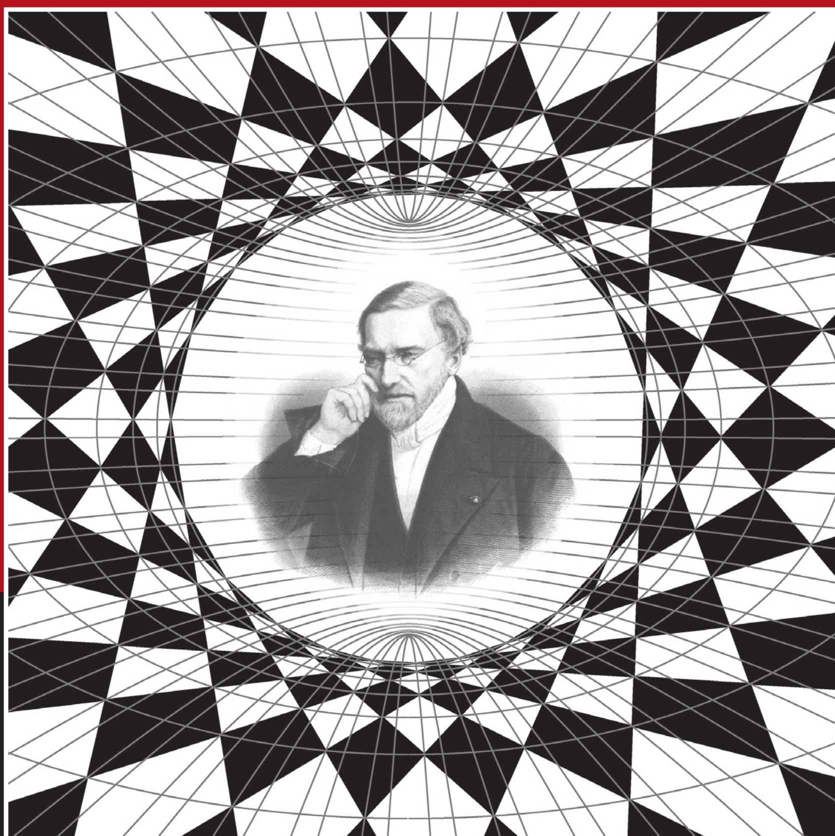
Poncelet's Theorem



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In Loving Memory

To Zehava, whose encouragement and spirit made this book possible

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Preface

One of the most important and beautiful theorems in projective geometry is that of Poncelet, concerning closed polygons which are inscribed in one conic and circumscribed about another (the exact statement is given in §1.1). The theorem is also of great depth in that it relates to a large body of mathematics. The aim of this book is to explore these relations, which provide much insight into several important mathematical topics.

The topics in question are Poncelet's theorem, billiards in an ellipse, and double queues. At first sight these topics seem unrelated, belonging to three distinct mathematical fields: geometry, dynamical systems, and probability. But there is a hidden thread tying these topics together: the existence of an underlying structure (we name it the Poncelet correspondence $\mathcal{M}$ (see §1.1)) which turns out to be an elliptic curve. As is well known, elliptic curves can be endowed with a group structure, and the exploitation of this structure sheds much light on the aforementioned topics.

The only prerequisites for reading this book are the following standard subjects covered in undergraduate and first year graduate mathematics courses: complex analysis, linear algebra, and some point set topology.

The book is organized as follows. Chapter 1 gives a description of the main topics of the book (these topics are discussed in Parts III and

IV). Chapters 2–14 are divided into four parts. These are followed by the supplementary Chapter 15, "Billiards and the Poncelet Theorem", by S. Tabachnikov. There are also five appendices. The purpose of these is to fill in several omissions from Parts I–IV.

Parts III and IV form the core of the book. Part III discusses the theorems of Poncelet and Cayley (the latter is explained in §1.1) and is based on the approach used in the papers of Griffiths and Harris [GH1], [GH2]), which relates these theorems to elliptic curves (over the complex field) and to their parameterization by elliptic functions. Another approach, using notions from dynamical systems, is also presented here. The papers [GH1], [GH2] take for granted various algebro-geometric notions. Part III explains and elaborates on these notions.

Part IV discusses billiards in an ellipse and double queues and is based on papers which I authored and co-authored ([Fl], [FH]). The ideas in these papers are reworked and further developed in Part IV. Furthermore, the presentation in Part IV displays a fundamental connection between these topics and Poncelet's theorem. Indeed the topic of double queues, which appears last in the book, could very well have been the first, for it is the study of double queues that led me to the surprising connection with Poncelet's theorem.

Chapter 15, written by S. Tabachnikov, gives an expository account of mathematical billiards and demonstrates how this theory provides an alternative proof of Poncelet's theorem. In addition, the theory provides another proof of the recent result by R. Schwartz on Poncelet grids (see §15.3).

Chapter 15 also discusses recent developments connecting Poncelet's theorem to other mathematical topics: geodesics on an ellipsoid and dual billiards.

The topics appearing in Parts I and II are collected from various sources. Most of the material is fairly standard, a notable exception being the discussion in Chapter 8 on division points on elliptic curves, the treatment being a modification of the one given in [GH2]. Part I deals with projective geometry, and Part II deals with complex analysis. The treatment of topics presented in Parts I and II is by no means complete, nor is it intended to be. Rather, the choice of material presented in these parts is dictated by the desire to make Parts III and IV accessible to as wide an audience as possible. The knowledgeable reader can immediately proceed to Parts III and IV. The same advice is offered to the less knowledgeable reader, who can occasionally turn back to Parts I and II, as the need arises.

It is of course possible to discuss projective geometry with coordinates coming from any preassigned field, but we consider only the complex projective plane and, occasionally, the real projective plane. The reason for this restriction is that the results of Parts III and IV make use of complex analysis.

In conclusion, many topics are treated in the book, all relating to Poncelet's theorem. In this sense, the approach of this book follows the maxim of the Talmudic sage Abaye "from topic to topic, yet always in the same topic" (Babylonian Talmud, Tractate Kiddushin, p. 6a). The proof of Poncelet's theorem reveals deep connections between the seemingly disparate subjects treated in this book. It is this aspect of Poncelet's theorem that has drawn me to a detailed study of it and its ramifications. The book demonstrates that Poncelet's theorem serves as a prism through which one can learn and appreciate a lot of beautiful mathematics.

Acknowledgments. I express my thanks and gratitude to the following people who contributed to the writing of this book, both on the personal and professional levels.

First and foremost, gratitude goes to my late beloved wife, Zehava, the driving force behind the book. Without her constant support and encouragement, the book would not have "seen the light of day". Then I thank my children, Sharon and David, for their interest and gentle goading, constantly inquiring, "So what is happening with the book?" Likewise, I thank my friends Joel Gribetz and Daniel Lasker, with whom I have renewed a friendship after a lapse of more than fifty years, due to cataclysmic circumstances beyond our control.

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The book is an outgrowth of a series of lectures given at the NSA, where I spent a sabbatical year during 1998–1999. I thank Mel Currie for organizing a seminar on the subject matter. Special thanks goes to Harvey Cohn and Donald Newman for participating in the entire lecture series and for offering insightful comments.

List of Commonly Used Symbols

$:=$	symbol used to indicate that the left-hand side is defined by the right-hand side
$\{p\colon p \text{ has property } A\}$	set of elements p having property A
iff	if and only if
$\blacksquare$	q.e.d.
$\mathbb{Z}$	set of integers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\widehat{\mathbb{C}}$	complex sphere
$\operatorname{Re} z$, $\operatorname{Im} z$	real and imaginary parts of complex number z
$\lvert z\rvert$	absolute value of complex number z
$\arg z$	argument of complex number $z \neq 0$
$\det A$ or $\lvert A\rvert$	determinant of square matrix A
$f\rvert_A$	restriction of function f to set A
$f \circ g$	composition of the two mappings f and g
f^{-1}	transformation inverse to f
$a \in A$	a is a member of set A
$a \notin A$	a is not a member of set A
$f(A)$	image of set A under the mapping f

$\sharp A$ or $\|A\|$	cardinality of A = number of elements in A
$A \subset B$	A is a subset of B
$A \cup B$	union of A and B = set of elements either in A or in B
$A \cap B$	intersection of A and B = set of elements in both A and B
$A \backslash B$	set of elements in A and not in B
$A \times B$	product of A and B = $\{(a, b) \colon a \in A, b \in B\}$
$\emptyset$	the empty set

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Poncelet's theorem is a famous result in algebraic geometry, dating to the early part of the nineteenth century. It concerns closed polygons inscribed in one conic and circumscribed about another. The theorem is of great depth in that it relates to a large and diverse body of mathematics. There are several proofs of the theorem, none of which is elementary. A particularly attractive feature of the theorem, which is easily understood but difficult to prove, is that it serves as a prism through which one can learn and appreciate a lot of beautiful mathematics.

This book stresses the modern approach to the subject and contains much material not previously available in book form. It also discusses the relation between Poncelet's theorem and some aspects of queueing theory and mathematical billiards.

The proof of Poncelet's theorem presented in this book relates it to the theory of elliptic curves and exploits the fact that such curves are endowed with a group structure. The book also treats the real and degenerate cases of Poncelet's theorem. These cases are interesting in themselves, and their proofs require some other considerations. The real case is handled by employing notions from dynamical systems.

The material in this book should be understandable to anyone who has taken the standard courses in undergraduate mathematics. To achieve this, the author has included in the book preliminary chapters dealing with projective geometry, Riemann surfaces, elliptic functions, and elliptic curves. The book also contains numerous figures illustrating various geometric concepts.

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