Intrinsic Geometry of Surfaces

A.D. Aleksandrov
and
V.A. Zalgaller

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А. Д. АЛЕКСАНДРОВ и В. А. ЗАЛГАЛЛЕР

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(Foundations of the intrinsic geometry of surfaces)

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INTRODUCTION

The present monograph is devoted to a systematic presentation of the intrinsic geometry of nonregular surfaces. The work of Gauss on intrinsic geometry deals only with regular surfaces. The Riemannian geometry of abstract metrized manifolds is also restricted to those admitting a regularly defined metric. The construction of the geometry of nonregular convex surfaces was achieved by A. D. Aleksandrov in [5]. In the concluding section of that monograph a program was indicated for the construction of the intrinsic geometry of general surfaces. What was wanted was a system of concepts and methods which would be applicable equally to the investigation of the intrinsic geometry of regular surfaces and of two-dimensional Riemannian manifolds, polyhedra and polyhedral evolutes, general convex surfaces, and also to the investigation of the widest possible class of nonconvex nonregular surfaces and metrized two-dimensional manifolds.

In the papers [6], [14], [17], [46], this program was worked out in more detail. The principal results were communicated without proofs. Under some additional hypotheses, A. D. Aleksandrov succeeded in investigating two-dimensional metric spaces by geometric methods, and in particular in introducing into them the concepts of shortest and geodesic curves, angles, integral curvature, area, direction and rotation of curves, and in establishing the most important properties of these concepts.

These spaces have been called "two-dimensional manifolds of bounded curvature." They are the natural closure of the class of Riemannian spaces. This extended class is obtained by adding to the two-dimensional Riemannian spaces all two-dimensional metrized manifolds whose metric in the neighborhood of each point may be uniformly approximated by Riemannian metrics such that the integrals of the absolute values of the Gaussian curvatures are uniformly bounded.

The types of spaces and surfaces enumerated above belong to this class. At the same time the above class of surfaces admits common methods of investigation. These extend techniques developed by A. D. Aleksandrov for convex surfaces. Among them are the following.

1. An axiomatic method, starting from the definition of such spaces by means of a minimal choice of the properties of their metrics.

2. An approximative method, based on approximation by polyhedral or Riemannian metrics. This method makes use of theorems on the

possibility of appropriate approximations of the spaces themselves and of figures in them by simpler spaces and figures, and also of general theorems on the connection between the numerical characteristics of the converging figures and of the limiting figure.

3. A synthetic method based on geometric constructions in such spaces and a study in them of curves, triangles and other figures. This method makes use in particular, of the comparison of figures in such a space with similar figures on the plane and of methods such as the cutting and pasting of new spaces from pieces of existing spaces.

The results obtained, for all their generality, retain their geometric intuitiveness. The fact that the class of spaces in question is closed makes it possible to state and solve extremal problems in that class in a natural way. The basic restriction adopted turns out to be completely natural. The integral curvature characterizes the deviation of the intrinsic geometry of the surface from Euclidean geometry, and the restriction on the integral curvature makes it in fact possible to retain the basic integral concepts of classical differential geometry.

The actual construction of a theory of two-dimensional manifolds of bounded curvature is the object of the present work.

An exposition beginning with a small number of initial axioms requires a gradual accumulation of facts. Analogously, if we make use only of the possibility of approximation by polyhedral metrics, we need information about those polyhedral metrics. Therefore the exposition is carried out cyclically. Certain results are established at first in less than their full extent or in special cases, and later they are extended to more general, definite results.

A large number of later papers by Soviet geometers deal with the material of this monograph. The authors intend to prepare for publication a collection of papers in the directions indicated in §6 of Chapter I.

COMPLEMENTS TO CHAPTER II

On Various Definitions of the Angle

We consider a series of possible definitions of angle. All of them characterize the rapidity of departure from one another of curves issuing from a common point, and in the case of regular curves in Euclidean (or Riemannian) space lead, as a rule, to the usual values of the angle. The situation is different in more complicated spaces. In Chapter II we needed only two concepts, the upper angle and the lower strong angle between shortest arcs. The material presented below will show why preference was given to these two definitions. Further, these materials may be used for other kinds of generalized expositions of the theory.

1. *Triangle on a K-plane.* Suppose in an arbitrary metric space that there issue from the point O two curves $L = X(t)$ and $M = Y(s)$. We select points X and Y on them distinct from O. Suppose further that

(1) $$\rho(O, X) = x, \quad \rho(O, Y) = y, \quad \rho(X, Y) = z.$$

In subsection 4 of Chapter I, in introducing the concept of the angle between L and M, we made use of the auxiliary angle $\gamma(X, Y)$, constructing on the plane a triangle T_0 with sides x, y and z and considering in it the angle γ opposite the side z. But we could have constructed instead of T_0 a triangle T_K with the sides x, y, z on a surface with an arbitrarily fixed constant curvature K. We shall call such a surface a K-plane.[1] Here the angle $\gamma_K(X, Y)$ opposite the side z will be quite different from $\gamma(X, Y) = \gamma_0(X, Y)$.

As is known from differential geometry, for the angles $\alpha_K, \beta_K, \gamma_K$ and $\alpha_0, \beta_0, \gamma_0$ of the triangles T_K and T_0 there is the equation

(2) $$(\alpha_K - \alpha_0) + (\beta_K - \beta_0) + (\gamma_K - \gamma_0) = \sigma K,$$

where σ is the area of the triangle T_K. We need to add to this that all three differences in parentheses on the left side of (2) are either simultaneously equal to zero (for $K = 0$ or $\sigma = 0$), or have the same sign as the quantity K. This last elementary assertion may be verified on the example of the angles γ_K and γ_0, starting from the explicit expressions for the

[1] If $K \leqq 0$ the triangle T_K exists, since $x, y,$ and z satisfy the triangle inequality. If $K = k^2 > 0$ it may be constructed under the conditions $kx, ky, kz < \pi$, $kx + ky + kz \leqq 2\pi$. These conditions will be supposed satisfied when we speak of constructing a triangle on the corresponding K-plane.

cosines of these angles. If $K = -k^2 < 0$ and x and y are fixed and arbitrary, then for all z in the interval $|x-y| \leqq z \leqq x+y$ we have

$$\frac{\cosh kx \cosh ky - \cosh kz}{\sinh kx \sinh ky} - \frac{x^2 + y^2 - z^2}{2xy} \geqq 0.$$

For $K = k^2 > 0$ and arbitrary fixed $0 < kx,\ ky < \pi$, for all z in the interval $|x-y| \leqq z \leqq x+y$ we have

$$\frac{\cos kz - \cos kx \cos ky}{\sin kx \sin ky} - \frac{x^2 + y^2 - z^2}{2xy} \leqq 0.$$

Equality here is attained only at the endpoints of the interval of variation of z. Therefore

(3) $$|\gamma_K - \gamma_0| \leqq \sigma |K|.$$

Thus it follows that if we are interested in the limiting values of the angles $\gamma_K(X, Y)$ for sequences of points X, Y for which the area $\sigma(T_K) \to 0$, then it makes no difference whether we consider the angles $\gamma_K(X, Y)$ or $\gamma_0(X, Y)$.

REMARK. Lemma 1 of Chapter I remains valid for the angles γ_K:

(4) $$\cos \gamma_K = \frac{y - z}{x} + \varepsilon,$$

where $\varepsilon \to 0$ as $x/y \to 0$, with the additional requirement that $\sigma(T_K) \to 0$. For $K > 0$ the condition $\sigma \to 0$ follows from $x/y \to 0$, since in this case y is supposed bounded ($y\sqrt{K} < \pi$). For $K < 0$, for $\sigma \to 0$ it suffices that not only x/y but also $x \to 0$.

2. ***Upper and lower angles.*** The lower, upper and ordinary angles $\alpha_-, \bar{\alpha}$ and α between L and M were defined in Chapter II respectively as the lower, upper and ordinary limits of the angles $\gamma(X, Y)$ as $X, Y \to O$, $X \in L$, $Y \in M$, $X \neq O$, $Y \neq O$. Evidently $0 \leqq \alpha_- \leqq \bar{\alpha} \leqq \pi$. The angle α exists when $\alpha_- = \bar{\alpha}$.

The properties of the angle $\bar{\alpha}$ were considered in § 1, Chapter I.

The essential difference between the lower angle α_- and the upper angle $\bar{\alpha}$ is connected with the asymmetry of the basic triangle inequality. For lower angles assertions of the type of the theorems of Chapter II do not hold. In connection with this, in Chapter II the more complicated concept $\alpha_{(-)s}$ was investigated. We restrict ourselves to an example connected with Theorems 5 and 6.

EXAMPLE. Suppose that in the plane curvilinear triangle depicted in Figure 127 the lengths of the convex curves AB, AC and of the straight side BC are equal to l, and angle $\phi<\pi/3$. We construct an abstract space, consisting of the three threads $AB=BC=CA=l$ and an infinite number of other threads joining pairwise the points $X\in AB$ and $Y\in AC$ and having the same lengths as the corresponding shortest arc XY in Figure 127. Along the lengths of the curves in this abstract space we introduce an intrinsic metric. We may vertify that in the resulting space ABC is a triangle. In it at the vertex A

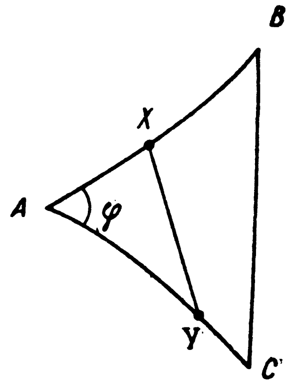

Figure 127.

$$\alpha_- = \phi, \quad \alpha_0 = \frac{\pi}{3}, \quad \alpha_- - \alpha_0 < 0.$$

At the same time for any triangle AXY

$$\delta_-(AXY) = (\phi + \pi + \pi) - \pi = \pi + \phi.$$

Thus for the quantity

$$\nu_A^- = \inf_{\substack{X\in AB \\ Y\in AC}} \delta_-(\overline{AXY})$$

we do not have a relation of the type of Theorem 6.[2]

$$\alpha_- - \alpha_0 \not\geqq \nu_A^-.$$

3. ***Angle in the weak sense.*** The limit of the angles $\gamma(X,Y)$ may be considered under additional restrictions of the possible situations of the points X, Y. Sometimes it is comparatively easy to follows the value of $\gamma(X,Y)$ under the condition that the ratio of the distances x and y of the points X and Y from O remains within limits:

$$0 < a \leqq \frac{x}{y} \leqq b < \infty.$$

We shall call the upper weak angle and the lower weak angle the following limits, which always exist and do not depend on a and b:

$$\bar{\alpha}_W = \lim_{\substack{a\to 0 \\ b\to\infty}} \lim_{\substack{X,Y\to O \\ 0<a\leqq x/y\leqq b<\infty}} \sup \gamma(X,Y), \tag{5}$$

$$\alpha_{(-)W} = \lim_{\substack{a\to 0 \\ b\to\infty}} \lim_{\substack{X,Y\to O \\ 0<a\leqq x/y\leqq b<\infty}} \inf \gamma(X,Y). \tag{6}$$

As before we consider only points $X\in L$, $Y\in M$, $X\neq O$, $Y\neq O$.

[2] This example answers the question set in [13], the footnote on page 8. However it is not clear whether one can find an analogous example in a space which is a two-dimensional manifold.

Suppose that $\alpha_{(-)W} = \bar{\alpha}_W$, *i.e.* for any $0 < a \leqq b < \infty$ there exists the limit

$$\alpha_W = \lim_{\substack{X, Y \to O \\ 0 < a \leqq x/y \leqq b < \infty}} \gamma(X, Y), \tag{7}$$

which thus will not depend on the choice of a and b. Then its value $\alpha_W = \alpha_{(-)W} = \bar{\alpha}_W$ is called the ***weak angle,*** or the angle in the weak sense.

We may further consider, so to speak, the "weakest" upper, lower, and simple angles $\bar{\alpha}_{WW}$, $\alpha_{(-)WW}$, α_{WW}, imposing the more rigid condition $x = y$. Evidently

$$0 \leqq \alpha_- \leqq \alpha_{(-)W} \leqq \alpha_{(-)WW} \leqq \bar{\alpha}_{WW} \leqq \bar{\alpha}_W \leqq \bar{\alpha} \leqq \pi.$$

THEOREM 1. ***If each of two curves has a definite direction, then the weak upper angle between them is equal to the upper angle*:**

$$\bar{\alpha}_W = \bar{\alpha}.$$

PROOF. Since always $\bar{\alpha}_W \leqq \bar{\alpha}$, it suffices to show that under the conditions of the theorem $\bar{\alpha} \leqq \bar{\alpha}_W$.

We choose on the curves L and M in question points X_n, Y_n with $\gamma(X_n, Y_n) \to \bar{\alpha}$ and converging to O. If in addition $0 < a \leqq x_n/y_n \leqq b < \infty$, then $\bar{\alpha} \leqq \bar{\alpha}_W$. Suppose that $X_n/Y_n \to O$ (if $X_n/Y_n \to \infty$, we change the names of x and y). On M we mark points Y_n' for which $x_n/y_n = a$, where $a > 0$ is an arbitrarily small number. We construct on the plane the corresponding triangle OXY' with sides x_n, y_n', z_n'. To its side OY' we adjoin another triangle $OY'Y$ with the sides $y_n', y_n, Y_n'Y_n$, as in Figure 128.

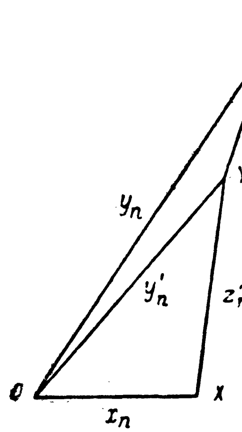

***Figure* 128.**

Now it is easy to explain why $\bar{\alpha} \leqq \bar{\alpha}_W$. The angle at the vertex O in the plane triangle OXY' cannot for large n essentially exceed $\bar{\alpha}_W$. The angle $Y'OY$ is small, since the curve M has a direction. Finally, $z_n = X_nY_n \leqq X_nY_n' + Y_n'Y_n$. But because of the smallness of $a = x_n/y_n$ even on rectification of the side $YY'X$ in the plane quadrilateral $OYY'X$ the angle at the vertex O cannot essentially increase, i.e., $\gamma(X_n, Y_n)$, and therefore also $\bar{\alpha}$ cannot essentially exceed $\bar{\alpha}_W$.

Let us make this more precise. By the choice of X_n, Y_n and Lemma 1 of Chapter II,

$$\cos \bar{\alpha} = \lim_{n \to \infty} \frac{y_n - z_n}{x_n}.$$

Since M has a direction, the angle $\gamma(Y_n, Y_n') \to 0$, *i.e.*, $(y_n - Y_n'Y_n)/y_n' \to 1$ or $Y_n'Y_n = y_n - y_n' + \varepsilon_n y_n'$, where $\varepsilon_n \to 0$. Finally,

$$z_n \leqq X_n Y_n' + Y_n' Y_n = z_n' + y_n - y_n' + \varepsilon_n (x_n/a),$$

so that

$$\frac{y_n - z_n}{x_n} \geqq \frac{y_n' - z_n'}{x_n} - \frac{\varepsilon_n}{a}.$$

Thus again using Lemma 1 of Chapter II, this time with the sharpening (6) of subsection 3 of Chapter II, we have:

$$\cos \bar{\alpha} \geqq \limsup_{n\to\infty} \frac{y_n' - z_n'}{x_n} \geqq \limsup_{n\to\infty} \left[\cos \gamma(X_n, Y_n') - \frac{1}{2}\frac{x_n}{y_n'}\right]$$

$$\geqq \liminf_{\substack{X\to O \\ a\leqq x/y\leqq b}} \cos \gamma(X, Y) - \frac{a}{2} = \cos\left[\limsup_{\substack{x\to 0 \\ a\leqq x/y\leqq b}} \gamma\ (X, Y)\right] - \frac{a}{2}.$$

But $a > 0$ may be taken arbitrarily small and b arbitrarily large. Therefore $\cos \bar{\alpha} \geqq \cos \bar{\alpha}_W$ and $\bar{\alpha} \leqq \bar{\alpha}_W$. The theorem is proved.

REMARKS. 1) Theorem 1 essentially complements Theorem 4 of Chapter II, establishing the still greater stability of the upper angle.

2) For the lower angle an assertion analogous to the theorem just proved does not hold. *Example.* Compare the plane sector bounded by the arcs L and M with the acute angle ϕ in a conical trough (Fig. 129). We join individual points of its boundary in space by segments A_1B_1, $A_2B_2, \cdots$, and suppose that the angle of inclination of these segments to the straight line L tends to zero as they approach the vertex O. In the metric space which the cone along with the adjoined threads A_iB_i represents, the arcs L and M are shortest arcs. The weak angle between them exists and is equal to the complete angle ϕ of the sector, and the lower angle may be obtained starting from the sequence of points A_iB_i. It is equal to the space angle between L and M, which is less than ϕ. In this example $\alpha_- < \alpha_{(-)W} = \bar{\alpha}_W = \bar{\alpha} = \phi$.

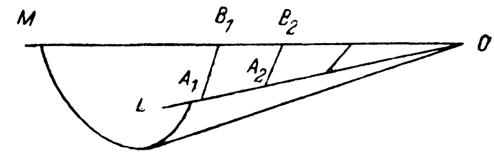

***Figure* 129.**

3) If the ordinary angle exists, then the weak angle exists and coincides with it. But curves may form a weak angle but not an ordinary one. This is shown by the last example. Here are other examples. Suppose that the plane spiral L (Figure 130) forms infinitely many loops as it

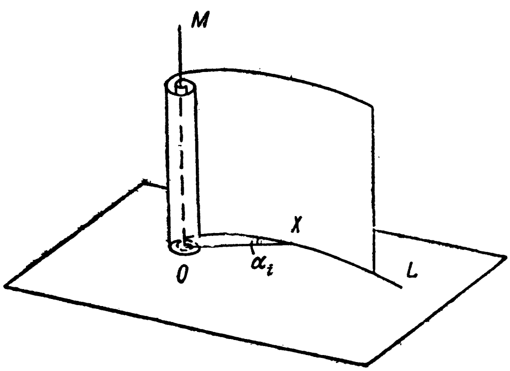

Figure **130.**

approaches the center O. Each ith loop is a piece of a logarithmic spiral on which the segment OX forms with L the angle α_i. If $\alpha_i \to 0$ as $i \to \infty$, then as is easily verified, the spiral L forms at O with itself the weak angle $\alpha_W = 0$. But in the ordinary sense L does not have a direction at O. The curve given on the plane by the equation $y = x \sin \ln |\ln x|$ has the same property at the point (0,0).

4) The concept of weak angle is used for example in the paper [31].

5) Theorem 1 generally speaking ceases to be true if one of the curves does not have a definite direction. *Example.* We erect perpendiculars at the points of the plane spiral L of Figure 130. In the metric space consisting of the plane in which the spiral L lies and the resulting cylindrical surface, we consider the angle at the point O between L and the perpendicular M. In this example, as is easily verified,

$$\pi/2 = \alpha_- = \alpha_{(-)W} = \bar{\alpha}_W < \bar{\alpha} = \pi.$$

4. *Extended angle.* One may consider the limit of the angles $\gamma(X, Y)$ under widened possibilities for the positions of X and Y. Suppose that X and Y do not necessarily lie on the curves L and M, but rather that as they approach zero the distances from X and Y to L and M respectively go down faster than the distance from O:

$$\frac{\rho(X, L)}{\rho(X, O)} \to 0, \quad \frac{\rho(Y, M)}{\rho(Y, O)} \to 0.$$

The upper and lower limits of the angles $\gamma(X, Y)$ for all possible such sequences $X, Y \to O$ will be called the upper and lower extended angles $\bar{\alpha}_E, \alpha_{(-)E}$. If $\alpha_{(-)E} = \bar{\alpha}_E$, their common value is called the extended angle between L and M. Obviously, it is always true that

$$0 \leqq \alpha_{(-)E} \leqq \alpha_- \leqq \bar{\alpha} \leqq \bar{\alpha}_E \leqq \pi.$$

REMARK. We give examples when $\alpha_{(-)E} < \alpha_-$ or $\bar{\alpha} < \bar{\alpha}_E$.

1) Suppose that L and M are rays issuing from O on the plane, forming

an acute angle ϕ, and N is a curve lying in the same plane tangent to M from within at the point O. From the point Y_1 on the curve N we drop a perpendicular Y_1A_1 onto M. We choose $X_1 \in L$ so that $X_1Y_1 + Y_1A_1 = X_1O + OA_1$. Then we choose the point $Y_2 \in N$ very much closer to O than X_1 and Y_1. We repeat this construction as in Figure 131. In the intrinsic geometry of the figure consisting only of the threads $L, M, N, A_1Y_1, A_2Y_2, \cdots$ we will have $\alpha_{(-)\mathrm{E}} \leqq \phi < \pi = \alpha_-$ for the angle between L and M.

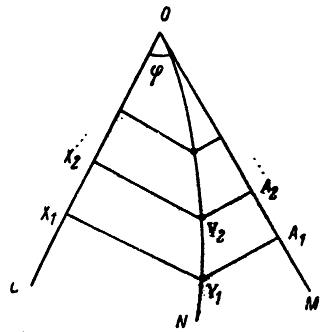

Figure **131.**

2) Consider the plane sector LOM with acute angle ϕ and a convex arc N lying in the same plane and tangent to M from outside at the point O, as in Figure 132. From the point $Y_1 \in N$ we drop a perpendicular Y_1A_1 onto M. On L we choose a point X_1 so close to O that $Y_1A_1 + A_1X_1 > \widehat{Y_1O} + OX_1$. This is possible since by the convexity of the arc $\widehat{Y_1O} < Y_1A_1 + A_1O$. Then we choose a point $Y_2 \in N$ very much closer to O, and repeat the construction, and so forth. In the intrinsic geometry of the figure made up of the plane sector LOM and the threads N, A_1Y_1, A_2Y_2, for the angle between L and M we will have $\bar{\alpha} = \phi < \pi = \bar{\alpha}_{\mathrm{E}}$.

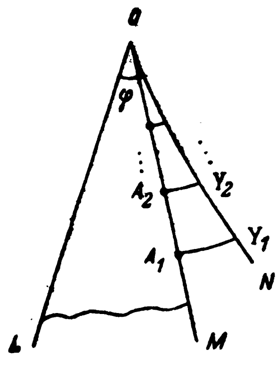

Figure **132.**

5. *Nonlocal characteristics of the angle between shortest arcs.* In the definition of the angle in the strong sense (§2 of Chapter II), we considered, for the shortest arcs $L = OX_0$, $M = OY_0$ the lower and upper limits $\alpha_{(-)\mathrm{S}}, \bar{\alpha}_{\mathrm{S}}$ of the angles $\gamma(X, Y)$, taken for all possible sequences of points X_n, Y_n for which

a) $X_n \in L$, $Y_n \in M$, $X_n \neq O$, $Y_n \neq O$, $X_n \to O$ or $Y_n \to O$;

b) if $X_n \to O$ there exist shortest arcs $\overline{X_nY_n}$ converging to a piece of the shortest arc M, and if $Y_n \to O$ to a piece of L (we suppose that at least one such sequence X_n, Y_n exists).

The upper weak angle coincides with $\bar{\alpha}$, and the lower $\alpha_{(-)\mathrm{S}}$ is a nonlocal characteristic of the angle.

This last insufficiency is also suffered by the always existing quantity $\alpha_{=}$, defined as the lower limit of the angles $\gamma(X, Y)$, taken over all possible sequences X_n, Y_n satisfying condition a) without condition b).

To a certain extent the abovementioned insufficiency is excluded if one turns from $\alpha_{(-)S}$ to the characteristic $\alpha_{(=)S}$, defined as the greatest lower bound of the values of $\alpha_{(-)S}$ for all possible pairs of shortest arcs, which on arbitrarily small initial segments coincide with L and M. This characteristic was used in [5].

Analogously one may define $\alpha_{\equiv}$ as the lower bound of $\alpha_{=}$ for various extensions of the initial segments of L and M.

The definitions of the quantities $\bar{\alpha}$, $\bar{\bar{\alpha}}_S$, $\bar{\bar{\alpha}}$ are obtained by a replacement of the lower limit by the upper limit and of the greatest lower bound by the least upper bound. But by Theorem 3 of Chapter II all these coincide with $\bar{\alpha}$.

REMARKS. 1) For the angle $\alpha_{=}$ an assertion holds which is similar to Lemma 3 of Chapter II. In fact,

$$\left(\frac{\partial z}{\partial x}\right)_L \leqq \cos \alpha_{=}. \tag{9}$$

2) Assertions of the type of Lemmas 5, 6 and 7 of Chapter II and Theorem 6 of Chapter II are also valid. But this time

$$\alpha_{=} - \alpha_0 \geqq \nu_{(\bar{=})A}, \tag{10}$$

where

$$\nu_{(\bar{=})A} = \inf_{\substack{X\in AB \\ Y\in AC}} \left[\sup_{\overline{XY}} \delta_{=} (\overline{AXY})\right]. \tag{11}$$

3) All these assertions are proved analogously, and even somewhat more simply than the corresponding theorems of Chapter II. However, the angle $\alpha_{=}$ cannot replace the angle $\alpha_{(-)S}$ in the construction of the theory of two-dimensional manifolds of bounded curvature. In these spaces there exists an angle in the strong sense between shortest arcs, but the characteristic $\alpha_{=}$ may fail to coincide with this angle.

4) We give a simple example when $\alpha_{=} < \alpha_{(-)S}$. Consider on the sphere two shortest arcs L and M issuing from the point O. Suppose that they form at O an acute angle ϕ, with the opposite ends of the shortest arcs coinciding and lying at a point diametrically opposite to O. For such shortest arcs $\alpha_{=} = 0 < \phi = \alpha_{(-)S}$.

If in this last example we somewhat shorten L and M, we will have an example in which $\alpha_{\equiv} = 0 < \phi = \alpha_{=}$.

5) In the case of the so-called manifolds of nonpositive curvature, in which all the excess $\bar{\delta} \leqq 0$ or of "negative curvature not greater than K" (see [13], § 4) for shortest arcs always $\alpha_{=} = \bar{\alpha}$, so that there exists an angle in this more extended sense.

6. *Relations of the various definitions of angle.*

LEMMA. *For the shortest arcs L and M in a locally compact space with intrinsic metric, always* $\alpha_{(-)\mathrm{E}} \leqq \alpha_{(-)\mathrm{S}}$.

PROOF. Consider a sequence X_n, Y_n for which $\gamma(X_n, Y_n) \to \alpha_{(-)\mathrm{S}}$, $\rho(O, X_n) = x_n \to 0$, $\overline{X_n Y_n} \to \overline{OY} \subset M$. If moreover $\rho(O, Y_n) = y_n \to 0$ then evidently $\alpha_{(-)\mathrm{E}} \leqq \alpha_{(-)\mathrm{S}}$. Suppose that $y_n \geqq a > 0$. Then on the shortest arcs $\overline{X_n Y_n}$ one may select respectively points Y_n' such that $\rho(Y_n', M)$ and x_n decrease faster than $y_n' = \rho(O, Y_n')$. From the triangle OY_nY_n' we have

$$y_n' + (z_n - z_n') \leqq y_n,$$

where $z_n = \rho(X_n, Y_n)$, $z_n' = \rho(X_n, Y_n')$. Therefore

$$\cos \alpha_{(-)\mathrm{S}} = \lim \frac{y_n - z_n}{x_n} \leqq \limsup \frac{y_n' - z_n'}{x_n} \leqq \cos \alpha_{(-)\mathrm{E}}.$$

This proves the lemma in question. Analogously one may verify that $\alpha_{(-)\mathrm{E}} \leqq \alpha_{(=)\mathrm{S}}$. Directly from the definitions and also from Theorem 3 of Chapter II and Theorem 1 of this supplement and the last lemma it results that the following theorem is valid.

THEOREM 2. *In a locally compact space with an intrinsic metric, for the various characteristics of the angle between two shortest arcs the following relations are valid:*

$$\left.\begin{matrix} 0 \leqq \alpha_{\equiv} \leqq \alpha_{=} \\ 0 \leqq \alpha_{(-)\mathrm{E}} \leqq \alpha_{(=)\mathrm{S}} \end{matrix}\right\} \leqq \alpha_{(-)\mathrm{S}} \leqq \alpha_{-} \leqq \alpha_{(-)\mathrm{W}} \leqq \alpha_{(-)\mathrm{WW}} \leqq \bar{\alpha}_{\mathrm{WW}} \leqq \bar{\alpha}_{\mathrm{W}} \tag{12}$$

$$= \bar{\alpha} = \bar{\alpha}_{\mathrm{S}} = \bar{\bar{\alpha}}_{\mathrm{S}} = \bar{\bar{\alpha}} = \bar{\bar{\bar{\alpha}}} \leqq \bar{\alpha}_{\mathrm{E}} \leqq \pi.$$

For each sign "$\leqq$" in the chain of relations (12) one may present an example in which the strict inequality is realized. In subsection 3 the example for $\alpha_{-} < \alpha_{(-)\mathrm{W}}$ was given, in subsection 4 examples for $\alpha_{(-)\mathrm{E}} < \alpha_{-}$, $\bar{\alpha} < \bar{\alpha}_{\mathrm{E}}$, and in subsection 5 $\alpha_{=} < \alpha_{(-)\mathrm{S}}$, $\alpha_{\equiv} < \alpha_{=}$. We shall give further an example in which $\alpha_{(-)\mathrm{S}} < \alpha_{-}$.

In the plane sector LOM with acute angle ϕ we mark points $A_i \to O$ which approach the side L faster than O, as in Figure 133. Along cuts along the segments YA_i we paste high, twice-covered partitions. On the resulting surface the angle at the point O between L and M will satisfy

$\phi = \alpha_{(-)S} < \alpha_- = \pi$.

The construction of the missing examples is left to the reader.

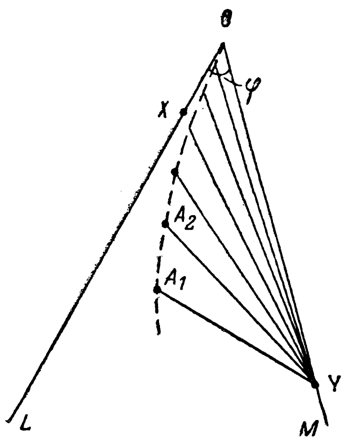

***Figure* 133.**

7. *Comparison with a triangle on a K-plane.* The excess of a triangle may be measured by the difference of the sum of its angles (in one or another of the definitions) from the sum of the angles of the triangle with sides of the same length on a K-plane. For such "relative" excesses $\bar{\delta}_K(T)$, $\delta_{(-)SK}(T)$, $\delta_{(=)K}(T)$ it is possible to define the corresponding quantities ν analogously to the definitions (20), (27) of Chapter II or (12) of this supplement. We shall denote them by the supplementary index K.

For angles γ_K on a K-plane Lemmas 4, 5 and 6 of Chapter II are valid with the following alterations.

1. In Lemma 4 formula (12) is replaced by the following:

$$\frac{\Delta z}{\Delta x} = \cos \xi_K + \frac{x}{\lambda} \frac{\Delta \gamma_K}{\Delta x} \sin \xi_K + \varepsilon, \tag{13}$$

where

$$\lambda = \begin{cases} \dfrac{kx}{\sin kx} & \text{if } K = k^2 > 0, \\ 1 & \text{if } K = 0, \\ \dfrac{kx}{\sinh kx} & \text{if } K = -k^2 < 0. \end{cases} \tag{14}$$

For the proof it suffices only in the infinitesimal discussion of Chapter II to have in mind a construction on a K-plane and to replace formula (13) of Chapter II by the law of sines on a K-plane, i.e., one of the three expressions

$$kl = \sin kx \sin \Delta\gamma_K, \quad l = x \sin \Delta\gamma_K, \quad kl = \sinh kx \sin \Delta\gamma_K.$$

We note that it follows from (14) that for any x, if $K < 0$ and all $0 < x < \pi/k$ the quantity λ has a positive minimum for $K = -k^2 < 0$.

2. In Lemma 5 inequalities (14) and (15) are replaced by

$$\left(\frac{\partial \gamma_K}{\partial x}\right)_{LL} \geqq \frac{\cos \bar{\xi} - \cos \xi_K}{\sin \xi_K} \cdot \frac{\lambda}{x}, \tag{15}$$

(16) $$\left(\frac{\partial \gamma_K}{\partial x}\right)_{\mathrm{LUS}} \leqq \frac{\cos \xi_{(-)\mathrm{S}} - \cos \xi_K}{\sin \xi_K} \cdot \frac{\lambda}{x},$$

(17) $$\left(\frac{\partial \gamma_K}{\partial x}\right)_{\mathrm{LU}} \leqq \frac{\cos \xi_{=} - \cos \xi_K}{\sin \xi_K} \cdot \frac{\lambda}{x},$$

where λ is determined by (14).

3. In Lemma 6 of Chapter II the angles γ and ξ_0 are replaced by γ_K and ξ_K. Moreover, the constant M depends this time not only on ε but also on the minimal value of λ, which in its turn depends on K and the upper estimate of the diameter of the triangle.

4. Theorems 4 and 5 of Chapter II take this time the following form:

$$\bar{\alpha} - \alpha_K \leqq \bar{\nu}^{+}_{KA},$$

$$\alpha_{=} - \alpha_K \geqq \nu^{-}_{(=)KA},$$

$$\alpha_{(-)\mathrm{S}} - \alpha_K \geqq \nu^{-}_{(-)\mathrm{S}KA}.$$

The proofs remain as before.

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