aforementioned topics.

The only prerequisites for reading this book are the following standard subjects covered in undergraduate and first year graduate mathematics courses: complex analysis, linear algebra, and some point set topology.

The book is organized as follows. Chapter 1 gives a description of the main topics of the book (these topics are discussed in Parts III and

IV). Chapters 2–14 are divided into four parts. These are followed by the supplementary Chapter 15, “Billiards and the Poncelet Theorem”, by S. Tabachnikov. There are also five appendices. The purpose of these is to fill in several omissions from Parts I–IV.

Parts III and IV form the core of the book. Part III discusses the theorems of Poncelet and Cayley (the latter is explained in §1.1) and is based on the approach used in the papers of Griffiths and Harris [GH1], [GH2]), which relates these theorems to elliptic curves (over the complex field) and to their parameterization by elliptic functions. Another approach, using notions from dynamical systems, is also presented here. The papers [GH1], [GH2] take for granted various algebro-geometric notions. Part III explains and elaborates on these notions.

Part IV discusses billiards in an ellipse and double queues and is based on papers which I authored and co-authored ([F1], [FH]). The ideas in these papers are reworked and further developed in Part IV. Furthermore, the presentation in Part IV displays a fundamental connection between these topics and Poncelet’s theorem. Indeed the topic of double queues, which appears last in the book, could very well have been the first, for it is the study of double queues that led me to the surprising connection with Poncelet’s theorem.

Chapter 15, written by S. Tabachnikov, gives an expository account of mathematical billiards and demonstrates how this theory provides an alternative proof of Poncelet’s theorem. In addition, the theory provides another proof of the recent result by R. Schwartz on Poncelet grids (see §15.3).

Chapter 15 also discusses recent developments connecting Poncelet’s theorem to other mathematical topics: geodesics on an ellipsoid and dual billiards.

The topics appearing in Parts I and II are collected from various sources. Most of the material is fairly standard, a notable exception being the discussion in Chapter 8 on division points on elliptic curves, the treatment being a modification of the one given in [GH2]. Part I deals with projective geometry, and Part II deals with complex analysis. The treatment of topics presented in Parts I and II is by no means complete, nor is it intended to be. Rather, the choice of

material presented in these parts is dictated by the desire to make Parts III and IV accessible to as wide an audience as possible. The knowledgeable reader can immediately proceed to Parts III and IV. The same advice is offered to the less knowledgeable reader, who can occasionally turn back to Parts I and II, as the need arises.

It is of course possible to discuss projective geometry with coordinates coming from any preassigned field, but we consider only the complex projective plane and, occasionally, the real projective plane. The reason for this restriction is that the results of Parts III and IV make use of complex analysis.

In conclusion, many topics are treated in the book, all relating to Poncelet's theorem. In this sense, the approach of this book follows the maxim of the Talmudic sage Abaye "from topic to topic, yet always in the same topic" (Babylonian Talmud, Tractate Kiddushin, p. 6a). The proof of Poncelet's theorem reveals deep connections between the seemingly disparate subjects treated in this book. It is this aspect of Poncelet's theorem that has drawn me to a detailed study of it and its ramifications. The book demonstrates that Poncelet's theorem serves as a prism through which one can learn and appreciate a lot of beautiful mathematics.

Acknowledgments. I express my thanks and gratitude to the following people who contributed to the writing of this book, both on the personal and professional levels.

First and foremost, gratitude goes to my late beloved wife, Zehava, the driving force behind the book. Without her constant support and encouragement, the book would not have "seen the light of day". Then I thank my children, Sharon and David, for their interest and gentle goading, constantly inquiring, "So what is happening with the book?" Likewise, I thank my friends Joel Gribetz and Daniel Lasker, with whom I have renewed a friendship after a lapse of more than fifty years, due to cataclysmic circumstances beyond our control.

Special thanks goes to the following people for their careful reading of various parts of the book and for their valuable criticisms in an effort to improve its quality: Seymour Haber, Henry Landau, Hans

Witsenhauser, and Yuli Baryshnikov. I also thank Joe Harris for several illuminating conversations regarding his joint papers with Phillip A. Griffiths. I express my gratitude to Serge Tabachnikov for his generous contribution of the supplementary chapter “Billiards and the Poncelet Theorem”.

Thanks also goes to Sergei Gelfand and Arlene O’Sean for their invaluable assistance in preparing the book for publication.

The book is an outgrowth of a series of lectures given at the NSA, where I spent a sabbatical year during 1998–1999. I thank Mel Currie for organizing a seminar on the subject matter. Special thanks goes to Harvey Cohn and Donald Newman for participating in the entire lecture series and for offering insightful comments.

List of Commonly Used Symbols

$:=$	symbol used to indicate that the left-hand side is defined by the right-hand side
$\{p: p \text{ has property } A\}$	set of elements p having property A
iff	if and only if
■	q.e.d.
\mathbb{Z}	set of integers
\mathbb{R}	set of real numbers
\mathbb{C}	set of complex numbers
$\widehat{\mathbb{C}}$	complex sphere
Re z , Im z	real and imaginary parts of complex number z
$ z $	absolute value of complex number z
arg z	argument of complex number $z \neq 0$
det A or $ A $	determinant of square matrix A
$f _A$	restriction of function f to set A
$f \circ g$	composition of the two mappings f and g
f^{-1}	transformation inverse to f
$a \in A$	a is a member of set A
$a \notin A$	a is not a member of set A
$f(A)$	image of set A under the mapping f

$\#A$ or $ A $	cardinality of A = number of elements in A
$A \subset B$	A is a subset of B
$A \cup B$	union of A and B = set of elements either in A or in B
$A \cap B$	intersection of A and B = set of elements in both A and B
$A \setminus B$	set of elements in A and not in B
$A \times B$	product of A and B = $\{(a, b) : a \in A, b \in B\}$
\emptyset	the empty set

References

Books

- [A] L. V. Ahlfors, *Complex Analysis*, McGraw Hill, 1953
- [Ap] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Second Edition, Springer-Verlag, 1990
- [B1] M. Berger, *Geometry I*, Springer-Verlag, 1987
- [B2] M. Berger, *Geometry II*, Springer-Verlag, 1987
- [B] G. D. Birkhoff, *Dynamical Systems*, A.M.S. Colloquium Publications, Vol. 9, 1960
- [FK] H. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag, 1980
- [FIM] G. Fayolle, R. Iasnogorodski, and V. Malyshev, *Random Walks in the Quarter Plane*, Springer-Verlag, 1999
- [Fo] O. Foster, *Lectures on Riemann Surfaces*, Springer-Verlag, 1981
- [HW] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Second Edition, Oxford Clarendon Press, 1945
- [K] F. Kirwan, *Complex Algebraic Curves*, Cambridge University Press, 1992
- [L] S. Lang, *Elliptic Functions*, Addison-Wesley Publishing Company Inc., 1973.
- [MKM] H. M. McKean and V. Moll, *Elliptic Curves*, Cambridge University Press, Cambridge, 1997
- [M] R. Miranda, *Algebraic Curves and Riemann Surfaces*, A.M.S., 1991

- [N] Z. Nehari, *Conformal Mapping*, McGraw Hill, First Edition, 1952
- [P] J. V. Poncelet, *Traité sur les Propriétés des Figures*, Paris, 1822 (Second Edition, two volumes, 1865–1866)
- [SZ] S. Saks and A. Zygmund, *Analytic Functions*, Elsevier Publishing Co., Amsterdam, NY, 1971
- [S] P. Samuel, *Projective Geometry*, Springer-Verlag, NY, 1988
- [Sc1] I. J. Schoenberg, *Mathematical Time Exposures*, M.A.A., 1982
- [ST] J. H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Springer-Verlag, NY, 1992
- [Sp] G. Springer, *Introduction to Riemann Surfaces*, Addison-Wesley Publishing Co., Reading, MA, 1957
- [W] R. J. Walker, *Algebraic Curves*, Princeton, 1959

Papers

- [BKOR] H. J. M. Bos, C. Kers, F. Oort, and D. W. Raven, Poncelet's closure theorem, *Expo. Math.* 5 (1987), 289–364
- [FI] L. Flatto, Billiards in an ellipse, *International Conference on Complex Analysis and Dynamical Systems II*, A.M.S. (2003), 141–148
- [FH] L. Flatto and S. Hahn, Two parallel queues created by arrivals with two demands I, *SIAM J. Appl. Math.*, Vol. 44, No. 5 (1984), 1041–1053
- [Ki] J. King, Three problems in search of a measure, *Amer. Math. Monthly* 101 (1994), 608–628
- [GH1] P. Griffiths and J. Harris, A Poncelet theorem in space, *Math. Helv.* 52 (1977), 145–166
- [GH2] P. Griffiths and J. Harris, On Cayley's explicit solution to Poncelet's porism, *L'Enseignement Mathématique* (ser. 2) 24 (1978), 31–40
- [Sc2] I. J. Schoenberg, On Jacobi-Bertrand's proof of a theorem of Poncelet, *Studies in pure mathematics*, 623–627, Birkhäuser, Basel, 1983

Index

- affine coordinates, 16
- affine plane, 16
- algebraic curve, 25
- algebraic model, 113
- analytic atlas, 62
- automorphism, 63

- Bezout's theorem, 43
- billiard ball map, 191
- branch point, 76

- caustic, 166
- Cayley's theorem, 4
- chart, 61
- complete sets of zeros and poles, 84
- complex structure, 62
- conic pencil, 53
- covering map, 77
- covering surface, 76

- degenerate conic, 32
- division point, 115
- double queues, 11

- elliptic curve, 111
- elliptic function, 83
- elliptic function $sn(z)$, 94

- fiber, 77
- fundamental parallelogram, 84
- fundamental region, 100

- genus, 73
- good measure, 157

- homogeneous coordinate ratio, 23
- homogeneous coordinates, 16
- homotopy, 223

- intersection number, 45
- invariant measure, 154
- inversion problem, 97
- isomorphism, 63

- lattice, 66
- lift of a curve, 79
- line at infinity, 17
- local coordinate, 61
- locally conformal, 64

- Möbius transformation, 23
- modular function, 98
- modular group, 99
- multiplicity type, 52

- natural projection, 67

- order of a polynomial, 25
- outer billiard transformation, 207

- parametric disk, 62
- pencil of lines, 18
- 1/2-periods, 94
- point at infinity, 16

- Poncelet correspondence, 5
- Poncelet grid, 198
- Poncelet map, 5
- Poncelet's theorem, 2
- projection map, 76
- projective line, 22
- projective plane, 16
- projectivity, 19

- quadratic parametrization, 42

- regular point, 76
- Riemann surface, 62
- Riemann-Hurwitz theorem, 77
- rotation number, 158

- smooth conic, 32
- smooth cover, 76
- Steiner theorem, 6
- string construction, 193

- tangent line, 26
- topological conjugacy, 158
- torus, 67
- translation, 82
- two-dimensional manifold, 61

- P_2 -version of Poncelet's theorem, 3

- Weierstrass \wp -function, 87

Poncelet's theorem is a famous result in algebraic geometry, dating to the early part of the nineteenth century. It concerns closed polygons inscribed in one conic and circumscribed about another. The theorem is of great depth in that it relates to a large and diverse body of mathematics. There are several proofs of the theorem, none of which is elementary. A particularly attractive feature of the theorem, which is easily understood but difficult to prove, is that it serves as a prism through which one can learn and appreciate a lot of beautiful mathematics.

This book stresses the modern approach to the subject and contains much material not previously available in book form. It also discusses the relation between Poncelet's theorem and some aspects of queueing theory and mathematical billiards.

The proof of Poncelet's theorem presented in this book relates it to the theory of elliptic curves and exploits the fact that such curves are endowed with a group structure. The book also treats the real and degenerate cases of Poncelet's theorem. These cases are interesting in themselves, and their proofs require some other considerations. The real case is handled by employing notions from dynamical systems.

The material in this book should be understandable to anyone who has taken the standard courses in undergraduate mathematics. To achieve this, the author has included in the book preliminary chapters dealing with projective geometry, Riemann surfaces, elliptic functions, and elliptic curves. The book also contains numerous figures illustrating various geometric concepts.

ISBN 978-0-8218-4375-8



9 780821 843758

MBK/56

AMS on the Web

www.ams.org



For additional information
and updates on this book, visit

www.ams.org/bookpages/mbk-56