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ТЕОРИЯ ВОЛЬТЕРРОВЫХ ОПЕРАТОРОВ
В ГИЛЬБЕРТОВОМ ПРОСТРАНСТВЕ И ЕЕ ПРИЛОЖЕНИЯ

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AUTHORS' PREFACE TO THE ENGLISH EDITION

This book is in some ways a continuation of our first book "Introduction to the theory of linear nonselfadjoint operators", which has already been published in English translation by the American Mathematical Society. Nevertheless, it can be read independently of the former.

We thank the American Mathematical Society for the interest which they have manifested in this book.

We wish to express our sincere thanks to Professor A. Feinstein for his considerable efforts in the translation of this book.

Odessa, Arcadia
June 1, 1969

I. C. GOHBERG
M. G. KREĬN

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PREFACE

This book may be read independently of our earlier book "Introduction to the theory of linear nonselfadjoint operators" (henceforth cited as [GK]). At the same time, the appearance of this book is inseparable from that of the first book, if only because in first getting down to work we had intended to write a review paper, and later, in the course of developing the theme, we had attempted to present it in one compact monograph.

In this initial period (Spring 1959) the abstract theory of triangular integration—the basis of this book—could not have figured in our plans; we had not the slightest inkling of it. In fact, this theory did not then exist.

But in roughly a year this theory had already assumed a leading position in our review paper, which was proposed for the journal Uspehi Matematičeskih Nauk. Although this paper was not published (cf. the preface of [GK]), copies of the manuscript made their way to a number of scientific centers. Evidently this circumstance raised some interest in the embryonic theory. At any rate, in 1960 the number of investigators working on this theory increased noticeably—from three... to four.

In the summer of 1963 we submitted to "Fizmatgiz" a manuscript totalling 37 printer's sheets. For reasons which we shall not go into here, in about a year, by means of a certain amount of "vivisection", we separated out from our manuscript our first book—[GK]. The second volume of our book, however, was delayed. A certain amount of time was needed for its regeneration. But this was only one of the reasons. During the time which had passed considerable progress had been made in the theory of nonselfadjoint operators with continuous spectra. We wanted to show the results of these methods and the ideas in these new investigations. To this end we had decided to write another chapter. When the size of this chapter (after much review) began to endanger the size of the book, we realized that we had actually written several chapters of a new, third book, which ought to be titled "The spectral analysis of operators, close to selfadjoint or unitary ones". At the same time it became clear that the second part of the "trilogy" was completed.

In this book we discuss a new theory. To master it, as any new theory, one has to overcome a certain psychological barrier. In this connection Academician N. N. Luzin wrote, in a letter to a young scientist: "One has to coax scientists from their attitudes in the same way that a brisk May morning coaxes a man from his bed... But such coaxing should be maturely deliberate. And it goes into your scientific work...".[1)]

We should frankly admit that we tried to follow this advice; to this end we adopted certain measures. Let us point out the two most important: 1) we chose the proper editor for the book and secured his consent, 2) we did not miss opportunities to show unexpected and fruitful connections of the new theory with questions from classical analysis.

Thus we have fulfilled our promise, made in the preface to [GK], to continue our account of how these first two books came to be written. Moreover, we hope that this account will be continued and—we would hope—concluded in our third book.

It is pleasent for us to mention that the group of colleagues who assisted us (M. S. Brodskiĭ, S. G. Kreĭn, B. Ja. Levin, V. B. Lidskiĭ, Ju. I. Ljubič, A. S. Markus, V. I. Macaev, L. A. Sahnovič), to whom we have already expressed our profound thanks, which we again confirm, has increased. We are very grateful for the valuable assistance of M. Š. Birman and M. Z. Solomjak, Ju. P. Ginzburg, G. È. Kisilevskiĭ, E. M. Semenov and Ju. L. Šmul'jan.

As in the first volume, our editor F. V. Širokov vigilantly and firmly stood guard over the reader's interests. He frequently made us build platforms on steep slopes to allow the reader a rest, not overburden his memory, and illuminate the path ahead with the pleasant light of heuristics. More generally, in taking upon himself the right to speak in the reader's name, our editor with his characteristic conscientiousness found for himself and for us varied and at times hard work.

However we must admit that ultimately the reader's interests coincide with our own. We wish to express to F. V. Širokov our deep and sincere gratitude for his considerable and friendly assistance.

We are also very much obliged to A. Z. Ryvkin for his considerable work in preparing this book for publication.

Odessa, Arcadia I. C. GOHBERG
July 1966 M. G. KREĬN

[1)] Cf. "Letters of Academician N. N. Luzin" in the collection *Paths in the unknown*, "Sovetskiĭ pisatel'", Moscow, 1960, pp. 522-525.

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APPENDIX

UNICELLULAR OPERATORS AND RELATED ANALYTIC PROBLEMS

Until recently the theory of unicellular operators and the theory of the decomposition of a linear operator into a "sum" of unicellular operators have been beset by a paucity of results and a host of problems. The situation has changed markedly for the better in recent years. Certain of these recent results are so remarkable that we have devoted this Appendix to them. Unfortunately, instead of complete proofs (which would take us far afield) we are forced to restrict ourselves to brief clarifications and comments.

Let us recall that a bounded linear operator A acting in $\mathfrak{H}$ is said to be *unicellular* if for any two invariant subspaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ ($\mathfrak{L}_1 \neq \mathfrak{L}_2$) of A we have either $\mathfrak{L}_1 \subset \mathfrak{L}_2$ or $\mathfrak{L}_2 \subset \mathfrak{L}_1$. As was already noted in §9, Chapter I, by virtue of a well-known theorem of F. Riesz (see [GK], Chapter I) the spectrum of a unicellular operator is always a connected set. From a bound obtained by V. I. Macaev [2] for the resolvent of a linear operator with imaginary component of class $\mathfrak{S}_\omega$, and from earlier results of Ju. I. Ljubič and V. I. Macaev [1] (on the existence, for a closed linear operator whose resolvent has a certain behavior, of invariant subspaces which split the spectrum of the operator) we immediately have the following result.

A) *Every unicellular operator A of the form $A = H + T$, where $H = H^*$ and $T \in \mathfrak{S}_\omega$, has a one-point spectrum, and this point lies on the real axis.*[1)]

[1)] This assertion does not require that the selfadjoint operator H be bounded. More generally, for any (not necessarily unicellular) operator $A = H + T$, where $H = H^*$ (with H, generally speaking, unbounded) and $T \in \mathfrak{S}_\infty$, every *condensation point* of the spectrum of H (i.e. a limit point of the spectrum of H or an eigenvalue of infinite multiplicity) will be a condensation point of the spectrum of A, and conversely (see, for example, Gohberg-Kreĭn [1], §5). If $T \in \mathfrak{S}_\omega$ and the spectrum of A has at least two distinct points, then there exist two subspaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ whose intersection consists solely of the zero vector, which lie entirely in the domain $\mathfrak{D}(H)$ ($= \mathfrak{D}(A)$), and are invariant with respect to A. We mention in passing a remarkable result of V. I. Macaev.

If $T \in \mathfrak{S}_\omega$, and $H = H^*$ has a discrete spectrum (i.e. a spectrum without finite condensation points), then the system of root vectors of the operator $A = H + T$ is complete. If the completely continuous operator T does not belong to $\mathfrak{S}_\omega$, then there exists a self-

On the other hand, if an operator A ($\in \Re$) is unicellular, then obviously so is the operator $A - aI$, where a is an arbitrary number. Therefore in studying the question of the unicellularity of an operator A ($\in \Re$) with $A_{\mathcal{J}} \in \mathfrak{S}_\omega$, we can assume in advance, without loss of generality, that the entire spectrum of A consists of the point $\lambda = 0$. We note further (see [GK], Theorem I.5.2) that for every operator A with imaginary component $A_{\mathcal{J}} \in \mathfrak{S}_\infty$ the set of condensation points of its spectrum coincides with the set of condensation points of its real component $A_{\mathcal{R}}$. Hence if $\lambda = 0$ is the only spectral point of A, then A is completely continuous, i.e. it is a Volterra operator.

In all the sections of this Appendix, except for the third, we shall consider unicellular operators with $A_{\mathcal{J}} \in \mathfrak{S}_1$ (§§1, 2, 4, 5) or with $A_{\mathcal{J}} \in \mathfrak{S}_p$ (§6). Therefore without loss of generality we may restrict ourselves at once, in these considerations, to Volterra operators, and indeed to simple operators (obviously a nonsimple operator is never unicellular).

In §3 we study unicellular contractions T which "differ" from a unitary operator by a nuclear operator: $I - T^*T \in \mathfrak{S}_1$. The results of §3 enable us to interpret the unicellularity of dissipative operators with nuclear imaginary component in a new way.

Let us emphasize that problems of the theory of unicellular operator turn out to be connected with certain deep analytic problems of the theory of inverse problems for canonical differential equations. These problems are of independent interest (see §§4 and 5).

§1. The decomposition of simple dissipative Volterra operators into sums of unicellular operators

1. A linear operator acting in a complex n-dimensional space E_n is unicellular if and only if it has exactly n distinct (nonzero) invariant subspaces.

Taking over the terminology from infinite-dimensional Hilbert spaces to finite-dimensional spaces, we can call an operator A in E_n a *Volterra* operator if its spectrum consists of the single point $\lambda = 0$.

---

adjoint operator H with a discrete spectrum such that the operator $A = H + T$ has no spectrum (i.e. the operator A^{-1} is a Volterra operator).

The first assertion is contained in Theorem 3 of a note by V. I. Macaev [2], and the second assertion was given in his doctoral dissertation.

We should mention that for a general (not necessarily selfadjoint) operator A the condensed spectrum is defined here as the complement of the set $\tilde{\rho}(A)$ of normal points of A (for whose definition see [GK], Chapter I, §2.3).

Let $\mathfrak{L}_1 \subset \mathfrak{L}_2 \subset \cdots \subset \mathfrak{L}_n = E_n$ be the n invariant subspaces of some unicellular Volterra operator A; then $A\mathfrak{L}_n = \mathfrak{L}_{n-1}$, $A\mathfrak{L}_{n-1} = \mathfrak{L}_{n-2}, \cdots, A\mathfrak{L}_2 = \mathfrak{L}_1$. Choosing an arbitrary vector $e_1 \in \mathfrak{L}_n - \mathfrak{L}_{n-1}$ and putting $e_k = A^k e_1$ $(k = 1, 2, \cdots, n)$, we obtain a basis $\{e_k\}_1^n$ of E_n which is *cyclic* with respect to A:

$$e_2 = Ae_1,\ e_3 = Ae_2, \cdots, e_n = Ae_{n-1},$$

and such that $Ae_n = 0$. The matrix of A relative to the basis $\{e_1, e_2, \cdots, e_n\}$ will have all elements equal to zero, with the exception of those lying just below the principal diagonal, which will all equal unity, i.e. it will be a matrix consisting of one Jordan cell with zero principal diagonal.[2)]

The converse is also obvious; if an operator A in E_n has a spectrum consisting solely of zero, and A is cyclic (i.e. there exists a basis of E_n which is cyclic with respect to A), then A is unicellular.

We now recall that an operator $A \in \mathfrak{R}$, acting in a separable infinite-dimensional Hilbert space $\mathfrak{H}$, is said to be *cyclic* if there exists a vector $e \in \mathfrak{H}$ such that the linear hull of the sequence of vectors $\{A^k e\}_0^\infty$ is *dense* in $\mathfrak{H}$. It was proved in §9, Chapter I (Theorem 9.1) that *every unicellular operator* A $(\in \mathfrak{R})$ *is cyclic*.

As was remarked there (without substantiation), this result does not admit a converse. It turns out that in every class $\mathfrak{S}_p$ $(p > 0)$ there exist cyclic Volterra operators which, nonetheless, are not unicellular. In fact, let us consider in the space $\mathfrak{H} = L_2^{(2)}(0,1)$ of vector-functions $f = \{f_1(x), f_2(x)\}$ the Volterra operator A defined by

$$g = \{g_1(x), g_2(x)\} = Af = \left\{ \int_0^x f_1(t)\,dt,\ -\int_0^x f_2(t)\,dt \right\} \quad (0 \leqq x \leqq 1).$$

Let u denote the vector-function from $L_2^{(2)}(0,1)$ whose components are identically equal to unity. Then

$$u^{(n)} = A^n u = \{x^n/n!, (-x)^n/n!\} \qquad (n = 1, 2, \cdots).$$

Since the even powers x^{2n} $(n = 0, 1, \cdots)$, as well as the odd powers x^{2n+1} $(n = 0, 1, \cdots)$, form a complete system of functions in $L_2(0,1)$, it follows that the system of vector-functions $u^{(n)}$ is complete in $L_2^{(2)}(0,1)$. Thus A is a cyclic operator (with generating element u); nevertheless this operator is obviously not unicellular (the subspaces consisting of all elements with first or with second component identically equal to

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[2)] This is the origin of the term "unicellular operator".

zero are invariant subspaces of A).[3)]

The operator A has a two-dimensional real component, and it belongs to the ideal $\mathfrak{S}_\Omega$, and hence to every ideal $\mathfrak{S}_p$ for $p > 1$. Each of its odd powers A^{2l+1} $(l = 0, 1, 2, \cdots)$ is also a Volterra operator, indeed cyclic with the same generating element u (by virtue of the fact that any system of powers x^{a+dn} $(n = 0, 1, \cdots)$ and, in particular, the systems with $a = 0,\ 2l+1,\ d = 2(2l+1)$ are dense in $L_2(0,1)$). Choosing the integer l sufficiently large, we obtain an operator which belongs to $\mathfrak{S}_p$ for any prescribed arbitrarily small $p > 0$.

Nevertheless, it may be that any cyclic *dissipative* Volterra operator is unicellular. At all events, this is the case for cyclic dissipative Volterra operators with nuclear imaginary component; i.e. the following result holds (compare with Theorem I.9.1).

THEOREM 1.1 (G. È. KISILEVSKIĬ [5]). *A simple dissipative Volterra operator with nuclear imaginary component is unicellular if and only if it is cyclic.*

2. As is known, every linear operator in a finite-dimensional space either is unicellular or can be broken up into the direct sum of unicellular operators. G. È. Kisilevskiĭ [5] has generalized this result to dissipative Volterra operators in a Hilbert space $\mathfrak{H}$.

Let $\{\mathfrak{L}_\nu\}_{\nu \in N}$ be a finite or countable set of subspaces of the space $\mathfrak{H}$. We shall say that $\mathfrak{H}$ *breaks up into the quasi-direct sum of the subspaces* $\mathfrak{L}_\nu$ $(\nu \in N)$, and write

$$\mathfrak{H} = \sum_{\nu \in N} \dot{+}\, \mathfrak{L}_\nu, \tag{1.1}$$

if the following two conditions are satisfied:

1) *The subspaces* $\mathfrak{L}_\nu$ $(\nu \in N)$ *are linearly independent* (i.e. for any finite collection of subspaces $\mathfrak{L}_1, \mathfrak{L}_2, \cdots, \mathfrak{L}_m$ from the system $\{\mathfrak{L}_\nu\}$ the equality $f_1 + f_2 + \cdots + f_m = 0$, where $f_k \in \mathfrak{L}_k$ $(k = 1, 2, \cdots, m)$, is possible only for $f_1 = f_2 = \cdots = f_m = 0$).

2) *The linear hull of the subspaces* $\mathfrak{L}_\nu$ $(\nu \in N)$ *is dense in* $\mathfrak{H}$.

If the space $\mathfrak{H}$ can be decomposed into a quasi-direct sum of invariant subspaces $\{\mathfrak{L}_\nu\}_{\tau \in N}$ of some operator A $(\in \mathfrak{R})$, and A_ν $(= A|\mathfrak{L}_\nu)$ is the restriction of A to $\mathfrak{L}_\nu$ $(\nu \in N)$, then we shall say that *the operator A breaks up into the quasi-direct sum of the operators A_ν* $(\nu \in N)$, and write

[3)] This example was communicated to us independently by L. A. Sahnovič and G. È. Kisilevskiĭ.

$$A = \sum_{\nu \in N} \dot{+} A_\nu. \tag{1.2}$$

If A is, say, a Volterra or dissipative operator, or has nuclear imaginary component, then obviously the same is true of each of the operators A_ν ($\nu \in N$) in the decomposition (1.2). If $A_{\mathcal{J}} \in \mathfrak{S}_1$, then it turns out that (1.2) implies

$$\operatorname{sp} A_{\mathcal{J}} = \sum_{\nu \in N} \operatorname{sp}(A_\nu)_{\mathcal{J}}.$$

THEOREM 1.2 (G. È. KISILEVSKIĬ [5]). *Every simple dissipative Volterra operator with nuclear imaginary component either is unicellular, or admits a decomposition into the quasi-direct sum of unicellular operators* A_k $(k = 1, 2, \cdots, \omega; \omega \leqq \infty)$:

$$A = \sum_{k=1}^{\omega} \dot{+} A_k. \tag{1.3}$$

The cardinal number ω *and the positive numbers*

$$\sigma_k = \operatorname{sp}(A_k)_{\mathcal{J}} \qquad (k = 1, 2, \cdots, \omega)$$

are invariants of the operator A.[4)]

If the imaginary component $A_{\mathcal{J}}$ *has finite dimension* n, *then* $\omega \leqq n$, *and the decomposition* (1.3) *can be chosen so that the imaginary component* $(A_k)_{\mathcal{J}}$ *of the operator* A_k *has dimension* $\leqq n - k + 1$ $(k = 1, 2, \cdots, \omega)$.[5)]

§2. A basic criterion for the unicellularity of a dissipative Volterra operator with nuclear imaginary component

1. The derivations of Theorems 1.1 and 1.2 make essential use of various facts from the theory of the characteristic function of a nonselfadjoint operator, and in particular use the criterion to be given below for the unicellularity of a dissipative Volterra operator, which is formulated in terms of its characteristic function.

Let us give the general definition of the characteristic operator-function $W_A(\lambda)$ of a nonselfadjoint operator A ($\in \mathfrak{R}$), in the form originally given by M. S. Livšic [2].

---

4) That is, the number ω and the numbers σ_k $(k = 1, 2, \cdots, \omega)$, up to their order, do not depend upon the choice of representation (1.3).

5) The first and third assertions of Theorem 1.2 are stated and proved in a paper by G. E. Kisilevskiĭ [5]; the second assertion was communicated by him in a letter.

We denote by $\mathfrak{J}(A)$ the closure of the range of the imaginary component $A_{\mathscr{J}}$ of A:

$$\mathfrak{J}(A) = \overline{\Re(A_{\mathscr{J}})} = \overline{\Re(A - A^*)}.$$

The operator-function $W_A(\lambda)$ is defined on the set $\rho(A^*)$ of all regular points λ of A^* by[6)]

(2.1) $W_A(\lambda) = [I + 2i(\operatorname{sign} A_{\mathscr{J}})|A_{\mathscr{J}}|^{1/2}(A^* - \lambda I)^{-1}|A_{\mathscr{J}}|^{1/2}]|\mathfrak{J}(A)$,

where the symbol $|\mathfrak{J}(A)$ denotes that the operator in the square brackets is restricted to the subspace $\mathfrak{J}(A)$. In order to clarify the formula (2.1), we note that

$$(A - \lambda I)(A^* - \lambda I)^{-1} = I + 2iA_{\mathscr{J}}(A^* - \lambda I)^{-1}.$$

On the other hand, we have

$$A_{\mathscr{J}} = |A_{\mathscr{J}}|^{1/2} \operatorname{sign} A_{\mathscr{J}} \, |A_{\mathscr{J}}|^{1/2};$$

therefore for the case in which $A_{\mathscr{J}}$ is invertible (and hence $\mathfrak{J}(A) = \mathfrak{H}$) formula (2.1) simplifies:

$$W_A(\lambda) = |A_{\mathscr{J}}|^{-1/2}(A - \lambda I)(A^* - \lambda I)^{-1}|A_{\mathscr{J}}|^{1/2}.$$

Let us also consider the case in which $A_{\mathscr{J}}$ is completely continuous and hence admits a spectral representation

$$A_{\mathscr{J}} = \sum_{j=1}^{r} \epsilon_j(\cdot, \psi_j)\psi_j, \tag{2.2}$$

where $\{\psi_j\}_1^r$ is some orthogonal (not normalized) system of elements,

---

6) If H is a selfadjoint operator and $H = \int_{-\infty}^{\infty} \lambda dE_\lambda$ is its spectral representation, then we put

$$\operatorname{sign} H = \int_{-\infty}^{\infty} \operatorname{sign} \lambda dE_\lambda = I - E(+0) - E(-0), \quad |H| = \int_{-\infty}^{\infty} |\lambda| dE_\lambda,$$

(in contrast with the norm we denote the "modulus" of the operator H by $|H|$). Obviously $|H| = H \operatorname{sign} H = (\operatorname{sign} H)H$. If, in particular, $H \in \mathfrak{S}_\infty$ and

$$H = \sum_j \lambda_j(\cdot, \phi_j)\phi_j \qquad ((\phi_j, \phi_k) = \delta_{jk}, \; j, k = 1, 2, \cdots)$$

is the spectral representation of H, then

$$|H| = \sum_j |\lambda_j|(\cdot, \phi_j)\phi_j, \qquad \operatorname{sign} H = \sum_j \operatorname{sign} \lambda_j(\cdot, \phi_j)\phi_j.$$

$\epsilon_j = \pm 1$ $(j = 1, 2, \cdots, r)$, and r is the dimension of the range of $A_{\mathscr{J}}$.

The matrix of the operator $W_A(\lambda)$ relative to the basis $\{\phi_j\}_1^r$ $(\phi_j = \psi_j / |\psi_j|)$ of $\mathfrak{J}(A)$ is

$$\mathscr{W}(\lambda) = \| w_{jk}(\lambda) \|_1^r,$$

where

$$w_{jk}(\lambda) = (W_A(\lambda)\phi_k, \phi_j) \qquad (j, k = 1, 2, \cdots, r).$$

By (2.1) we have

$$\begin{aligned} w_{jk}(\lambda) &= \delta_{jk} + 2i((\operatorname{sign} A_{\mathscr{J}})\,|A_{\mathscr{J}}|^{1/2}(A^* - \lambda I)^{-1}|A_{\mathscr{J}}|^{1/2}\phi_k, \phi_j) \\ &= \delta_{jk} + 2i((A^* - \lambda I)^{-1}|A_{\mathscr{J}}|^{1/2}\phi_k, (\operatorname{sign} A_{\mathscr{J}})\,|A_{\mathscr{J}}|^{1/2}\phi_j), \end{aligned}$$

and since the representation (2.2) implies that

$$|A_{\mathscr{J}}|^{1/2}\phi_k = \psi_k, \qquad \operatorname{sign} A_{\mathscr{J}}\,|A_{\mathscr{J}}|^{1/2}\phi_j = \epsilon_j \psi_j,$$

we have

$$w_{jk}(\lambda) = \delta_{jk} + 2i\epsilon_j((A^* - \lambda I)^{-1}\psi_k, \psi_j) \qquad (j, k = 1, 2, \cdots, r).$$

We see that the matrix $\mathscr{W}(\lambda)$ is none other than the characteristic matrix-function of the operator A (see Chapter V, §4.3).[7]

In considering Volterra operators A it is convenient to go over from the characteristic operator-function $W_A(\lambda)$ to the *monodromy operator-function* $U_A(\lambda)$ (see Chapter V, §4.3), related to the former by

$$U_A(\mu) = W_A(1/\mu). \tag{2.3}$$

An independent definition of the function $U_A(\mu)$ is the following:

$$U_A(\mu) = [I - 2i\mu \operatorname{sign} A_{\mathscr{J}}\,|A_{\mathscr{J}}|^{1/2}(1 - \mu A^*)^{-1}|A_{\mathscr{J}}|^{1/2}]|\mathfrak{J}(A).$$

If A is a Volterra operator, then obviously $U_A(\mu)$ will be an entire operator-function of the complex variable μ.

For the case in which $A_{\mathscr{J}} \in \mathfrak{S}_1$ $(A \in \mathfrak{R})$, the determinant

$$\begin{aligned} \det U_A(\mu) &= \det(I - 2\mu i \operatorname{sign} A_{\mathscr{J}}\,|A_{\mathscr{J}}|^{1/2}(I - \mu A^*)^{-1}|A_{\mathscr{J}}|^{1/2}) \\ &= \det(I - 2\mu i(I - \mu A^*)^{-1}|A_{\mathscr{J}}|^{1/2}\operatorname{sign} A_{\mathscr{J}}\,|A_{\mathscr{J}}|^{1/2}) \\ &= \det(I - 2\mu i(I - \mu A^*)^{-1}A_{\mathscr{J}}) = \det((I - \mu A)(I - \mu A^*)^{-1}) \end{aligned}$$

[7] We remark that in the definition of the characteristic matrix-function $\mathscr{W}(\lambda) = \| w_{jk}(\lambda) \|_1^n$ given in Chapter V, §4.3, it was not required that the basis $\{\psi_j\}$ be orthogonal.

is meaningful.[8)] Thus, by the notation from [GK], Chapter IV, §3.1,

$$\det U_A(\mu) = D_{A/A^*}(\mu). \tag{2.4}$$

For a Volterra operator A the perturbation determinant $D_{A/A^*}(\mu)$ has the quite simple value (see [GK], Chapter IV, §6.2)[9)]

$$D_{A/A^*}(\mu) = \exp(-2i\mu\,\mathrm{sp}\,A_{\mathscr{J}}). \tag{2.5}$$

Thus

$$|\det U_A(\mu)| = \exp(2\,\mathrm{Im}\,\mu\,\mathrm{sp}\,A_{\mathscr{J}}). \tag{2.6}$$

If A is dissipative, then the expressions for $W_A(\lambda)$ and $U_A(\mu)$ $(\lambda = 1/\mu)$ simplify:

$$W_A(\lambda) = [I + 2iA_{\mathscr{J}}^{1/2}(A^* - \lambda I)^{-1}A_{\mathscr{J}}^{1/2}]|\mathfrak{J}(A)$$

and

$$U_A(\mu) = [I - 2i\mu A_{\mathscr{J}}^{1/2}(I - \mu A^*)^{-1}A_{\mathscr{J}}^{1/2}]|\mathfrak{J}(A).$$

From the easily verified identity (see, for example, [GK], Chapter IV, §5)

$$W_A^*(\lambda)W_A(\lambda) = [I - 2\,\mathrm{Im}\,\lambda A_{\mathscr{J}}^{1/2}(A - \bar{\lambda}I)^{-1}(A^* - \lambda I)^{-1}A_{\mathscr{J}}^{1/2}]|\mathfrak{J}(A) \tag{2.7}$$

it follows that

$$W_A^*(\lambda)W_A(\lambda) \leqq I \quad \text{for } \mathrm{Im}\,\lambda > 0, \tag{2.8}$$

$$W_A^*(\lambda)W_A(\lambda) \geqq I \quad \text{for } \mathrm{Im}\,\lambda < 0. \tag{2.9}$$

Let us denote by $s_j(\mu)$ $(j = 1, 2, \cdots)$ the sequence of s-numbers of the operator $U_A(\mu)$:

$$s_j^2(\mu) = \lambda_j(U_A^*(\mu)U_A(\mu)) \qquad (j = 1, 2, \cdots).$$

Assuming that $A_{\mathscr{J}} \in \mathfrak{S}_1$, we can write

---

8) If $A_{\mathscr{J}} \in \mathfrak{S}_1$, then $|A_{\mathscr{J}}|^{1/2} \in \mathfrak{S}_2$, and consequently

$$C = -2\mu i\,\mathrm{sign}\,A_{\mathscr{J}}\,|A_{\mathscr{J}}|^{1/2} \in \mathfrak{S}_2,\ D = (I - \mu A^*)^{-1}|A_{\mathscr{J}}|^{1/2} \in \mathfrak{S}_2;$$

we have also used the fact that if $C, D \in \mathfrak{S}_2$, then $\det(I + CD) = \det(I + DC)$ (see [GK], Chapter IV, §§1 and 3).

9) If we use a canonical model of a dissipative operator, then the equality $\det U_A(\mu) = \exp(-2i\mu\,\mathrm{sp}\,A_{\mathscr{J}})$ can be obtained as a corollary of the formula of Liouville-Ostrogradskiĭ. We have already used this equality in Chapter VI for the derivation of Theorem VI.2.2. We also note that formula (2.5) can be deduced from the general formula (3.21).

$$|\det U_A(\mu)|^2 = \det U_A^*(\mu) \det U_A(\mu) = \det(U_A^*(\mu) U_A(\mu)) = \prod_{j=1}^{\infty} s_j^2(\mu).$$

On the other hand, by (2.9) $s_j(\mu) \geqq 1$ $(j = 1, 2, \cdots)$ for $\operatorname{Im} \mu \geqq 0$. Thus

$$|\det U_A(\mu)| \geqq s_1(\mu) = | U_A(\mu) | \quad \text{for } \operatorname{Im} \mu \geqq 0.$$

Recalling (2.6), we obtain the bound

$$| U_A(\mu) | \leqq \exp(2 \operatorname{Im} \mu \operatorname{sp} A_{\mathscr{J}}) \qquad (\operatorname{Im} \mu \geqq 0). \tag{2.10}$$

As regards the lower halfplane $\operatorname{Im} \mu \leqq 0$, by (2.8) we have there

$$| U_A(\mu) | \leqq 1 \qquad (\operatorname{Im} \mu \leqq 0). \tag{2.11}$$

The bound (2.10) is familiar from the work of V. P. Potapov [1], and M. S. Brodskiĭ and M. S. Livšic [1].[10)]

2. We can now formulate the following fundamental criterion for the unicellularity of a Volterra operator of the class being considered.

THEOREM 2.1 (M. S. BRODSKIĬ AND G. È. KISILEVSKIĬ [1]). *In order that a simple dissipative Volterra operator A with nuclear imaginary component be unicellular, it is necessary and sufficient that*

$$\overline{\lim_{\rho \to \infty}} \left\{ \frac{1}{\rho} \ln | U_A(i\rho) | \right\} = 2 \operatorname{sp} A_{\mathscr{J}}. \tag{2.12}$$

The sufficiency of condition (2.12) had already been established by M. S. Brodskiĭ [1] in 1956. Then G. È. Kisilevskiĭ [2] proved the necessity of this condition for the case in which the imaginary component $A_{\mathscr{J}}$ is two-dimensional. For the general case ($A_{\mathscr{J}} \in \mathfrak{S}_1$) the necessity of (2.12) was established in a joint paper [1] of these authors.

Condition (2.12) is obviously a requirement that the bound (2.10) be exact. According to this bound, for the general case of a dissipative Volterra operator A with nuclear imaginary component we will have, for the quantity

$$\hbar(A) = \overline{\lim_{\rho \to \infty}} \left\{ \frac{1}{\rho} \ln | U_A(i\rho) | \right\},$$

---

10) The method of obtaining a bound for the norm of a matrix-function through its determinant was already used in the work of V. P. Potapov [1]. The bound (2.10), for the case of a finite-dimensional imaginary component $A_{\mathscr{J}}$, also follows almost immediately from the multiplicative representation (4.1). Since the latter can be extended to the case of dissipative Volterra operators with an infinite-dimensional component $A_{\mathscr{J}} \in \mathfrak{S}_1$ (see, in this connection, the end of §3), the bound (2.10) can be obtained by this route.

the inequality

$$\hbar(A) \leqq 2 \operatorname{sp} A_{\mathscr{J}} .$$

If the strict inequality holds here, then the bound (2.10) can be improved, namely:

$$|U_A(\mu)| \leqq \exp(\hbar(A)\operatorname{Im}\mu) \quad \text{for } \operatorname{Im}\mu \geqq 0. \tag{2.13}$$

In fact, according to the bounds (2.10) and (2.11), for any unit vectors $f, g \in \mathfrak{H}$ the entire function $F(\mu) = F_{f,g}(\mu) = (U(\mu)f, g)$ satisfies the conditions

$$\sup_{-\infty<\mu<\infty} |F(\mu)| \leqq 1, \qquad \overline{\lim_{\rho\to\infty}} \left\{\frac{1}{\rho} \ln \max_{0\leqq\theta\leqq 2\pi} |F(\rho e^{i\theta})|\right\} < \infty .$$

But then, by a familiar theorem of the theory of entire functions (see B. Ja. Levin [1], Chapter 1, §14)

$$\ln|F(\mu)| \leqq h_F^+ \operatorname{Im}\mu \quad \text{for } \operatorname{Im}\mu \geqq 0,$$

where

$$h_F^+ = \overline{\lim_{\rho\to\infty}} \left\{\frac{1}{\rho} \ln|F(i\rho)|\right\}.$$

Since $h_F^+ \leqq \hbar(A)$ for any choice of the unit vectors $f, g \in \mathfrak{H}$, (2.13) follows.

On the basis of the foregoing and also of the relation (2.20) below, it is natural to call the functional $\hbar(A)$ the *exponential type* of the dissipative Volterra operator A.

3. We shall show that the criterion for unicellularity can be reformulated in terms of the Fredholm resolvent $A(\mu) = A(I - \mu A)^{-1}$ of the operator A as follows.

THEOREM 2.2. *In order that a simple dissipative Volterra operator A with nuclear imaginary component be unicellular, it is necessary and sufficient that*

$$\overline{\lim_{\rho\to\infty}} \left\{\frac{1}{\rho} \ln|A(-i\rho)|\right\} = 2 \operatorname{sp} A_{\mathscr{J}} . \tag{2.14}$$

To establish the equivalence of conditions (2.14) and (2.12), we shall show that

$$\hbar(A) = \overline{\lim_{\rho\to\infty}} \left\{\frac{1}{\rho} \ln|A(-i\rho)|\right\}. \tag{2.15}$$

It follows from the identity (2.7) that

$$|W_A(\bar{\lambda})|^2 = 1 + 2\operatorname{Im}\lambda |T(\lambda)|^2 \quad \text{for } \operatorname{Im}\lambda \geqq 0,$$

where

$$T(\lambda) = (A^* - \bar{\lambda}I)^{-1}A_{\mathcal{J}}^{1/2}.$$

On the other hand,

$$|T(\lambda)|^2 = |T^*(\lambda)|^2 = |T(\lambda)T^*(\lambda)| = |(A^* - \bar{\lambda}I)^{-1}A_{\mathcal{J}}(A - \lambda I)^{-1}|,$$

and therefore

$$|W_A(\bar{\lambda})|^2 = |I + 2\operatorname{Im}\lambda(A^* - \bar{\lambda}I)^{-1}A_{\mathcal{J}}(A - \lambda I)^{-1}|.$$

Representing $2A_{\mathcal{J}}$ in the form

$$2A_{\mathcal{J}} = \frac{1}{i}\left[(A - \lambda I) - (A^* - \bar{\lambda}I) + 2\operatorname{Im}\lambda I\right],$$

we find that

$$\begin{aligned}(2.16)\qquad \max(|S(\lambda)|^2, (\operatorname{Im}\lambda)^2|(A - \lambda I)^{-1}|^2) &\leqq |W_A(\bar{\lambda})|^2 \\ &\leqq |S(\lambda)|^2 + (\operatorname{Im}\lambda)^2|(A - \lambda I)^{-1}|^2 \qquad (\operatorname{Im}\lambda \geqq 0),\end{aligned}$$

where

$$S(\lambda) = I + i\operatorname{Im}\lambda(A - \lambda I)^{-1} = (A - \operatorname{Re}\lambda I)(A - \lambda I)^{-1}.$$

For $\lambda = -1/i\rho$ $(\rho > 0)$ we have

$$S(\lambda) = -i\rho A(I + i\rho A)^{-1} = -i\rho A(-i\rho),$$

$$\begin{aligned}(\operatorname{Im}\lambda)^2(A^* - \bar{\lambda}I)^{-1}(A - \lambda I)^{-1} &= (I - i\rho A^*)^{-1}(I + i\rho A)^{-1} \\ &= (I - i\rho A(-i\rho))^*(I - i\rho A(-i\rho)).\end{aligned}$$

Thus, the substitution $\lambda = -1/i\rho$ $(\rho > 0)$ in (2.16) yields

$$(2.17)\qquad |\rho A(-i\rho)|^2 \leqq |U_A(i\rho)|^2 \leqq |\rho A(-i\rho)|^2 + |I - i\rho A(-i\rho)|^2.$$

In particular, it follows that

$$|\rho A(-i\rho)| \leqq |U_A(i\rho)| \leqq 1 + \sqrt{2}\rho|A(-i\rho)|,$$

and so (2.15) holds.

REMARK 2.1. If an operator A $(\in \mathfrak{R})$ is dissipative, then

$$|(A - \lambda I)^{-1}| \leqq 1/|\operatorname{Im}\lambda| \qquad (\operatorname{Im}\lambda < 0).$$

Putting $\lambda = 1/\mu$, we find that the Fredholm resolvent $A(\mu)$ of A has the bound

(2.18) $$|A(\mu)| \leqq |A| |(I - \mu A)^{-1}| \leqq |A| |\mu| / \operatorname{Im} \mu \quad \text{for } \operatorname{Im} \mu > 0.$$

Thus $A(\mu)$ is bounded on each interior ray $\mu = \rho \exp(i\theta)$ $(0 < \theta < \pi)$ of the upper halfplane. Conversely, for a dissipative Volterra operator A with $A_{\mathcal{J}} \in \mathfrak{S}_1$ the resolvent $A(\mu)$ has exponential growth on each interior ray $\mu = \rho \exp(i\theta)$ $(-\pi < \theta < 0)$ of the lower halfplane, namely

(2.19) $$\overline{\lim_{\rho \to \infty}} \left\{ \frac{1}{\rho} \ln |A(\rho e^{i\theta})| \right\} = \hbar(A) |\sin \theta| \quad (-\pi < \theta < 0).$$

This is easily obtained from relations (2.13) and (2.16).

We can also assert that the relation (2.16) is satisfied uniformly in θ, and therefore it is also valid for $\theta = 0, \pi$.

It will be shown below (see (2.26)) that the resolvent $A(\mu)$ admits the representation

(2.20) $$A(\mu) = i \int_0^{\hbar(A)} e^{i\mu t} T_t \, dt,$$

where T_t $(0 \leqq t < \infty)$ is a strongly continuous semigroup of contractions.

It follows at once from this representation that

$$|A(\mu)| \leqq \begin{cases} \hbar(A) & \text{for } \operatorname{Im} \mu \geqq 0, \\ \hbar(A) e^{|\operatorname{Im} \mu| \hbar(A)} & \text{for } \operatorname{Im} \mu \leqq 0. \end{cases}$$

The representation (2.20) enables us to show that the lim sup in (2.12), (2.14), (2.15) and (2.19) can be replaced by lim. Thus, for a dissipative Volterra operator A with nuclear imaginary component $A_{\mathcal{J}}$ the quantity $\hbar(A)$, defined by (2.15), is the *exponential type of the entire function* $A(\mu)$.

In conclusion we make a few comparisons with the finite-dimensional case.

If A is a Volterra operator in an n-dimensional space E_n, then $A^n = 0$ and

$$A(\mu) = A(I - \mu A)^{-1} = A + \mu A^2 + \cdots + \mu^{n-2} A^{n-1}.$$

The operator A will be unicellular if and only if $A^{n-1} \neq 0$. Thus for A to be unicellular it is necessary and sufficient that the resolvent $A(\mu)$ be a polynomial of degree precisely $n - 2$.

If the degree of $A(\mu)$ (and hence, the order of growth of A) is lower, then A ceases to be unicellular.

The condition (2.14) is also a requirement of the maximality of the growth of the Fredholm determinant $A(\mu)$ of A as an entire operator-function, in other words, the condition of the maximality of its exponential type (or in the terminology of S. N. Bernšteĭn, of its *degree*).

This is the analogy with the finite-dimensional case. However, this analogy was established for a simple dissipative Volterra operator, *for which there is no analog* among the Volterra operators in E_n!

4. As we know (see §1.1), a Volterra operator A in E_n is unicellular if and only if the smallest value of p for which $A^p = 0$ is n.

It turns out that one can formulate a criterion for the unicellularity of a dissipative Volterra operator in $\mathfrak{H}$, which has points of contact with this elementary criterion.

To do this we need certain new concepts.

An operator B with domain $\mathfrak{D}(B)$ dense in $\mathfrak{H}$ is said to be *dissipative*, if

$$\operatorname{Im}(Bf,f) \geqq 0 \quad \text{for } f \in \mathfrak{D}(B). \tag{2.21}$$

A dissipative operator B is said to be *maximal* if it has no dissipative extensions.

A maximal dissipative operator is always closed. It is characterized by the property that for at least one λ (and then, for all λ) from the open lower halfplane $\operatorname{Im}\lambda < 0$ it has a bounded resolvent $(B - \lambda I)^{-1} \in \mathfrak{R}$ (see R. S. Phillips [1]).

The dissipative operators A $(\in \mathfrak{R})$ considered above, being everywhere defined, are obviously maximal dissipative operators.

Just as for bounded dissipative operators, one has for maximal dissipative operators the bound

$$|(B - \lambda I)^{-1}| \leqq 1/|\operatorname{Im}\lambda| \quad \text{for } \operatorname{Im}\lambda < 0.$$

If $\lambda = 0$ is not an eigenvalue of the dissipative operator A $(\in \mathfrak{R})$ (and hence of A^*), then the range $\mathfrak{R}(A)$ is dense in $\mathfrak{H}$. In this case, as is easily seen, the operator $B = -A^{-1}$ (which, generally speaking, is unbounded) will be maximal dissipative and

$$(B - \lambda I)^{-1} = -A(-\lambda). \tag{2.22}$$

Now let T_t $(0 \leqq t < \infty)$ be a one-parameter strongly continuous semigroup of uniformly bounded operators in $\mathfrak{H}$.[11] Then by a well-known theorem of K. Yosida (see E. Hille and R. S. Phillips [1]) this semigroup has an infinitesimal generating operator Γ, i.e. a closed linear operator Γ such that the set of all $f \in \mathfrak{H}$ for which the strong limit

[11] That is, 1) $T_0 = I$; 2) $T_{t+s} = T_t T_s$ $(0 \leqq s, t < \infty)$; 3) $\text{s-lim}_{t \downarrow 0} T_t = I$ and 4) $\sup |T_t| < \infty$. We refer the reader to the book by E. Hille and R. S. Phillips [1] for all facts regarding the theory of semigroups.

$$\operatorname{s-lim}_{t \downarrow 0} \{t^{-1}(T_t - I)f\}$$

exists coincides with $\mathfrak{D}(\Gamma)$, and this strong limit (s-lim) coincides with Γf. One writes $T_t = \exp(t\Gamma)$. Among the many justifications of this notation we point out that the entire left halfplane $\operatorname{Re}\zeta < 0$ consists of regular points of Γ, and for the resolvent $(\Gamma - \zeta I)^{-1}$ one has the formula

$$-(\Gamma - \zeta I)^{-1} f = \int_0^\infty e^{-\zeta t} T_t f \, dt. \tag{2.23}$$

It turns out (a theorem of Phillips [1]) that every maximal dissipative operator B, when multiplied by i, is the infinitesimal generator of some strongly continuous one-parameter semigroup of contractions T_t $(0 \leq t < \infty)$, and conversely every such semigroup has infinitesimal generator Γ of the form $\Gamma = iB$, where B is a maximal dissipative operator. We shall say that a strongly continuous semigroup of contractions T_t $(0 \leq t < \infty)$ is *nilpotent*, if for some value $t = a$ (and hence for all $t \geqq a$) we have $T_t = 0$. The smallest value of a for which $T_a = 0$ is called the *exponent of nilpotence* of the semigroup T_t.

The following result holds.

THEOREM 2.3. *Let A be a simple dissipative Volterra operator with nuclear imaginary component. Then the operator $-iA^{-1}$ is the generator of some nilpotent semigroup of contractions $T_t = \exp(-itA^{-1})$ with exponent of nilpotence s_A equal to $\hbar(A)$ $(\leqq 2 \operatorname{sp} A_{\mathcal{J}})$.*

Hence if we make use of Theorem 2.1 (or Theorem 2.2) we immediately obtain the following corollary.

COROLLARY. *The operator A of the preceding theorem will be unicellular if and only if the exponent of nilpotence s_A of the semigroup* $\exp(-itA^{-1})$ *is equal to* $2 \operatorname{sp} A_{\mathcal{J}}$.

Let us turn to the proof of the theorem.

For the semigroup $T_t = \exp(-itA^{-1})$ the relations (2.22) and (2.23) (for $\Gamma = iB$, $B = -A^{-1}$, $\lambda = -\mu$, $\zeta = -i\mu$) yield

$$A(\mu) f = i \int_0^\infty e^{i\mu t} T_t f \, dt. \tag{2.24}$$

Hence for $f, g \in \mathfrak{H}$

$$(A(\mu) f, g) = i \int_0^\infty e^{i\mu t} (T_t f, g) \, dt. \tag{2.25}$$

Since

$$|(A(\mu)f,g)| \leqq |A(\mu)| \, |f| \, |g|,$$

the left side of (2.25) is an entire function of exponential type, and its type does not exceed $\hbar(A)$. By the well-known Paley-Wiener theorem (see, for example, N. I. Ahiezer [1]) it follows that $(T_t f,g) = 0$ for $t \geqq \hbar(A)$. Since f and g are arbitrary vectors in $\mathfrak{H}$, we conclude that $T_t = 0$ for $t \geqq \hbar(A)$, so that $s_A \leqq \hbar(A)$.

On the other hand, if $T_t = 0$ for $t \geqq a$ $(= s_A)$, then by (2.24)

$$A(\mu)f = i\int_0^a e^{i\mu t} T_t f \, dt, \tag{2.26}$$

and so

$$|A(\mu)| \leqq \int_0^a |e^{i\mu t}| \, |T_t| \, dt \leqq a e^{|\mathrm{Im}\mu| a} \quad (\mathrm{Im}\,\mu < 0).$$

Recalling (2.15), we conclude that $\hbar(A) \leqq a = s_A$.

Thus the theorem is proved. Let us now comment on this theorem.

Choosing an arbitrary positive integer n, we put $T = \exp(-i\tau A^{-1})$, where $\tau = ([2\,\mathrm{sp}\,A_{\mathcal{J}}] + 1)/n$. The spectrum of A is concentrated at zero, that of A^{-1}—at infinity, and that of T—again at zero, since T is nilpotent: $T^n = 0$. Considering the powers T^p $(= \exp(-ip\tau A^{-1}))$ with an arbitrary positive exponent p, we can assert that $p = \hbar(A)/\tau$ is the smallest exponent of nilpotency of the operator T. This exponent does not exceed $2\,\mathrm{sp}\,A_{\mathcal{J}}/\tau$. For the case in which it achieves this value, and only for this case, is A unicellular. Nevertheless the operator T itself is not unicellular. Indeed, given any vector $f \in \mathfrak{H}$ we obtain an invariant subspace $\mathfrak{L}_f$, spanned by the vectors $f, Tf, \cdots, T^{n-1}f$, of dimension not greater than n. Obviously we can always find vectors $f,g \in \mathfrak{H}$ such that neither of the subspaces $\mathfrak{L}_f$ and $\mathfrak{L}_g$ contains the other.

One can show that the operators $T_t = \exp(-itA^{-1})$ are not Volterra operators.

To conclude this section we make a few bibliographical remarks.

Under the assumption that $A_{\mathcal{J}}$ is finite-dimensional, the nilpotence of the semigroup $T_t = \exp(-itA^{-1})$, where A is a simple dissipative Volterra operator, was established by B. Sz.-Nagy and C. Foiaş [**10**]. These authors showed (see [**10**], p. 322) that the exponent of nilpotency s_A of this semigroup does not exceed a_T, where a_T is a number for whose computation the authors indicated a procedure in terms of the operator $T = (iI + A)(iI - A)^{-1}$, and the exponent s_A is equal to a_T if and only if A is unicellular. We now know that the number a_T coincides with $2\,\mathrm{sp}\,A_{\mathscr{J}}$ (see (3.6)).

Sz.-Nagy and Foiaş obtained their results from an entirely different circle of ideas (see §3.4).

Following these authors, we consider by way of example a unicellular dissipative Volterra operator A with a one-dimensional imaginary component. This operator, the simplest of the class being considered, when normalized so that $\mathrm{sp}\,A_{\mathscr{J}} = \frac{1}{2}$ is unitarily equivalent to the operator of integration $\frac{1}{2}J$ in $L_2(0,1)$ (see Theorem I.8.1):

$$(\tfrac{1}{2}Jf)(x) = i\int_x^1 f(s)\,ds.$$

The domain of the operator $\Gamma = -i(\frac{1}{2}J)^{-1}$ is the set of all absolutely continuous functions $g(t) \in L_2(0,1)$ for which $g'(t) \in L_2(0,1)$ and $g(1) = 0$, and $\Gamma g = g'$. It is not hard to show that the operator Γ generates the semigroup $\exp(s\Gamma)$ of contractions defined by

$$(\exp(t\Gamma)f)(x) = \begin{cases} f(x+t) & \text{for } x+t \in [0,1], \\ 0 & \text{for } x+t \notin [0,1] \end{cases} \quad (0 \leqq t \leqq 1).$$

The exponent s_A of nilpotence of this semigroup obviously equals 1, which is in complete accord with the corollary to Theorem 2.3.

§3. Unicellular contractions

1. Recall that an operator T $(\in \mathfrak{R})$ is called a *contraction* if $|T| \leqq 1$. A contraction T is said to be *simple* if it has no invariant subspace on which its restriction is unitary. By a theorem of Langer—Sz.-Nagy—Foiaş (see [GK], Theorem V.3.1) a contraction can be split into the orthogonal sum of a simple contraction and a unitary operator. Obviously only a simple contraction can be unicellular.

From these results and the arguments of Ju. I. Ljubič and V. I. Macaev [1] and V. I. Macaev [2], from which the result A) formulated in the introduction was obtained, we also have the following result.

THEOREM 3.1. *An invertible*[12] *unicellular contraction T for which*

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[12] The requirement that T be invertible in this theorem (as well as in other theorems where it appears) could have been replaced by a formally more general requirement, namely, that at least one point ζ $(|\zeta| < 1)$ not be in the spectrum of T. If this condition is satisfied and $I - T^*T \in \mathfrak{S}_\infty$, then T can have only a discrete spectrum inside the unit disc, consisting of normal eigenvalues (see [GK], Chapter I). If the contraction T is unicellular, this spectrum must be absent, and hence T is invertible.

We also remark that Theorem 3.1 enables one to strengthen result A) (see the relation (3.3)).

$I - T^*T \in \mathfrak{S}_\omega$ has a one-point spectrum, and this point lies on the unit circle.

If T is a contraction with one-point spectrum α ($|\alpha| = 1$), then $T_\alpha = \bar{\alpha} T$ will be a contraction with one-point spectrum $\lambda = 1$. Therefore wherever it is convenient, we may assume without loss of generality that T has the one-point spectrum $\lambda = 1$.

2. If A ($\in \mathfrak{R}$) is a dissipative operator, then its Cayley transform

$$T = (iI - A)(iI + A)^{-1}, \tag{3.1}$$

as is easily seen, will be a contraction ($|T| \leq 1$), for which the point $\lambda = -1$ is regular.

If, in turn, T is a contraction for which the point $\lambda = -1$ is regular, then the transformation inverse to (3.1),

$$A = i(I - T)(I + T)^{-1} \tag{3.2}$$

yields a bounded dissipative operator A.

From the relations

$$I - T^*T = 4(-iI + A^*)^{-1} A_{\mathcal{J}} (iI + A)^{-1}, \tag{3.3}$$

$$A_{\mathcal{J}} = (I + T^*)^{-1}(I - T^*T)(I + T)^{-1} \tag{3.4}$$

it follows that if $A_{\mathcal{J}}$ belongs to some two-sided ideal $\mathfrak{S}$ ($\subset \mathfrak{S}_\infty$), then $I - T^*T$ belongs to the same ideal, and conversely. If, moreover, we consider the relations

$$I - T = 2A(iI + A)^{-1} = 2iA(i),$$

$$A = iS(2I - S)^{-1} = 2iS(\tfrac{1}{2}) \quad (S = I - T),$$

where $S(\mu)$ is the Fredholm resolvent of the operator S, then we can conclude that the dissipative operator A will be a Volterra operator if and only if $S = I - T$ is a Volterra operator.

Thus the transformation (3.1) maps the class of dissipative operators with nuclear imaginary component, being considered above, onto the class of all contractions T having the following properties:

1) *$T - I$ is a Volterra operator*;

2) *$I - T^*T$ is a nuclear operator.*

Henceforth for convenience we shall call the operator $I - T^*T$ the deviation of T from unitarity or, briefly, the *deviation operator*.

We remark that property 1) (regarded as a condition) is more restrictive than the following condition:

1′) *T has a one-point spectrum* $\lambda = 1$.

Nevertheless, as is not hard to show, in the class of all operators T for which the deviation operator $I - T^*T$ is completely continuous (and a fortiori in the class of those operators for which the deviation operator $I - T^*T$ is nuclear), the properties 1) and 1′) are equivalent. Therefore the criterion established above for the unicellularity of a dissipative Volterra operator A with $A_{\mathcal{J}} \in \mathfrak{S}_1$ can be transformed into a criterion for unicellularity in the indicated class of contractions T. For example, Theorem 2.2 (and the inequality $\hbar(A) \leqq 2\,\mathrm{sp}\,A_{\mathcal{J}}$) enables us to state the following result.

THEOREM 3.2. *Let T be a simple contraction whose spectrum consists of the single point $\lambda = 1$, and whose deviation operator $I - T^*T$ is nuclear. Then*

$$\overline{\lim_{\rho \uparrow 1}}\{(1-\rho)\ln|(T-\rho I)^{-1}|\} \leqq -\ln\det T^*T. \tag{3.5}$$

Equality holds here if and only if T is unicellular.

Since, by (3.1),

$$(1-\rho)(T-\rho I)^{-1} = (I - iA)(I + irA)^{-1} = I + i(r-1)A(ir)$$
$$(r = (1+\rho)/(1-\rho)),$$

the left side of (3.5) coincides with twice the left side of (2.14). We shall show that the right sides of (3.5) and (2.14) are similarly related, i.e.

$$-\ln\det T^*T = 4\,\mathrm{sp}\,A_{\mathcal{J}}. \tag{3.6}$$

To do so, we put $\mu = -i$ in (2.5); then we find that

$$\det((I + iA)(I + iA^*)^{-1}) = \exp(-2\,\mathrm{sp}\,A_{\mathcal{J}}). \tag{3.7}$$

Since the right side is real, this equality will persist if we take the conjugate of the operator in the determinant, i.e.

$$\det((I - iA)^{-1}(I - iA^*)) = \exp(-2\,\mathrm{sp}\,A_{\mathcal{J}}). \tag{3.8}$$

Let us compute the product of the determinants in the left sides of (3.7) and (3.8). Obvious calculations yield[13]

$$\det((I - iA^*)(I - iA)^{-1}(I + iA)(I + iA^*)^{-1})$$
$$= \det((iI - A^*)^{-1}(iI + A^*)(iI + A)^{-1}(iI - A)) = \det(T^*T),$$

from which (3.6) follows.

We further note that for a contraction T satisfying the hypotheses of Theorem 3.2 one has

[13] We have used here the fact that if $X, Y, X^{-1} \in \mathfrak{R}$ and $XY - I \in \mathfrak{S}_1$, then $\det XY = \det YX$ (see [GK], Chapter IV, §1).

$$\overline{\lim_{\rho\uparrow 1}}(1-\rho)\ln M_T(\rho) = \overline{\lim_{\rho\uparrow 1}}(1-\rho)\ln|(T-\rho I)^{-1}|, \tag{3.9}$$

where

$$M_T(\rho) = \max_{0\leq\theta\leq 2\pi} |(T-\rho e^{i\theta}I)^{-1}| \quad (0<\rho<1). \tag{3.10}$$

In fact, for $A = i(I-T)(I+T)^{-1}$ we have

$$|(I-\mu A)^{-1}| = |I+\mu A(\mu)| \leqq 1 + \hbar(A)|\mu|\exp(\hbar(A)|\mathrm{Im}\,\mu|) \quad (\mathrm{Im}\,\mu \leqq 0).$$

Hence for the operator T with $\mu = i(1+\zeta)/(1-\zeta)$ we obtain

$$\begin{aligned} |(T-\zeta I)^{-1}| &= \left|\frac{2}{1-\zeta}(I+T)^{-1}(I-\mu A^{-1})\right| \\ &\leqq \frac{\mathrm{const}}{1-\rho}\left(1+\frac{\hbar(A)}{1-\rho}\exp\left(\frac{2\hbar(A)}{1-\rho}\right)\right) \quad (\rho = |\zeta|<1). \end{aligned} \tag{3.11}$$

We have here used the inequality $|\mathrm{Im}\,\mu| \leqq |\mu| \leqq 2(1-\rho)^{-1}$.

From (3.11) we can conclude that the left side of (3.9) does not exceed $2\hbar(A)$; on the other hand it is not less than the right side, which equals $2\hbar(A)$.

For completeness we also note that when the point ζ approaches the unit circle from the outside, the growth of the norm of the resolvent of any contraction T satisfies the bound

$$|(T-\zeta I)^{-1}| = \left|\sum_{k=0}^{\infty}\frac{1}{\zeta^{k+1}}T^k\right| \leqq \sum_{k=0}^{\infty}\frac{1}{|\zeta|^{k+1}} = \frac{1}{|\zeta|-1}. \tag{3.12}$$

3. We shall omit formulating the result for contractions which corresponds to Theorem 2.1. To formulate this result we would have to introduce the concept of the characteristic function of a contraction (see M. S. Livšic and V. P. Potapov [1], Ju. L. Šmul′jan [1, 2], B. Sz.-Nagy and C. Foiaş [6] and V. T. Poljackiĭ [1]), which we have not used up till now.

Instead of this we present a criterion for the unicellularity of a contraction T which in certain connections seems to be more complete and sounds more effective than Theorem 3.2.

THEOREM 3.3. *Let T be a simple invertible contraction with nuclear deviation operator $(I - T^*T \in \mathfrak{S}_1)$. In order that T be unicellular, it is necessary and sufficient that*

1) *T have no eigenvalues inside the unit disc;*
2) *the relation*

$$(3.13)\qquad \overline{\lim_{\rho\uparrow 1}}\{(1-\rho)\ln M_T(\rho)\} = -\ln\det T^*T$$

hold, where

$$M_T(\rho) = \max_{|\zeta|=\rho} |(T-\zeta I)^{-1}| \quad (\rho < 1).$$

If the first condition is satisfied and T is not unicellular, then instead of the equal sign in (3.13) *one has* $<$.

We shall prove this theorem, omitting certain details. We first present certain facts (which are of independent interest) whose derivation is not very difficult.

First of all, if for some invertible operator T ($\in \mathfrak{R}$) one has $I - T^*T \in \mathfrak{S}_1$, then

$$T^{*-1} - T = T^{*-1}(I - T^*T) \in \mathfrak{S}_1.$$

Hence in this case the perturbation determinant

$$(3.14)\qquad \begin{aligned} D_{T/T^{*-1}}(\zeta) &= \det((T-\zeta I)(T^{*-1}-\zeta I)^{-1}) \\ &= \det(T^*(T-\zeta I)(I-\zeta T^*)^{-1}) \end{aligned}$$

exists (see [GK], Chapter IV). Here ζ belongs to $\rho(T^*)$, the resolvent set of T^*. For brevity we put $\Delta_T(\zeta) = D_{T/T^{*-1}}(\zeta)$.

It will be more convenient for us to deal with the normalized function

$$(3.15)\qquad d_T(\zeta) = \Delta_T(\zeta)/\Delta_T^{1/2}(0)$$

in place of $\Delta_T(\zeta)$. We can write this function in another way. Since by hypothesis $T^*T = I - H_1$, where $H_1 \in \mathfrak{S}_1$, it follows that also $(T^*T)^{1/2} = I - H$, where $H \in \mathfrak{S}_1$. We have

$$\Delta_T^{1/2}(0) = [\det(T^*T)]^{1/2} = \det((T^*T)^{1/2}) \quad (= \det(I-H)).$$

From the polar representation $T = U(T^*T)^{1/2}$, where U is some unitary operator, we have $T^* = (T^*T)^{1/2}U^*$ and, by virtue of (3.14),

$$\Delta_T(\zeta) = \det(T^*T)^{1/2}\det(U^*(T-\zeta I)(I-\zeta T^*)^{-1}).$$

Thus[14] we obtain a new way of writing $d_T(\zeta)$:

[14] The relation (3.16) is important, besides everything else, in that it enables us to define the function $d_T(\zeta)$ for the case in which $\det T^*T = 0$ (i.e., when an inverse T^{-1} ($\in \mathfrak{R}$) does not exist, and hence T^{*-1} does not exist). In order to define the function $d_T(\zeta)$ by (3.16) it is essential that besides the condition $I - T^*T \in \mathfrak{S}_1$ the condition which we spoke of in footnote 12 be satisfied (the operator T should have at least one regular point inside the unit disc). One can show that when this condition is satisfied the function $d_T(\zeta)$ does not depend upon the choice of the unitary operator U in the polar representation $T = U(T^*T)^{1/2}$ (this choice is nonunique when $\det T^*T = 0$).

$$d_T(\zeta) = \det(U^*(T - \zeta I)(I - \zeta T^*)^{-1}). \tag{3.16}$$

We shall show, secondly, that for every contraction T with nuclear deviation operator the function $d_T(\zeta)$ satisfies the inequality

$$|d_T(\zeta)| \leqq 1 \qquad (|\zeta| < 1). \tag{3.17}$$

In fact, we have

$$\overline{d_T(\zeta)} = \det((I - \bar{\zeta}T)^{-1}(T^* - \bar{\zeta}I)U)$$

and

$$\overline{d_T(\zeta)}d_T(\zeta) = \det((I - \bar{\zeta}T)^{-1}(T^* - \bar{\zeta}I)(T - \zeta I)(I - \zeta T^*)^{-1}).$$

It can be shown that the value of the determinant on the right does not change if we interchange the first two factors. Thus

$$\begin{aligned}|d_T(\zeta)|^2 &= \det((T^* - \bar{\zeta}I)(I - \bar{\zeta}T)^{-1}(T - \zeta I)(I - \zeta T^*)^{-1}) \\ &= \det((I - \zeta T^*)^{-1}(T^* - \bar{\zeta}I)(I - \bar{\zeta}T)^{-1}(T - \zeta I)),\end{aligned}$$

i.e.

$$|d_T(\zeta)|^2 = \det(T_\zeta^* T_\zeta), \tag{3.18}$$

where

$$T_\zeta = (T - \zeta I)(I - \bar{\zeta}T)^{-1}. \tag{3.19}$$

From the easily verified identity

$$I - T_\zeta^* T_\zeta = (1 - |\zeta|^2)(I - \zeta T^*)^{-1}(I - T^*T)(I - \bar{\zeta}T)^{-1}$$

it follows that for every contraction T the operator T_ζ will also be a contraction for any ζ with $|\zeta| < 1$.

This proves the inequality (3.17).

We note, thirdly, that by a well-known theorem of the theory of functions (see I. I. Privalov [1], Chapter II, §6.2), inequality (3.17) and the inequality

$$d_T(0) = (\det(T^*T))^{1/2} > 0 \tag{3.20}$$

imply that $d_T(\zeta)$ can be represented in the following form:

$$d_T(\zeta) = B_T(\zeta)\exp\left(-\int_0^{2\pi} \frac{e^{it} + \zeta}{e^{it} - \zeta}\, d\sigma_T(t)\right), \tag{3.21}$$

where

$$B_T(\zeta) = \prod_j \frac{\alpha_j - \zeta}{1 - \bar{\alpha}_j \zeta} \frac{\bar{\alpha}_j}{|\alpha_j|} \tag{3.22}$$

is the Blaschke product formed from all the zeros of the function $d_T(\zeta)$, and $\sigma_T(t)$ $(0 \leqq t \leqq 2\pi)$ is some nondecreasing function (for definiteness we normalize this function by the conditions $\sigma_T(0) = 0$, $\sigma_T(t) = \sigma_T(t-0)$ for $0 < t \leqq 2\pi$). We now proceed directly to the proof of the theorem. We begin with the second assertion.

Recalling that the function $d_T(\zeta)$ differs from the perturbation determinant $\Delta_T(\zeta)$ only by a constant factor, we conclude (see [GK], Chapter IV, §2) that the sequence $\{\alpha_j\}$ in the representation (3.22) yields precisely the complete sequence of all eigenvalues of the contraction T lying inside the unit disc (if T is a simple contraction, that it cannot have any other eigenvalues).[15]

We note a relation which is obtained from (3.20) and (3.21) for $\zeta = 0$:[16]

$$(\det(T^*T))^{1/2} = \prod_j |\alpha_j| \exp\left(-\int_0^{2\pi} d\sigma_T(t)\right). \tag{3.23}$$

If the contraction T has no eigenvalues inside the unit disc (i.e. $B_T(\zeta) = 1$), this relation transforms into

$$\int_0^{2\pi} d\sigma_T(t) = -\tfrac{1}{2}\ln\det(T^*T). \tag{3.24}$$

Let us denote this expression by a_T:

$$a_T = -\tfrac{1}{2}\ln\det(T^*T).$$

We remark that this functional in fact coincides with the functional a_T introduced by B. Sz.-Nagy and C. Foiaş (see the end of §2).

We now rewrite relation (3.18) in the following form:

$$\det(T_\zeta^{-1}T_\zeta^{*-1}) = |d_T^{-1}(\zeta)|^2$$

and use it to estimate the resolvent $(T - \zeta I)^{-1}$. From this relation there follows (compare the arguments on pp. 356-357)

$$|T_\zeta^{-1}| \leqq |d_T^{-1}(\zeta)|. \tag{3.25}$$

Further, according to (3.21) and (3.24), if $B_T(\zeta) \equiv 1$, then

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[15] In the report by M. G. Kreĭn [25] there is given a spectral interpretation, obtained by the authors, of the function $\sigma_T(t)$ in terms of a chain of invariant subspaces of T.

[16] If the system of root vectors of a simple invertible contraction T is complete in $\mathfrak{H}$, then and only then do we have $\det T^*T = \prod|\alpha_j|$ (see [GK], Theorem V.3.2), i.e., by (3.23) in this case and only in this case do we have $d_T(\zeta) = B_T(\zeta)$. This deduction can be obtained immediately by applying the indicated criterion of completeness to the operator T_ζ.

$$(3.26)\qquad \ln|d_T^{-1}(\zeta)| = \mathrm{Re}\int_0^{2\pi}\frac{e^{it}+\zeta}{e^{it}-\zeta}\,d\sigma_T(t) \leqq \frac{2a_T}{1-|\zeta|}.$$

Taking into account now the bounds

$$\left.\begin{aligned} |T_\zeta^{-1}| &= |(I-\bar{\zeta}T)(T-\zeta I)^{-1}| \leqq 2|(T-\zeta I)^{-1}|, \\ |(T-\zeta I)^{-1}| &= |(I-\bar{\zeta}T)^{-1}T_\zeta^{-1}| \leqq \frac{1}{1-|\zeta|}|T_\zeta^{-1}|, \end{aligned}\right. \qquad (|\zeta|<1)$$

and the bounds (3.25) and (3.26), we find that

$$(3.27)\qquad \ln|(T-\zeta I)^{-1}| \leqq \ln\frac{1}{1-|\zeta|} + \frac{2a_T}{1-|\zeta|} \quad (|\zeta|<1).$$

From this bound on the growth of the resolvent it follows that

$$(3.28)\qquad \begin{aligned} \overline{\lim_{\rho\uparrow 1}}\{(1-\rho)\ln M_T(\rho)\} &= \overline{\lim_{|\zeta|\uparrow 1}}\{(1-|\zeta|)\ln|T_\zeta^{-1}|\} \\ &\leqq \overline{\lim_{|\zeta|\uparrow 1}}\left\{(1-|\zeta|)\mathrm{Re}\int_0^{2\pi}\frac{e^{it}+\zeta}{e^{it}-\zeta}\,d\sigma_T(t)\right\}. \end{aligned}$$

It now follows from (3.26) that always

$$(3.29)\qquad \overline{\lim_{\rho\uparrow 1}}\{(1-\rho)\ln M_T(\rho)\} \leqq 2a_T.$$

Thus the second assertion of Theorem 3.3 is proved.

To prove the first (basic) assertion of Theorem 3.3 we have to use the preceding Theorems 3.1 and 3.2.

The necessity of the first condition of the theorem is obvious, and the necessity of the second condition (equality (3.13)) follows from Theorems 3.1 and 3.2.

It remains to clarify the sufficiency of these conditions. When they are fulfilled, then by (3.28) and (3.29), which we now have to take with the equal signs, we have

$$\overline{\lim_{|\zeta|\uparrow 1}}\left\{(1-|\zeta|)\mathrm{Re}\int_0^{2\pi}\frac{e^{it}+\zeta}{e^{it}-\zeta}\,d\sigma_T(t)\right\} = 2\int_0^{2\pi}d\sigma_T(t) = 2a_T.$$

It is not hard to show that this relation is valid if and only if the function $\sigma_T(t)$ has only one point of increase. If this point is $t=t_0$, then the entire spectrum of the simple contraction T reduces to the point $\alpha = \exp(it_0)$.[17)]

[17)] One can show that in the general case the spectrum of a simple contraction T consists of the sequence of points $\{\alpha_j\}$, the limit points of this sequence, and the image points $\exp(it)$ on the unit circle of all points of increase t of the function $\sigma_T(t)$.

Without loss of generality we may suppose that $\alpha = 1$. But then the operator T will satisfy the conditions of Theorem 3.2, and (3.9) will hold for T. Hence, by (3.13) equality will hold in (3.5).

Thus Theorem 3.3 is proved.

4. As was already mentioned in §2.4, the investigations of B. Sz.-Nagy and C. Foiaş [**5**—**11**] have made it possible to obtain a criterion for the unicellularity of a contraction from another circle of ideas.

Let us denote by $H^{(\infty)}$ the class of all functions $u(\zeta)$, defined and holomorphic in the open disc $|\zeta| < 1$. If $u(\zeta) \in H^{(\infty)}$ and T is a contraction, then, as the aforementioned authors have shown [**7**], there is a natural way to give meaning to the expression $u(T)$. Let

$$u(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k \quad (|\zeta| < 1)$$

be the Maclaurin series of the function $u(\zeta)$; if $|T| < 1$, then in accordance with the usual definition of a function of an operator we put

$$u(T) = \sum_{k=0}^{\infty} c_k T^k;$$

in the general case $u(T)$ is defined as the following strong limit:

$$u(T) = \operatorname{s-lim}_{r \uparrow 1} u(rT).$$

It turns out that this strong limit always exists for any function $u(\zeta) \in H^{(\infty)}$ and any contraction T ($|T| \leqq 1$).

Let us denote by $\mathbf{C}_0$ the class of all contractions T, for each of which there exists a function $d(\zeta) \in H^{(\infty)}$, $d(\zeta) \not\equiv 0$, which annihilates this contraction, i.e. such that $d(T) = 0$.

The following important result (B. Sz.-Nagy and C. Foiaş [**5**]) holds.

1. *Let $T \in \mathbf{C}_0$. Then there exists a unique, up to a constant factor c ($|c| = 1$), function $m_T(\zeta)$ such that*

1) *$m_T(\zeta)$ is an inner function;*[18]

2) $m_T(T) = 0$;

3) *$m_T(\zeta)$ is a divisor in the ring $H^{(\infty)}$ of every function $u(\zeta) \in H^{(\infty)}$ for which $u(T) = 0$.*

The function $m_T(\zeta)$, whose existence is asserted by 1) for $T \in \mathbf{C}_0$, is called the *minimal function* of T.

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[18] A function $u(\zeta) \in H^{(\infty)}$ is called an *inner function* if $\lim_{\rho \uparrow 1} |u(\rho e^{i\theta})| = 1$ for almost every $\theta \in [0, 2]$ or, what is equivalent, if the function $u(\zeta)$ admits a representation of the type (3.21) in which the nondecreasing function $\sigma(t)$ is singular (i.e., has a derivative which equals zero almost everywhere).

B. Sz.-Nagy and C. Foiaş [**6**] have shown that if $T \in \mathbf{C}_0$, then

$$\text{s-}\lim_{n\to\infty} T^n = \text{s-}\lim_{n\to\infty} T^{*n} = 0. \tag{3.30}$$

It turns out that this condition is also necessary for T to belong to the class $\mathbf{C}_0$ for the important case, already considered above, of a contraction with nuclear deviation operator. Namely:

2) *Let T be a simple invertible contraction for which $I - T^*T \in \mathfrak{S}_1$. Then the condition* (3.30) *is necessary and sufficient for T to belong to the class $\mathbf{C}_0$. When this condition is satisfied, the above constructed function $d_T(\zeta)$ can be taken as an annihilating function, i.e. $d_T(T) = 0$.*

This result can be regarded as an analog of the Hamilton-Cayley theorem for matrices.

We have the following result.

THEOREM 3.4. *Let T be a simple invertible contraction with $I - T^*T \in \mathfrak{S}_1$. In order that T be unicellular, it is necessary and sufficient that the following two conditions be satisfied*:

1) *the spectrum of T consists of only one point, which lies on the unit circle*;
2) $m_T(\zeta) \equiv \text{const}\, d_T(\zeta)$.

For this theorem, as for Theorem 3.3, it is essential that the Hilbert space $\mathfrak{H}$ in which T acts be infinite-dimensional.

It is useful to make a parallel with what can be asserted for simple contractions T acting in n-dimensional space E_n.

A contraction T acting in E_n will be simple if and only if its entire spectrum lies inside the unit circle. Therefore in the finite-dimensional case condition 1) is *absurd* where it requires that the single spectral point shall lie on the unit circle. However the first part of condition 1) (the existence of a one-point spectrum) is a necessary condition for unicellularity in the finite-dimensional case.

For a simple invertible contraction T in E_n the function $d_T(\zeta)$ is a rational function

$$d_T(\zeta) = \text{const}\frac{\det(T - \zeta I)}{\det(I - \zeta T^*)},$$

whose numerator is the characteristic polynomial, and whose denominator is the mirror image of the numerator; these polynomials are mutually prime. Therefore the relation $d_T(T) = 0$ will always be satisfied on the basis of the familiar Hamilton-Cayley theorem, which is in complete accord with result 2.

For a simple contraction T in E_n the function $d_T(\zeta)$ can be used in place of the characteristic polynomial, and the function $m_T(\zeta)$ can be used in place of the minimal polynomial. Therefore condition 2) together with the first part of condition 1) is, for the case being considered, a necessary and sufficient condition for the unicellularity of T.

Let us remark also that in the formulation of condition 2) it appeared that it was first necessary to require that $T \in C_0$. However, the operator T, as can be shown, will always belong to the class C_0 whenever the preceding condition of the theorem is satisfied. If we denote by α ($|\alpha| = 1$) the unique spectral point of T, whose existence is asserted in condition 1), then condition 2) could have been formulated as follows:

2′) *the minimal function* $m_T(\zeta)$ *of* T *has the form*

$$m_T(\zeta) = \exp\left(a_T \frac{\alpha + \zeta}{\alpha - \zeta}\right).$$

This follows from the general representation (3.21) of the function $d_T(\zeta)$, in which under the hypotheses of the theorem and condition 1) the Blaschke product $B_T(\zeta)$ is identically equal to 1, and the function $\sigma_T(t)$ has a single point of increase $t_0 = \arg\alpha$ $(0 \leqq t_0 < 2\pi)$.

Result 2 and Theorem 3.4 were established by Sz.-Nagy and Foiaş, but with a narrower formulation: in place of the *nuclearity* of the operator $I - T^*T$ these authors required that it be finite-dimensional. The validity of these results in the extended formulation given here was established by Ju. P. Ginzburg.[19)]

We shall show that Theorem 3.4 follows (formally) very simply from Theorem 2.3.

By virtue of our previous results (for example, the general Theorem 3.1), we may suppose at once that the contraction T being considered $(I - T^*T \in \mathfrak{S}_1)$ has the one-point spectrum $\lambda = 1$. It remains to show that condition 2) of Theorem 3.4 is a necessary and sufficient condition for the unicellularity of T.

Under the assumptions which have been made,

$$d_T(\zeta) = \exp(a_T(1+\zeta)(1-\zeta)^{-1}),$$

the operator $A = i(I - T)(I + T)^{-1}$ is a dissipative Volterra operator, and finally $a_T = 2\,\mathrm{sp}\,A_{\mathscr{J}}$. By Theorem 2.3

$$d_T(T) = \exp(a_T(I+T)(I-T)^{-1}) = \exp(-2i(\mathrm{sp}\,A_{\mathscr{J}})A^{-1}) = 0.$$

19) In a report in a seminar by M. G. Kreĭn. A recent note by Sz.-Nagy and Foiaş [12] supports the assertion that they have also been able to obtain result 2 and Theorem 3.4 in the extended form given here.

Consequently, T has a minimal function $m_T(\zeta)$. It must be a divisor of the function $d_T(\zeta)$, and therefore $m_T(\zeta) = c\exp(a(1+\zeta)(1-\zeta)^{-1})$ ($|c| = 1$), where a is the smallest positive number such that $m_T(T) = 0$. Since, then, $m_T(T) = c\exp(-iaA^{-1})$, by Theorem 2.3 we have $a \leqq 2\,\mathrm{sp}\,A_{\mathscr{I}} = a_T$, and equality holds if and only if $a = a_T$, i.e. $m_T(\zeta) = cd_T(\zeta)$.

We also mention that in the work of Sz.-Nagy and Foiaş [**6**—**10**] the function $d_T(\zeta)$ is defined not by (3.16), but by means of the characteristic operator-function $\theta_T(\zeta)$ of the contraction T. This function, in turn, is defined by

$$\theta_T(\zeta) = [-T + \zeta(I - TT^*)^{1/2}(T - \zeta I)(I - \zeta T^*)^{-1}(I - T^*T)^{1/2}]|\mathfrak{D}_T \qquad (|\zeta| < 1),$$

where $\mathfrak{D}_T$ is the closure of the range of the operator $I - T^*T$. If $I - T^*T$ is finite-dimensional (n-dimensional), then $\theta_T(\zeta)$ will be n-dimensional. It is easily shown that in this case $\det\theta_T(\zeta) = \epsilon d_T(\zeta)$ ($|\epsilon| = 1$).

If the contraction T is related to a dissipative operator A by (3.1), then, as can be shown, the function $\theta_T(\zeta)$ coincides up to constant unitary factors (on the right and left)—which map $\mathfrak{J}(A)$, the closure of the range of $A_{\mathscr{I}}$, onto $\mathfrak{D}_T$—with the function $W_A(z)$, where $\zeta = (i - z)/(i + z)$.

Similarly to the way in which Theorems 2.1 and 2.2 can be obtained one from the other by replacing $|U(i\rho)|$ by $|A(-i\rho)|$, Theorems 3.2 and 3.3 also have duals, which are obtained from these theorems by replacing the resolvent $(T - \zeta I)^{-1}$ by the function $\theta_T(\zeta)$. In many cases this is very convenient, for example when $I - T^*T$ is finite-dimensional.

§4. The problem of unicellularity and inverse problems for a canonical equation

1. The criterion for unicellularity of M. S. Brodskiĭ and G. È. Kisilevskiĭ (Theorem 2.1) together with certain other results of the theory of characteristic operator-functions leads to a number of important deductions in the theory of canonical differential equations.

In his investigations in the theory of analytic matrix-functions V. P. Potapov [1] obtained (among other more general results) the following result.

THEOREM 4.1 (V. P. POTAPOV [1]).[20] *Let $\mathscr{Y}$ be a signature matrix*

[20] Let us recall (Chapter V, §3) that an H-matrix $\mathscr{H}(t)$ ($0 \leqq t \leqq l$) (i.e. a Hermitian matrix-function with elements from $L_1(0, l)$) is called an H-matrix of positive type if

of order n ($\mathscr{Y}^* = -\mathscr{Y}$, $\mathscr{Y}^2 = -I_n$), *and let* $\mathscr{U}(\mu) = \| u_{jk}(\mu) \|_1^n$ *be an entire matrix-function. In order that* $\mathscr{U}(\mu)$ *admit a representation in terms of a multiplicative integral*

$$\mathscr{U}(\mu) = \int_0^{l} \curvearrowleft e^{-\mu \mathscr{Y} \mathscr{H}(t)\, dt}, \tag{4.1}$$

where $\mathscr{H}(t) = \| h_{jk}(t) \|_1^n$ $(0 \leqq t \leqq l)$ *is an H-matrix of positive type, it is necessary and sufficient that the matrix-function* $\mathscr{U}(\mu)$ *have the following properties*:

1) $\mathscr{U}(0) = I_n$, *and* $\mathscr{Y}\mathscr{U}'(0)$ *is a positive-definite matrix*;
2) *for every real* μ *the matrix* $\mathscr{U}(\mu)$ *is* $\mathscr{Y}$*-unitary, i.e.* $\mathscr{U}^*(\mu)\mathscr{Y}\mathscr{U}(\mu) = \mathscr{Y}$;
3) *for* $\operatorname{Im}\mu > 0$ *the matrix* $\mathscr{U}(\mu)$ *is* $(1/i)\mathscr{Y}$*-contractive, i.e.*

$$\frac{1}{i}\left(\mathscr{Y} - \mathscr{U}^*(\mu)\mathscr{Y}\mathscr{U}(\mu)\right) \geqq 0 \quad (\operatorname{Im}\mu > 0). \tag{4.2}$$

Without going into other possibilities for defining a multiplicative integral, let us clarify that for a matrix $\mathscr{A}(t) = \| a_{jk}(t) \|_1^n$ with elements from $L_1(a,b)$ *the multiplicative integral*

$$\Pi(t) = \int_a^{t} \curvearrowleft e^{\mathscr{A}(s)\, ds} \quad (a \leqq t \leqq b)$$

furnishes the solution $\Pi(t)$ $(a \leqq t \leqq b)$ of the matrix differential equation

$$d\,\Pi/dt = \mathscr{A}(t)\,\Pi(t)$$

which is singled out by the initial condition $\Pi(0) = I_n$.

To put it another way, $\Pi(t)$ $(a \leqq t \leqq b)$ is the solution of the integral equation

$$\Pi(t) = I_n + \int_a^t \mathscr{A}(s)\,\Pi(s)\, ds.$$

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the following two conditions are satisfied:

a) the matrix-function $\mathscr{H}(t)$ is Hermitian-nonnegative (for almost all t);

b) its average, $\mathscr{H}_{\text{av}}$, is a Hermitian-positive matrix.

Potapov's theorem remains valid for more general representations than (4.1); these representations require that the H-matrix $\mathscr{H}(t)$ be only Hermitian-nonnegative (and the requirement b) is omitted); for this only the requirement, in condition 1), that $\mathscr{U}(0) = I_n$ is maintained for $\mathscr{U}(\mu)$. It was in precisely this general form that Theorem 4.1 was proved by Potapov [1, 2].

Among various properties of a multiplicative integral we note a fundamental property which clarifies its name:

$$\overset{\frown}{\int_a^b} e^{\mathscr{A}(s)\,ds} = \overset{\frown}{\int_c^b} e^{\mathscr{A}(s)\,ds} \overset{\frown}{\int_a^c} e^{\mathscr{A}(s)\,ds};$$

here c is any number from the interval $a < c < b$.

In particular, (4.1) simply says that the matrix-function $\mathscr{U}(\mu)$ is the monodromy matrix of the corresponding canonical differential equation, i.e. that $\mathscr{U}(\mu) = \mathscr{U}(l;\mu)$, where the matrizant $\mathscr{U}(t;\mu)$ $(0 \leqq t \leqq l)$ is in turn defined as the solution of the Cauchy problem

$$\mathscr{Y}\frac{d\mathscr{U}}{dt} = \mu\mathscr{A}(t)\mathscr{U}, \quad \mathscr{U}(0;\mu) = I_n. \tag{4.3}$$

Thus, Potapov's theorem completely solves the problem, *what analytic properties characterize the monodromy matrix-function of a canonical equation with a given signature matrix* $\mathscr{Y}$?

Let us now indicate that the necessity of condition 1) follows trivially from the expansion (1.6) which was given in Chapter VI, §1, and according to which

$$\mathscr{U}(t;\mu) = I_n - \mu\mathscr{Y}\int_0^t \mathscr{A}(s)\,ds + O(\mu^2) \quad (\mu \to 0).$$

Indeed, by this expansion

$$\mathscr{Y}\mathscr{U}'(0) = \mathscr{Y}\frac{d\mathscr{U}(l;\mu)}{d\mu}\bigg|_{\mu=0} = \int_0^l \mathscr{A}(s)\,ds.$$

Property 2) of the monodromy matrix $\mathscr{U}(\mu)$ was already noted in Chapter VI, §1.2. An immediate generalization of the simple argument given there enables us to obtain property 3) at the same time as property 2). Indeed, let us form the matrix-function

$$\Omega(t;\mu) = \frac{1}{i}(\mathscr{Y} - \mathscr{U}^*(t;\mu)\mathscr{Y}\mathscr{U}(t;\mu))$$

and differentiate it with respect to t, recalling (4.3).

This yields

$$\frac{d\Omega(t;\mu)}{dt} = \frac{1}{i}(\mu - \bar{\mu})\mathscr{U}^*(t;\mu)\mathscr{A}(t)\mathscr{U}(t;\mu),$$

and since $\Omega(0;\mu) = 0$ we obtain

$$\Omega(l;\mu) = 2\,\mathrm{Im}\,\mu \int_0^l \mathcal{U}^*(t;\mu)\mathcal{H}(t)\mathcal{U}(t;\mu)\,dt. \tag{4.4}$$

This equality immediately implies both properties 2) and 3).[21]

What is deep and difficult about this result is the proof of the sufficiency of conditions 1), 2) and 3). V. P. Potapov obtained this result by a long chain of analytic constructions.

The sufficiency of these conditions can at the present time be obtained from general theorems on the multiplicative representation of generalized characteristic operator-functions (see M. S. Brodskiĭ [7]).

2. Without loss of generality the H-matrix $\mathcal{H}(t)$ in the representation (4.1) may be supposed normalized,[22] i.e.

$$\mathrm{sp}\,\mathcal{H}(t) \equiv 1 \qquad (0 \leqq t \leqq l). \tag{4.5}$$

Indeed, this can always be accomplished by making the change of parameter $s = \mathrm{sp}\mathcal{H}(t)$ $(0 \leqq t \leqq l)$.

We will call a representation (4.1) with a normalized H-matrix $\mathcal{H}(t)$ $(0 \leqq t \leqq l)$ of positive type a *$\mathcal{J}$-canonical representation.*

The question naturally arises,

Under what conditions will an entire matrix-function $\mathcal{U}(\mu)$ have a unique $\mathcal{J}$-canonical representation?

In other words, under what conditions is the normalized H-matrix $\mathcal{H}(t)$ $(0 \leqq t \leqq l)$ in a $\mathcal{J}$-canonical representation (4.1) (with a previously prescribed signature matrix $\mathcal{J}$) defined uniquely almost everywhere?

A nontrivial answer to this question can first be given in the framework of the theory of unicellular operators.

From the results of M. S. Livšic [2] and M. S. Brodskiĭ [6] (see also the joint paper [1] of these authors) there follows an important result.

THEOREM 4.2. *Suppose that the matrix-function $\mathcal{U}(\mu)$ admits a canonical representation* (4.1). *In order that this representation be unique, it is necessary and sufficient that the simple part $A_{\mathcal{U}}$*[23] *of the model operator*

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[21] Moreover, we can conclude from (4.4) that the matrix $-i(\mathcal{J} - \mathcal{U}^*(\mu)\mathcal{J}\mathcal{U}(\mu))$ is positive (negative) definite for $\mathrm{Im}\,\mu > 0$ (< 0).

[22] This remark was already made in the theory of models (Chapter V, §3, Remark 3.2).

[23] The notation $A_{\mathcal{U}}$ can be justified by the fact that this operator is determined by the function $\mathcal{U}(\mu)$ (having the properties enumerated in Theorem 4.1) up to a unitary transformation. This operator is characterized by the property that $\mathcal{U}(\mu)$ is its characteristic matrix-function (in the sense of the definition given in Chapter V, §4).

(4.6) $$(A_M f)(t) = \mathscr{Y} \int_t^l \mathscr{A}^{1/2}(t)\mathscr{A}^{1/2}(s)f(s)\,ds \quad (f \in L_2^{(n)}(0,l))$$

be unicellular.

3. We shall be interested below in the *definite* case, in which $\mathscr{Y} = iI_n$, and hence the representation (4.1) has the form

(4.7) $$\mathscr{U}(\mu) = \overset{\frown}{\int_0^l} e^{-i\mu \mathscr{A}(t)\,dt},$$

where $\mathscr{A}(t)$ $(0 \leqq t \leqq l)$ is a normalized H-matrix of positive type.

We will call a representation (4.7) a *definite canonical representation*. By Potapov's theorem an entire matrix-function $\mathscr{U}(\mu)$ which admits a definite canonical representation (4.7) is characterized by the following properties:

I) $\mathscr{U}(0) = I_n$, $i\mathscr{U}'(0)$ *is a positive-definite matrix*;[24]

II) *for real* μ *the matrix* $\mathscr{U}(\mu)$ *is unitary*;

III) *for* $\operatorname{Im}\mu < 0$ *the matrix* $\mathscr{U}(\mu)$ *is contractive*:

$$I_n - \mathscr{U}^*(\mu)\mathscr{U}(\mu) \geqq 0.$$

This result should be regarded as a generalization of a well-known result of the theory of functions, according to which the function $u(\mu) = \exp(-ih\mu)$ $(h > 0)$ is singled out among all other entire functions by the following properties:

1) $u(0) = 1$, $iu'(0) > 0$; 2) $|u(\mu)| = 1$ for $\operatorname{Im}\mu = 0$, and 3) $|u(\mu)| < 1$ for $\operatorname{Im}\mu < 0$.

Comparing Theorems 2.1 and 4.2 leads to the following:

THEOREM 4.3 (M. S. BRODSKIĬ AND G. È. KISILEVSKIĬ [1]).[25] *In order that the matrix-function* $\mathscr{U}(\mu)$ *(having the properties* I), II), III)) *admit a unique definite canonical representation* (4.6), *it is necessary and sufficient that the condition*

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[24] It can be shown that the relation $\mathscr{U}(0) = I_n$ and conditions II) and III) for an entire matrix-function $\mathscr{U}(\mu)$ imply that the matrix $i\mathscr{U}'(0)$ is nonnegative. If the matrix $\mathscr{U}'(0)$ were singular, the space E_n could be broken up into the orthogonal sum of invariant subspaces of $\mathscr{U}(\mu)$, $E_n = Z \oplus N$, where Z is the set of all vectors annihilated by $\mathscr{U}'(0)$, and then $\mathscr{U}(\mu)$ would induce the identity operator on Z, and an operator-function $\hat{\mathscr{U}}(\mu)$ with the properties I), II) and III) on N.

[25] The sufficiency of condition (4.8) had already been obtained by M. S. Brodskiĭ [1, 6].

$$\overline{\lim_{\rho\to\infty}} \left\{ \frac{1}{\rho} \ln |\mathscr{U}(i\rho)| \right\} = \operatorname{sp}(i\mathscr{U}'(0)) \tag{4.8}$$

be satisfied.

Let us clarify that by (4.5) and (4.6)

$$l = \int_0^l \operatorname{sp}\mathscr{H}(t)\,dt = 2\operatorname{sp}(A_M)_{\mathscr{I}} = 2\operatorname{sp}(A_{\mathscr{U}})_{\mathscr{I}} = \operatorname{sp}(i\mathscr{U}'(0)).$$

Theorem 4.3 has an interesting corollary; to obtain this corollary we shall use the expansion of the function $\mathscr{U}(\mu)$ in a power series in μ (compare with Chapter VI, §1.2):

$$\mathscr{U}(\mu) = I_n - i\mu\mathscr{A}_1(l) + (-i\mu)^2\mathscr{A}_2(l) + \cdots, \tag{4.9}$$

where

$$\mathscr{A}_1(t) = \int_0^t \mathscr{H}(s)\,ds, \quad \mathscr{A}_n(t) = \int_0^t \mathscr{H}(s)\mathscr{A}_{n-1}(s)\,ds \quad (n = 2, 3, \cdots).$$

Let us denote by $h(t)$ the largest eigenvalue of the Hermitian matrix $\mathscr{H}(t)$ $(0 \leqq t \leqq l)$. Obviously

$$0 < h(t) \leqq \operatorname{sp}\mathscr{H}(t) \quad (\equiv 1). \tag{4.10}$$

If by the norm of an nth order matrix we understand the norm of the linear operator corresponding to it in the n-dimensional Hilbert space E_n, then we have $h(t) = |\mathscr{H}(t)|$. Therefore

$$|\mathscr{A}_1(t)| \leqq \int_0^t h(s)\,ds,$$

and arguing by induction we obtain the bounds

$$|\mathscr{A}_n(t)| \leqq \frac{1}{n!}\left(\int_0^t h(s)\,ds\right)^n \quad (n = 1, 2, \cdots).$$

Thus it follows from the expansion (4.9) that

$$|\mathscr{U}(\mu)| \leqq \exp\left(|\mu| \int_0^l h(s)\,ds\right).$$

If (4.8) holds, then a comparison with the preceding bound yields

$$l \leqq \int_0^l h(s)\,ds.$$

Taking (4.10) into account, we conclude that $h(t) \equiv 1 \equiv \operatorname{sp}\mathscr{H}(t)$.

Thus, all the eigenvalues of the matrix $\mathscr{A}(t)$, besides $h(t)$, equal zero.

COROLLARY 4.1. *If the matrix-function $\mathscr{U}(\mu)$ admits a unique definite canonical representation* (4.7), *then the H-matrix $\mathscr{A}(t)$ in this representation has rank equal to unity almost everywhere.*

4. Let us denote by $r(t)$ the rank of the H-matrix $\mathscr{A}(t)$ $(0 \leqq t \leqq l)$. We see that the relation $r(t) = 1$ (almost everywhere) is a necessary condition for the representation (4.7) to be unique. However this condition is *by no means* sufficient, as can be concluded from the following result.

THEOREM 4.4. *For any H-matrix $\mathscr{A}(t)$ $(0 \leqq t \leqq l)$ of positive type there exists an H-matrix $\mathscr{A}_0(t)$ $(0 \leqq t \leqq l)$ of positive type and with rank 1 $(r_0(t) \equiv 1)$ such that*[26)]

$$(4.11) \qquad \overset{\frown}{\int_0^l} \exp(-i\mu\mathscr{A}_0(t)\,dt) = \overset{\frown}{\int_0^l} \exp(-i\mu\mathscr{A}(t)\,dt).$$

This theorem is a simple consequence of the results noted above. In fact, let us form the simple dissipative Volterra operator A (with n-dimensional imaginary component) which has $\mathscr{A}(t)$ $(0 \leqq t \leqq l)$ as its H-matrix. This operator can obviously be defined, for example, as the simple part of the model operator

$$(A_M f)(t) = i\int_t^l \mathscr{A}^{1/2}(t)\mathscr{A}^{1/2}(s)f(s)\,ds \quad (f \in L_2^{(n)}(0,l)).$$

Then the right side of (4.11) will give us the monodromy matrix of the operator A (i.e. $\mathscr{U}(1/\lambda)$ will be the characteristic matrix-function of A).

On the other hand, according to Remark 3.1 of Chapter V, following Theorem V.3.1, on models of a dissipative operator A, there exists at least one canonical model with an H-matrix $\mathscr{A}_1(t)$ of rank $r(t) \equiv 1$. The multiplicative integral corresponding to $\mathscr{A}_1(t)$ will yield the characteristic matrix-function $\mathscr{W}_A(\lambda)$ $(\lambda = 1/\mu)$ in some orthonormal basis and, consequently, will be equal to $\mathscr{S}\mathscr{U}_A(\mu)\mathscr{S}^{-1}$, where $\mathscr{S}$ is some constant unitary matrix. Therefore if we put $\mathscr{A}_0(t) = \mathscr{S}\mathscr{A}_1(t)\mathscr{S}^{-1}$, the H-matrix $\mathscr{A}_0(t)$ will now have all the necessary properties.

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26) Thus Corollary 4.1 is trivially contained in Theorem 4.4. We formulated it separately, since it enables us to prove Theorem I.7.3 (of G. È. Kisilevskiĭ), which will be discussed below, and on which the derivation of Theorem 4.4 is based.

Remark V.3.1 to Theorem V.3.1 on models was obtained on the basis of Theorem I.7.3, which states that *every dissipative Volterra operator A with nuclear imaginary component has a maximal eigenchain of rank $r = 1$.*

We can now clarify how Theorem I.7.3 is obtained.

Let us first assume that A is a unicellular operator.[27] Then in the representation (4.7) for $\mathscr{U}(\mu) = \mathscr{U}_A(\mu)$ the normalized H-matrix $\mathscr{A}(t)$ of positive type will be determined uniquely almost everywhere and has rank $r(t) \equiv 1$ (almost everywhere). Hence, we will have

$$\mathscr{A}(t) = \xi(t)\xi^*(t) \qquad (0 \leqq t \leqq l), \tag{4.12}$$

where $\xi(t)$ $(0 \leqq t \leqq l)$ is some vector-function from $L_2^{(n)}(0,l)$.

We form the Hermitian-nonnegative scalar kernel

$$\mathscr{K}(t,s) = \xi^*(s)\xi(t) \qquad (0 \leqq s,t \leqq l) \tag{4.13}$$

and the corresponding model operator in $L_2(0,l)$:

$$(A_{\mathscr{K}} f)(t) = i\int_t^l \mathscr{K}(t,s)f(s)\,ds \quad (f \in L_2(0,l)).$$

The operator A, being unitarily equivalent to the simple part of A_M, will be unitarily equivalent to the simple part of $A_{\mathscr{K}}$. Taking into account that $A_{\mathscr{K}}$ has a complete eigenchain consisting of the truncation operators $\hat{P}(t)$ $(0 \leqq t \leqq l)$, which chain has rank 1, we conclude that the operator A has a maximal eigenchain $\mathfrak{P} = \{P\}$ of rank 1.

We now consider the case in which A is not unicellular. By Theorem 2.1, it can be decomposed into the quasidirect sum

$$A = \sum_{k=1}^{\omega} \dot{+}\, A_k \qquad (\omega \leqq n)$$

of unicellular operators A_k $(k = 1,2,\cdots,\omega)$. Since, as was just proved, each of the operators A_k has a maximal eigenchain $\mathfrak{P}_k$ $(k = 1,2,\cdots,\omega)$ of rank 1, the same will be true of the operator A.

Thus Theorem I.7.3 is proved.

We remark that to every ordering of the chains $\mathfrak{P}_k$ $(k = 1,2,\cdots,\omega)$ there corresponds a maximal eigenchain $\mathfrak{P}$ of rank 1 of A. Therefore the operator A will have at least $\omega!$ distinct maximal eigenchains of rank 1.

5. The property of unicellularity for a dissipative Volterra operator is very delicate. To support this assertion we give the following two results.

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[27] For this case the theorem was proved in a paper by M. S. Brodskiĭ and G. È. Kisilevskiĭ [1].

A) (M. S. Brodskiĭ [1]). *Suppose that the matrix-function* $\mathscr{U}(\mu)$ *admits a definite canonical representation* (4.7) *with a piecewise constant H-matrix* $\mathscr{A}(t)$ *of rank* 1, *i.e. an H-matrix of the form* $\mathscr{A}(t) = \xi(t)\xi^*(t)$ $(0 \leqq t \leqq l)$, *where* $\xi(t)$ $(0 \leqq t \leqq l)$ *is a piecewise constant vector-function with values in* E_n. *Then the representation* (4.7) *will be unique if and only if no two vectors* $\xi(t-0)$ *and* $\xi(t+0)$ $(0 < t < l)$ *are mutually orthogonal*:

$$\xi^*(t-0)\xi(t+0) \neq 0 \qquad (0 < t < l). \tag{4.14}$$

This result can be proved by directly verifying that for the special H-matrix $\mathscr{A}(t)$ being considered the conditions (4.8) and (4.14) are equivalent. As this verification is instructive, we shall carry it out.

In fact, suppose that the H-matrix $\mathscr{A}(t)$ has the constant value $\mathscr{A}_0 = \xi\xi^*$ $(\xi \in E_n,\ \xi^*\xi = 1)$ on some interval (a,b); then

$$\begin{aligned}\overset{\frown}{\int_a^b} \exp(-i\mu\mathscr{A}(t)\,dt) &= \exp(-i\mu(b-a)\mathscr{A}_0)\\ &= I_n + (\exp(-i\mu(b-a)) - 1)\mathscr{A}_0\\ &= I_n + (\exp(-i\mu(b-a)) - 1)\xi\xi^*.\end{aligned}$$

We have used here the relation $\mathscr{A}_0^n = \mathscr{A}_0$ $(n = 1, 2, \cdots)$. Therefore, if the interval $[0, l]$ can be broken up into subintervals

$$[t_{j-1}, t_j] \quad (j = 1, 2, \cdots, p;\ 0 = t_0 < t_1 < \cdots < t_p = l)$$

such that

$$\mathscr{A}(t) = \xi_j\xi_j^* \quad (t_{j-1} < t < t_j;\ j = 1, 2, \cdots, p),$$

where the ξ_j are constant unit vectors from E_n $(\xi_j^*\xi_j = 1;\ j = 1, 2, \cdots, p)$, then

$$\begin{aligned}\mathscr{U}(\mu) = \overset{\frown}{\int_0^l} e^{-i\mu\mathscr{A}(t)\,dt} &= \overset{\frown}{\int_{t_{p-1}}^l} e^{-i\mu\mathscr{A}(t)\,dt} \cdots \overset{\frown}{\int_0^{t_1}} e^{-i\mu\mathscr{A}(t)\,dt}\\ &= \overset{\frown}{\prod_{j=1}^p} (I_n + (\exp(-i\mu l_j) - 1)\xi_j\xi_j^*).\end{aligned} \tag{4.15}$$

Here $l_j = t_j - t_{j-1}$ $(j = 1, 2, \cdots, p)$, and the curved arrow indicates that the successive matrix factors are written from right to left.

Carrying out the multiplication in (4.15), we see that $\mathscr{U}(\mu)$ has the following form:

$$\mathscr{U}(\mu) = I_n + \sum_{k=1}^{p} C_k \exp(-i\mu a_k),$$

where

$$0 < a_1 < a_2 < \cdots < a_p = \sum_{j=1}^{p} l_j = l.$$

The leading coefficient C_p is obviously found from the formula

$$C_p = \gamma \xi_1 \xi_n^*, \quad \text{where } \gamma = \prod_{j=1}^{p-1} \xi_{j+1}^* \xi_j. \tag{4.16}$$

Thus $C_p \neq 0$ if and only if $\gamma \neq 0$, i.e. when the condition (4.14) is fulfilled.

On the other hand, in the case being considered the limit

$$\lim_{\rho \to \infty} \left\{ \frac{1}{\rho} \ln |\mathscr{U}(i\rho)| \right\} \tag{4.17}$$

will always exist and, obviously, will be equal to l if and only if $C_p \neq 0$.

6. The limit superior in (4.8) can be regarded as a functional $\Phi(\mathscr{H})$ defined on the convex cone $\mathfrak{K}_{n\times n}$ of H-matrices $\mathscr{H}(t) = \| h_{jk}(t) \|_1^n$ $(0 \leq t \leq l)$ of positive type. The cone $\mathfrak{K}_{n\times n}$ can be regarded in a natural way as a cone in the real Banach space $\mathfrak{B}_{n\times n}$ of all H-matrices $\mathscr{H}(t) = \| h_{jk}(t) \|_1^n$ of order n with elements from $L_1(0,l)$ (l is some fixed positive number) with the norm defined by

$$\| \mathscr{H}(t) \| = \int_0^l | \mathscr{H}(t) | \, dt.$$

Result A) shows that *the functional* $\Phi(\mathscr{H})$ *is discontinuous*. In fact, let the H-matrix $\mathscr{H}(t)$ have the form (4.12) with a piecewise constant function $\xi(t)$ and with the quantity γ, defined in (4.16), equal to zero. For this H-matrix the limit (4.17) will be $< l$. On the other hand, an arbitrary small change in the value of the vectors ξ_j $(j = 1, 2, \cdots, p)$ can lead to $\gamma \neq 0$, and then the limit (4.17) for the corresponding H-matrix will equal l.

Following what has been said, the following result assumes special interest.

B) (G. È. Kisilevskiĭ [5]).[28] *If the matrix function* $\mathscr{U}(\mu)$ *admits a*

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[28] This result is not explicitly formulated in Kisilevskiĭ's paper [5], although it is essentially proved if one takes note of the remark used below on the continuity of the vector-function $\xi(t)$ in the representation (4.12) of the H-matrix $\mathscr{H}(t)$ under the hypothesis of result B).

definite canonical representation (4.7) *with a continuous H-matrix* $\mathscr{A}(t)$ *of rank* 1 (*i.e.* $r(t) \equiv 1$), *then this representation is the unique definite canonical representation of the function* $\mathscr{U}(\mu)$.

We omit the proof of this result. We shall only prove that it is implied by the following result.[29]

C) (G. È. Kisilevskiĭ [5]). *Let* $\mathscr{K}(t,s)$ $(0 \leqq t,s \leqq l)$ *be a continuous Hermitian-nonnegative kernel, with* $\mathscr{K}(t,t) > 0$ $(0 \leqq t \leqq l)$. *Then the simple part of the integral operator*

$$(Kf)(t) = i \int_t^l \mathscr{K}(t,s) f(s)\, ds$$

is unicellular.

In fact, if the H-matrix $\mathscr{A}(t)$ in the representation (4.7) has rank $r(t) \equiv 1$ and is continuous, then, as is not hard to show, it will admit a representation (4.12) with a continuous vector-function $\xi(t)$ $(0 \leqq t \leqq l)$. Therefore the kernel $\mathscr{K}(t,s)$, defined by (4.13), will be a continuous Hermitian-nonnegative kernel with $\mathscr{K}(t,t) = \xi^*(t)\xi(t) = \operatorname{sp}\mathscr{A}(t) = 1$. Thus result B) is a consequence of result C) and Theorem 4.2.

In conclusion we point out that Theorem 4.1 of V. P. Potapov was generalized by Ju. P. Ginzburg [1,2] to the case of infinite-dimensional normalized H-matrices $\mathscr{A}(t)$ of positive type.[30] Theorems 4.2, 4.3 and 4.4 were proved by their authors for this more general case.

§5. Real unicellular operators and related analytic problems

1. Suppose that there is defined in the separable Hilbert space $\mathfrak{H}$ an involution S, which enables one to introduce the concepts of S-real elements, S-real operators, etc. (see Chapter V, §5).

An S-real operator A is said to be *S-unicellular* if, for any two of its invariant subspaces $\mathfrak{L}_1, \mathfrak{L}_2$ which have S-real kernels, at least one of the relations $\mathfrak{L}_1 \subset \mathfrak{L}_2$, $\mathfrak{L}_2 \subset \mathfrak{L}_1$ holds. Obviously every S-unicellular operator is simple.

In order to distinguish S-unicellularity from the concept of unicellularity introduced in §1, we shall refer to unicellularity in the sense of §1 as *complex* unicellularity.

29) In essence, result B) is equivalent to result C) for the case of degenerate kernels.

30) The first generalization of Theorem 4.1 to this case was obtained by M. S. Livšic [1] by operator methods (in all completeness only for $\mathscr{Y} = \pm iI$). Later an elegant derivation of Theorem 4.1 by means of the abstract triangular integral was given by M. S. Brodskiĭ [7].

If an S-real operator is *complex-unicellular*, then it is a fortiori *S-unicellular*. On the other hand, even in the finite-dimensional case an operator can be S-unicellular without being complex-unicellular. In a finite-dimensional space this occurs if and only if the S-real operator has, in all, only two distinct nonreal (complex conjugate) eigenvalues, to each of which there corresponds just one Jordan cell.

It is highly noteworthy that for Volterra operators in an infinite-dimensional space one observes situations for which there is no analog in a finite-dimensional space.

As is easily seen, an S-real Volterra operator in a finite-dimensional space will be S-unicellular if and only if it is complex-unicellular.

In contrast to this, in an infinite-dimensional Hilbert space $\mathfrak{H}$ there exist Volterra operators which are not complex-unicellular, but which turn out to be S-unicellular for a suitable choice of the involution S. We give an example of such an operator in §5.4.

Just as was the case for dissipative operators (see §4), the question of the unicellularity of an S-real operator with a finite-dimensional skewsymmetric component reduces to the delicate question of uniqueness for a certain inverse problem in the theory of differential equations. The solution of this problem requires, however, considerably more tools from the theory of entire functions than did the preceding problem. At the same time it is interesting in certain respects, since it concerns differential equations which are frequently encountered in problems of mathematical physics.

The questions treated in this section have a very direct relation to the theory of spectral functions of Sturm-Liouville differential operators and others, and also to the theory of the extrapolation of stationary random processes. However, due to lack of space these relations will not be revealed here.

2. According to Theorem V.5.3, every S-real simple Volterra operator A with a $2m$-dimensional skewsymmetric component A_α is S-unitarily equivalent[31] to the simple part of a certain model operator A_M, defined in $L_2^{(2m)}(0,l)$ by

$$(5.1) \qquad (A_M f)(t) = \mathscr{A}^{1/2}(t)\mathscr{Y}_m \int_t^l \mathscr{A}^{1/2}(s) f(s)\,ds.$$

In this model $\mathscr{Y}_m$ is the Hamiltonian signature matrix of order $2m$:

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[31] An operator A ($\in \mathfrak{R}$) is *S-unitarily equivalent* to an operator A' acting in $L_2^{(m)}(0,l)$, if $UAU^{-1} = A'$, where U is a unitary mapping of $\mathfrak{H}$ onto $L_2^{(m)}$ for which $USf = \overline{Uf}$ (the bar indicates complex conjugation).

$$\mathscr{Y}_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

and $\mathscr{H}(t)$ $(0 \leqq t \leqq l)$ is a Hamiltonian (a real H-matrix) of positive type. Without loss of generality we may suppose that this Hamiltonian is normalized:

$$\text{(5.2)} \qquad \operatorname{sp}\mathscr{H}(t) = 1 \qquad (0 \leqq t \leqq l).$$

In §5 of Chapter V (see also §4, Chapter V) we also clarified that the multiplicative integral

$$\text{(5.3)} \qquad \mathscr{U}(\mu) = \overset{\curvearrowleft}{\int_0^l} \exp(-\mu\mathscr{Y}_m\mathscr{H}(t)\,dt)$$

will be a monodromy matrix of the operator A (and hence $\mathscr{U}(1/\lambda)$ will be its characteristic matrix-function).

The representation of an entire matrix-function $\mathscr{U}(\mu)$ in the form of a multiplicative integral (5.3) will be called a *real canonical representation* (normalized, if the condition (5.2) is satisfied).

A function $\mathscr{U}(\mu)$ which has such a representation is obviously a real entire function, i.e. admits an everywhere convergent expansion

$$\mathscr{U}(\mu) = I + \mu C_1 + \mu^2 C_2 + \cdots$$

with real matrix coefficients C_k $(k = 1, 2, \cdots)$.

The theorem of V. P. Potapov (Theorem 4.1) can be supplemented by the following assertion: a real entire matrix-function $\mathscr{U}(\mu)$ which satisfies the hypotheses of that theorem for $\mathscr{Y} = \mathscr{Y}_m$, admits a real canonical representation.

Thus we can formulate the following result.

THEOREM 5.1. *In order that a matrix-function* $\mathscr{U}(\mu) = \| u_{jk}(\mu) \|_1^{2m}$ *of a complex variable admit a real canonical representation, it is necessary and sufficient that it be a real entire function and that it satisfy the following conditions*:

1) $\mathscr{U}(0) = I_{2m}$, *and* $\mathscr{Y}_m\mathscr{U}'(0)$ *is positive definite*;

2) $\mathscr{U}(\mu)$ *is a symplectic matrix*,[32] i.e.

$$\text{5.4)} \qquad \mathscr{U}^\tau(\mu)\mathscr{Y}_m\mathscr{U}(\mu) = \mathscr{Y}_m;$$

3) *for* $\operatorname{Im}\mu > 0$ *the matrix*

[32] We recall that $\mathscr{U}^\tau$ denotes the transpose of $\mathscr{U}$.

(5.5) $$i(\mathscr{U}^*(\mu)\mathscr{Y}_m\mathscr{U}(\mu) - \mathscr{Y}_m)$$

is Hermitian-positive.

Let us clarify that since, for real μ, $\mathscr{U}(\mu)$ is a real matrix, for real μ the condition (5.4) coincides with the condition of the $\mathscr{Y}_m$-unitarity of $\mathscr{U}(\mu)$ (i.e. condition 2) of Theorem 4.1). On the other hand, if (5.4) holds for real μ, then by virtue of the analyticity of $\mathscr{U}(\mu)$ it will hold for all complex μ. By (5.4),

$$\det(\mathscr{U}^\tau(\mu)\mathscr{U}(\mu)) = (\det\mathscr{U}(\mu))^2 = 1.$$

Recalling that $\mathscr{U}(0) = I_{2m}$, we conclude that

(5.6) $$\det\mathscr{U}(\mu) = 1.$$

For the case being considered, Theorem 4.2 can similarly be supplemented (compare with M. S. Brodskiĭ [8]).

THEOREM 5.2 *Suppose that the entire matrix-function* $\mathscr{U}(\mu) = \|u_{jk}(\mu)\|_1^{2m}$ *admits a real normalized canonical representation* (5.3) *with a Hamiltonian* $\mathscr{H}(t)$ $(0 \leqq t \leqq l)$. *In order that this Hamiltonian be the unique one*[33] *yielding the required representation of* $\mathscr{U}(\mu)$, *it is necessary and sufficient that the simple part of the model operator* (5.1) *corresponding to* $\mathscr{H}(t)$ $(0 \leqq t \leqq l)$ *be a real-unicellular operator in* $L_2^{(2m)}(0,1)$.

Let us clarify that in speaking of the real-unicellularity of an operator in $L_2^{(2m)}(0,1)$ we have in mind its S_0-unicellularity with respect to the involution S_0 of ordinary complex conjugation: $S_0 f = \bar{f}$.

3. In the language of the theory of differential equations, the relation (5.3) says that $\mathscr{U}(\mu)$ is the monodromy matrix of the Hamiltonian equation

(5.7) $$\mathscr{Y}_m dx/dt - \mu\mathscr{H}(t)x - 0$$

of phase dimension $2m$.

An important special case of equation (5.7) is that in which the Hamiltonian $\mathscr{H}(t)$ is quasi-diagonal, i.e.

(5.8) $$\mathscr{H}(t) = \begin{pmatrix} \mathscr{H}_1(t) & 0 \\ 0 & \mathscr{H}_2(t) \end{pmatrix},$$

33) Up to its values on a set of measure zero.

where $\mathscr{H}_1(t)$ and $\mathscr{H}_2(t)$ $(0 \leqq t \leqq l)$ are real H-matrices of positive type.

Putting $x = y \dot{+} z$, where y and z are m-dimensional vector-functions, we can write (5.7) in the form of a system:

$$dy/dt = -\mu\mathscr{H}_2(t)z, \qquad dz/dt = \mu\mathscr{H}_1(t)y. \tag{5.9}$$

In particular, the second-order equation

$$d^2y/dt^2 + \mu^2\mathscr{P}(t)y = 0 \tag{5.10}$$

can be transformed (see Chapter VI, §5.1) to such a system. In this case $\mathscr{H}_2(t) = I_m$, $\mathscr{H}_1(t) = \mathscr{P}(t)$.

We further recall that the differential equation of an arbitrary weighted string (see Chapter VI, §8) reduces to a system of the form (5.9) (with $m = 1$).

In view of all this, the following result is of interest.

THEOREM 5.3. *In order that a matrix-function $\mathscr{U}(\mu)$ of a complex variable μ admit a real canonical representation* (5.3) *with a quasi-diagonal Hamiltonian $\mathscr{H}(t)$ $(0 \leqq t \leqq l)$, it is necessary and sufficient that it be a real entire function, that it satisfy conditions* 1), 2) *and* 3) *of Theorem* 5.1, *and that when it is written in the form of four square blocks*

$$\mathscr{U}(\mu) = \begin{pmatrix} \mathscr{U}_{11}(\mu) & \mathscr{U}_{12}(\mu) \\ \mathscr{U}_{21}(\mu) & \mathscr{U}_{22}(\mu) \end{pmatrix} \tag{5.11}$$

the matrix-functions $\mathscr{U}_{11}(\mu)$ and $\mathscr{U}_{22}(\mu)$ be even, and the matrix-functions $\mathscr{U}_{12}(\mu)$ and $\mathscr{U}_{21}(\mu)$ be odd.

Since for the general case of the representation (5.3) there is no uniqueness theorem, Theorem 5.3 cannot be obtained from Theorem 5.1; it has to be deduced independently (of course, this remark also applies to the relation of Theorem 5.1 to Theorem 4.1).

In various problems of mathematical physics and the theory of spectral functions of differential equations (see M. G. Kreĭn [**9, 23**] one encounters differential equations of the form

$$\mathscr{Y}_m dy/dt - \mathscr{G}(t)y - \mu y = 0 \qquad (0 \leqq t \leqq l), \tag{5.12}$$

where $\mathscr{G}(t)$ is a real symmetric matrix-function with elements from $L_1(0,l)$.[34)]

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34) In the papers [**9, 23**] by Kreĭn the coefficient $\mathscr{G}(t)$ also satisfies the anticommutation condition $\mathscr{Y}_m\mathscr{G}(t) = -\mathscr{G}(t)\mathscr{Y}_m$. This can always be accomplished by transforming equation (5.12) by means of the substitution $y \to \mathscr{V}(t)y$, where $\mathscr{V}(t)$ is the unitary matrizant orthogonal to $\mathscr{Y}_m$: $d\mathscr{V}(t)/dt = -\frac{1}{2}(\mathscr{Y}_m\mathscr{G} + \mathscr{G}\mathscr{Y}_m)\mathscr{V}$, $\mathscr{V}(0) = I_{2m}$.

In §8, Chapter VII we showed that under certain conditions on the Hamiltonian $\mathscr{A}(t)$ equation (5.7) can be transformed to equation (5.12). If one starts with equation (5.12), it can be transformed to equation (5.7) in the following way.

We denote by $\mathscr{S}(t)$ the matrizant of equation (5.12) for $\mu = 0$, i.e. the solution of the matrix differential equation

$$\mathscr{Y}_m d\mathscr{S}/dt - \mathscr{G}(t)\mathscr{S} = 0, \tag{5.13}$$

singled out by the initial condition $\mathscr{S}(0) = I_{2m}$.

The substitution

$$y = \mathscr{S}(t)x \tag{5.14}$$

transforms equation (5.12) to equation (5.7), where

$$\mathscr{A}(t) = \mathscr{Y}_m\mathscr{S}^{-1}(t)\mathscr{Y}_m^{-1}\mathscr{S}(t) = \mathscr{S}^{\tau}(t)\mathscr{S}(t) \qquad (0 \leqq t \leqq l). \tag{5.15}$$

We have used here the fact that the solution $\mathscr{S}(t)$ of the Cauchy problem for the Hamiltonian equation (5.7) is always a symplectic matrix-function: $\mathscr{S}^{\tau}(t)\mathscr{Y}_m\mathscr{S}(t) = \mathscr{Y}_m$ $(0 \leqq t \leqq l)$ (see §1, Chapter VI).

Thus equation (5.12) can be transformed to a Hamiltonian equation (5.7) with a Hamiltonian $\mathscr{A}(t)$ of positive type, and having the additional properties:

1) *the elements of* $\mathscr{A}(t)$ *are absolutely continuous, and,* above all,
2) *the Hamiltonian* $\mathscr{A}(t)$ *is a symplectic matrix-function, i.e.*

$$\mathscr{A}(t)\mathscr{Y}_m\mathscr{A}(t) = \mathscr{Y}_m \quad (0 \leqq t \leqq l); \tag{5.16}$$

from which follows $\det\mathscr{A}(t) = 1$, so that the Hamiltonian $\mathscr{A}(t)$ is positive-definite.

We shall show that the preceding computations are reversible.

It is not hard to see that for a positive-definite real matrix $\mathscr{A}$ of order $2m$ the following three assertions are equivalent: a) the matrix $\mathscr{A}$ is symplectic, b) the spectrum of the matrix $\mathscr{Y}_m\mathscr{A}$ consists of the two points $\pm i$, and c) the matrix $\mathscr{A}$ can be represented in the form $\mathscr{A} = \mathscr{S}^{\tau}\mathscr{S}$, where $\mathscr{S}$ is a real symplectic matrix.

If now $\mathscr{A} = \mathscr{A}(t)$ $(0 \leqq t \leqq l)$ is a matrix-function with certain smoothness properties (if its elements are absolutely continuous or, say, p times continuously differentiable, etc.) and for each $t \in [0, l]$ one of the conditions a), b), c) is fulfilled, then there exists a representation (5.15) of the Hamiltonian $\mathscr{H}(t)$ in which the real symplectic matrix-function $\mathscr{S}(t)$ $(0 \leqq t \leqq l)$ will have the same smoothness properties as the Hamiltonian itself. The substitution $x = \mathscr{S}^{-1}(t)y$, effected with this matrix-function $\mathscr{S}(t)$, transforms the Hamiltonian equation (5.7) to equation (5.12).

Thus, properties 1) and 2) above of a Hamiltonian $\mathscr{H}(t)$ are characteristic for Hamiltonian equations (5.7) which can be transformed to equation (5.12) with a real symmetric matrix coefficient $\mathscr{G}(t)$, whose elements belong to the space $L_1(0,l)$.

It turns out that if the Hamiltonian $\mathscr{H}(t)$ in a Hamiltonian equation is symplectic, then it can be uniquely reconstructed from the monodromy matrix $\mathscr{U}(\mu)$ of this equation.[35]

However, a stronger result holds. In order to formulate it in the language of the theory of multiplicative integrals, we introduce the following definition.

A real canonical representation (5.3) of a function $\mathscr{U}(\mu)$ will be called *totally symplectic* if the Hamiltonian $\mathscr{H}(t)$ is *symplectic* for almost all $t \in [0,l]$.

We will call this representation *densely symplectic* if there is a nowhere dense set in the interval $[0,l]$ such that the Hamiltonian $\mathscr{H}(t)$ is almost everywhere symplectic on its complement.

THEOREM 5.3. *If the entire matrix-function $\mathscr{U}(\mu) = \| u_{jk}(\mu)\|_1^{2m}$ admits a dense symplectic representation* (5.3), *then this representation is unique.*

Moreover, in this case $\mathscr{U}(\mu)$ admits a unique normalized representation (5.3).

This result was obtained rather long ago by M. G. Kreĭn (for $m = 1$, in 1951) as a corollary of his investigations on entire Hermitian operators[36] and the problem of extending Hermitian-positive functions and matrix-functions, and also on inverse problems of the theory of spectral functions of differential operators (see his papers [**12–23**]). Theorem 5.3 is given here for the first time.

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[35] In particular, from the mondromy matrix of equation (5.12) one can always reconstruct uniquely the real symmetric matrix-function $\mathscr{G}(t)$ which anticommutes with $\mathscr{Y}_m$ (compare with the results in the papers by L. A. Sahnovic [**1, 2**]).

[36] Unfortunately, we cannot dwell here even on the definition of an *entire* operator.

4. For the case $m = 1$ we can formulate a more definitive result. In the fourth memoir of his outstanding studies in the theory of Hilbert spaces of entire functions, L. de Branges[37] (see [5], Theorem VI) established, in somewhat different form, the following result.

THEOREM 5.4. *If the entire matrix-function* $\mathscr{U}(\mu) = \|u_{jk}(\mu)\|_1^2$ *of second order admits a real canoncial representation*

$$\mathscr{U}(\mu) = \overset{l}{\widehat{\int_0}} \exp(-\mu \mathscr{Y}_1 \mathscr{H}(t)\, dt), \tag{5.17}$$

then it admits a unique representation of this kind with a normalized Hamiltonian.

We shall write the second-order Hamiltonian $\mathscr{H}(t)$ in question in the same form as in §§5 and 6 of Chapter VI:

$$\mathscr{H}(t) = \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix}. \tag{5.18}$$

That $\mathscr{H}(t)$ is a Hamiltonian of positive type means that the real functions $a(t)$, $b(t)$ and $c(t)$ (from $L_1(0,l)$) satisfy the conditions

$$\begin{gathered} a(t) \geqq 0, \quad c(t) \geqq 0, \quad a(t)c(t) - b^2(t) \geqq 0; \\ \int_0^l a(t)\,dt \int_0^l c(t)\,dt - \left(\int_0^l b(t)\,dt\right)^2 > 0. \end{gathered} \tag{5.19}$$

Normalization means that

$$a(t) + c(t) = 1 \quad (0 \leqq t \leqq l).$$

It is elementary to verify the relation

$$\mathscr{H}(t)\mathscr{Y}_1\mathscr{H}(t) = \det \mathscr{H}(t)\mathscr{Y}_1. \tag{5.20}$$

Thus, for the case being considered, that $\mathscr{H}(t)$ is symplectic is equivalent to its being unimodular: $\det \mathscr{H}(t) = 1$.

Therefore, if the condition

α) $\det \mathscr{H}(t)$ $(\equiv a(t)c(t) - b^2(t)) \neq 0$ *almost everywhere in* $[0,l]$

is satisfied, then the representation (5.17) can be transformed to a totally symplectic representation by means of a simple change of parameter.

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[37] We mention that, apparently, the work of M. G. Kreĭn on the theory of entire Hermitian operators and more generally the work in his entire series [12-24], to which de Branges' studies have direct relationship, remained unknown to de Branges (who in his investigations repeated many results from this series).

In fact, the substitution

$$s = \int_0^t \sqrt{\det \mathscr{A}(t)}\, dt \tag{5.21}$$

in the representation (5.17) yields

$$\mathscr{U}(\mu) = \overset{\hat{l}}{\underset{0}{\widehat{\int}}} \exp(-\mu \mathscr{Y}_1 \hat{\mathscr{A}}(s)\, ds),$$

where

$$\hat{\mathscr{A}}(s) = \mathscr{A}(t) / \sqrt{\det \mathscr{A}(t)}$$

$$\left(0 \leqq s \leqq \hat{l};\ \ \hat{l} = \int_0^l \sqrt{\det \mathscr{A}(t)}\, dt\right)$$

Obviously $\det \hat{\mathscr{A}}(s) = 1 \ \ (0 \leqq s \leqq \hat{l})$.

It is now not difficult to see that the representation (5.17) can be transformed to a dense symplectic representation if (and only if) the following condition is satisfied.

β) *The interval* $[0, l]$ *contains no interval* Δ *on which* $\det \mathscr{A}(t) = 0$ *almost everywhere.*

In this case the substitution (5.21) should be replaced by the substitution

$$s = \int_0^t p(\tau)\, d\tau \quad (0 \leqq t \leqq l),$$

where $p(t)$ is any positive function from $L_1(0, l)$ which satisfies the condition

$$p(t) = (\det \mathscr{A}(t))^{1/2} \quad \text{if} \quad \det \mathscr{A}(t) \neq 0.$$

5. Comparison of Theorems 5.2 and 5.4 leads to the following result.[38)]

THEOREM 5.5. *Every simple S-real Volterra operator A with a two-dimensional skewsymmetric component is S-unicellular.*

In fact, such an operator is S-unitarily equivalent to the simple part of the real model operator A_M,

$$(A_M f)(t) = \mathscr{A}^{1/2}(t) \mathscr{Y}_1 \int_t^l \mathscr{A}^{1/2}(s) f(s)\, ds. \tag{5.22}$$

[38)] This result, as a conjecture by M. G. Kreĭn, was stated in the joint report by M. S. Brodskiĭ, I. C. Gohberg, M. G. Kreĭn and V. I. Macaev [1], and also in a paper by M. S. Brodskiĭ [8]. M. G. Kreĭn was led to this conjecture several years earlier by comparing his earlier conjecture (Theorem 5.4) with the investigations of M. S. Livšic and M. S. Brodskiĭ. L. de Branges [7] also noted that Theorem 5.5 can be obtained on the basis of Theorems 5.2 and 5.4.

Here $\mathscr{A}(t)$ $(0 \leqq t \leqq l)$ is a normalized Hamiltonian of positive type of order 2 ($l = |A_\alpha|_1$; see p. 243).

On the other hand, on the basis of Theorems 5.2 and 5.3 the simple part of A_M is real-unicellular.

Theorem 5.5 implies the following corollary.

COROLLARY 5.1. *A real model operator* A_M *in* $L_2^{(2)}(0,l)$ *is real-unicellular if and only if*

$$\det \mathscr{A}(s) \neq 0 \quad \textit{almost everywhere in} \quad [0,l]. \tag{5.23}$$

In fact, it remains for us to show that (5.23) is a necessary and sufficient condition for A_M to be a simple operator.[39] If $\det \mathscr{A}(t) = 0$ on some set $\mathscr{S}$ of positive measure, then there exists a vector-function $f_0(t)$ $(\in L_2^{(2)}(0,l))$, different from zero on $\mathscr{S}$, such that $\mathscr{A}^{1/2}(t) f_0(t) = 0$ everywhere. Then it follows from (5.23) and from the expression for the adjoint,

$$(A_M^* f)(t) = -\mathscr{A}^{1/2}(t)\mathscr{Y}_1 \int_0^t \mathscr{A}^{1/2}(s) f(s)\, ds,$$

that $A_M^* f_0 = A_M f_0 = 0$, while $f_0 \neq 0$:

$$(f_0, f_0) = \int_0^l f_0^*(t) f_0(t)\, dt = \int_{\mathscr{S}} f_0^*(t) f_0(t)\, dt > 0 \quad (\operatorname{mes} \mathscr{S} > 0).$$

Thus A_M is not a simple operator.

It is easy to verify that the condition (5.23) is also sufficient for $A_M f = A_M^* f = 0$ to imply $f = 0$.

According to Corollary 5.1, the model operator with $\mathscr{A}(t) \equiv I_2$, i.e. the operator defined by

$$(J_{(2)} f)(t) = \mathscr{Y}_1 \int_t^l f(s)\, ds \quad (f \in L_2^{(2)}(0,l)),$$

will be real-unicellular. From this example we can see that *an operator which is not complex-unicellular can turn out to be real-unicellular.*

In fact, the operator $J_{(2)}$ has as invariant subspaces $\mathscr{L}_+$ and $\mathscr{L}_-$, consisting respectively of all vector-functions of the form

$$\{f(t), if(t)\} \in L_2^{(2)}(0,l) \quad \text{and} \quad \{f(t), -if(t)\} \in L_2^{(2)}(0,l);$$

at the same time, these subspaces are mutually orthogonal.

Thus the operator $J_{(2)}$ is not complex-unicellular.

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[39] This assertion is of a general nature: it concerns all model operators A_M. The trivial subspace of the operator A_M (the set of common zeros of A_M and A_M^*) consists of all $f \in L_2^{(n)}(0,l)$ for which $\mathscr{A}(t) f(t) = 0$ almost everywhere.

6. Theorem 4.1 for $n = 2$ yields necessary and sufficient conditions for a given matrix-function $\mathscr{U}(\mu) = \| u_{jk}(\mu) \|_1^2$ to have a representation (5.20). These conditions can be formulated in a new interesting form, which we will obtain with the help of the following elementary lemma.

LEMMA 5.1.[40] *In order that the fractional-linear function*

$$w = (\alpha z + \beta)(\gamma z + \delta)^{-1}, \tag{5.24}$$

corresponding to the complex unimodular matrix

$$\mathscr{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (\alpha\delta - \beta\gamma = 1),$$

map the upper halfplane Π_+ $(\operatorname{Im} z \geqq 0)$ *onto a disc lying inside* Π_+, *it is necessary and sufficient that the matrix*

$$i(\mathscr{A}^* \mathscr{Y}_1 \mathscr{A} - \mathscr{Y}_1) \tag{5.25}$$

be Hermitian-positive.

For the linear transformation

$$w_1 = \alpha z_1 + \beta z_2, \quad w_2 = \gamma z_1 + \delta z_2$$

the Hermitian-positivity of the matrix (5.25) means that for any z_1 and z_2, not simultaneously zero,

$$\frac{1}{i}(w_1 \bar{w}_2 - w_2 \bar{w}_1) > \frac{1}{i}(z_1 \bar{z}_2 - z_2 \bar{z}_1).$$

Hence for $z_1 = 1$, $z_2 = z$ $(\operatorname{Im} z < 0)$ it follows that $w_2 \neq 0$ and

$$\operatorname{Im}(w_1/w_2) > \operatorname{Im} z |w_2|^2,$$

which proves the sufficiency of the conditions of the lemma.

We note that the unimodularity of the matrix $\mathscr{A}$ was not used anywhere. However it is essential for the proof of the necessity of the conditions. We leave this proof to the reader (it is somewhat complicated, but still elementary).

We note that if the fractional-linear transformation (5.24) maps Π_+ onto a disc lying inside Π_+, then the same will be true of the fractional-linear transformation obtained from (5.24) by interchanging α and δ or by replacing β and γ by $-\gamma$ and $-\beta$.

We now introduce the following definition.

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[40] This lemma was recently generalized by M. G. Kreĭn and Ju. L. Šmul'jan [1] to the case in which w, z, α, β, γ and δ are matrices of arbitrary order (or even bounded linear operators, acting in a Hilbert space).

We will call an entire matrix-function

$$\mathscr{U}(\mu) = \begin{pmatrix} u_{11}(\mu) & u_{12}(\mu) \\ u_{21}(\mu) & u_{22}(\mu) \end{pmatrix}$$

an *entire ρ-matrix*[41] if its elements $u_{jk}(\mu)$ $(j,k = 1,2)$ are real entire functions satisfying the following conditions:

a) $\mathscr{U}(0) = I_2$;

b) $\det\mathscr{U}(\mu) = 1$;

c) *for any* μ *from the upper halfplane* $(\operatorname{Im}\mu > 0)$ *the fractional-linear transformation*

$$w = \frac{u_{11}(\mu)z + u_{12}(\mu)}{u_{21}(\mu)z + u_{22}(\mu)}$$

maps the upper halfplane Π_+ *onto a disc lying inside* Π_+.

We note that an entire ρ-matrix-function remains such if we interchange the elements u_{11} and u_{22} or replace u_{12} and u_{21} by $-u_{21}$ and $-u_{12}$.

We can now state the following result.

THEOREM 5.6. *In order that a matrix-function* $\mathscr{U}(\mu)$ *of order two admit a real canonical representation* (5.17), *it is necessary and sufficient that it be an entire* ρ-*matrix.*

This theorem is a corollary of Theorem 5.1 and Lemma 5.1. To show this, it remains to verify that conditions a), b) and c) are equivalent to conditions 1), 2) and 3) from Theorem 5.1 for $m = 1$. Conditions a) and 1) simply coincide. Since for $m = 1$

$$\mathscr{U}^{\tau}(\mu)\mathscr{Y}_1\mathscr{U}(\mu) = \det\mathscr{U}(\mu)\mathscr{Y}_1,$$

condition b) is equivalent to condition 2).

Finally, on the basis of Lemma 5.1 condition c) is equivalent to condition 3).

Recalling the connection between the equation of a string and a canonical equation (Chapter VI, §8), it is natural to conjecture that Theorems 5.5 and 5.6 should imply various results concerning inverse

[41] In M. G. Kreĭn's paper [19] these matrix-functions were called *special* matrix-functions. Various special cases of ρ-matrix-functions have been studied by Stieltjes and Hamburger in moment problems and by H. Weyl in the theory of singular differential equations (see Ju. M. Berezanskiĭ [1] and N. I. Ahiezer [2]; in [2] they are called Nevanlinna matrices). To every entire Hermitian operator there corresponds an entire ρ-matrix (M. G. Kreĭn [13]) with the property $u_{11}(ir)/u_{21}(ir) = O(1/r)$ for $r \to +\infty$, and conversely. In all of these investigations, entire ρ-matrices are used to describe the spectral functions and resolvents of certain Hermitian operators with deficiency indices (1,1). This circumstance clarifies the nomenclature "ρ-matrix".

problems for a string. In particular, from these theorems one can obtain a theorem stating that the mass distribution along a string with a given tension is completely determined by two of its frequency spectra: the spectrum obtained when both ends are fixed, and the spectrum obtained when one end is fixed and the other is free (M. G. Kreĭn [**3, 18, 24**]).

§6. Examples of unicellular and nonunicellular operators

1. The familiar operators from analysis J_α $(0 < \alpha < \infty)$ of fractional integration are instructive examples of unicellular operators; they are defined in the space $L_2(0,1)$ by

$$(J_\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds.$$

The operator J_α for $\alpha = 1$ is related to the operator

$$(Jf)(t) = 2i \int_t^1 f(s)\, ds,$$

considered in §§8 and 9 of Chapter I, by

$$J_1 = (i/2) J^*.$$

Since by Theorem I.9.1 the operator J is unicellular, it follows that J_1 is unicellular.

We have the following more complete result.

THEOREM 6.1. *The operators* J_α $(0 < \alpha < \infty)$ *are unicellular Volterra operators which are pairwise nonsimilar.*

We shall give below a purely analytic proof of the unicellularity of the operators J_α, based on a well-known theorem of Titchmarsh (see Theorem I.9.3).[42]

We will precede this proof by a simple lemma. To simplify its statement, we associate with every function $f \in L_2(0,1)$ a number l_f $(\geqq 0)$, defined by

$$\int_0^{l_f} |f(t)|^2 dt = 0, \qquad \int_{l_f}^{l_f+\epsilon} |f(t)|^2 dt > 0 \quad (\epsilon > 0).$$

[42] In §9, Chapter I it was shown that Titchmarsh's theorem is a consequence of Theorem I.9.1 on the unicellularity of the operator J. Now, in particular, it will be proved that conversely Titchmarsh's theorem implies Theorem I.9.1 (thus, these theorems are equivalent).

LEMMA 6.1. *Let* $\mathfrak{L}$ *be a closed subspace of* $L_2(0,1)$. *If for every* $f \in \mathfrak{L}$ *we have* $l_f > 0$, *then*

$$\inf_{f \in \mathfrak{L}} l_f > 0.$$

PROOF. Let us assume the contrary; then we can choose a sequence $f_n \in \mathfrak{L}$ $(n = 1, 2, \cdots)$ for which the sequence $l_n = l_{f_n}$ $(n = 1, 2, \cdots)$ is decreasing and tends to zero. Obviously

$$\int_{l_{n+1}}^{l_n} |f_{n+1}(t)|^2 dt > 0 \qquad (n = 1, 2, \cdots).$$

Let us consider the sequence of functions $\psi_n = b_n f_n$ $(n = 1, 2, \cdots)$, where $b_1 = 1$, and the further b_n are successively defined so that

$$\int_0^1 |\psi_{n+1}(t)|^2 dt < \frac{1}{3^2} \int_{l_n}^{l_{n-1}} |\psi_n(t)|^2 dt \qquad (n = 1, 2, \cdots).$$

For the functions ψ_n thus defined we obviously have

$$\int_0^1 |\psi_{n+k}(t)|^2 dt \leqq \frac{1}{3^{2k}} \int_{l_n}^{l_{n-1}} |\psi_n(t)|^2 dt \qquad (n = 1, 2, \cdots).$$

Form the function

$$\psi(t) = \sum_{n=1}^{\infty} \psi_n(t).$$

Obviously $\psi \in \mathfrak{L}$, and moreover

$$\begin{aligned} &\left(\int_{l_{n+1}}^{l_n} |\psi(t)|^2 dt\right)^{1/2} \\ &\qquad \geqq \left(\int_{l_{n+1}}^{l_n} |\psi_{n+1}(t)|^2 dt\right)^{1/2} - \sum_{j=n+2}^{\infty} \left(\int_{l_{n+1}}^{l_n} |\psi_j(t)|^2 dt\right)^{1/2} \\ &\qquad \geqq \left(\int_{l_{n+1}}^{l_n} |\psi(t)|^2 dt\right)^{1/2} - \sum_{j=n+2}^{\infty} \left(\int_0^1 |\psi_{n+1}(t)|^2 dt\right)^{1/2}. \end{aligned}$$

Hence

$$\int_{l_{n+1}}^{l_n} |\psi(t)|^2 dt \geqq \frac{1}{2} \int_{l_{n+1}}^{l_n} |\psi_{n+1}(t)|^2 dt > 0 \qquad (n = 1, 2, \cdots).$$

This shows that $l_\psi = 0$, which contradicts the hypothesis of the lemma. The lemma is proved.

PROOF OF THEOREM 6.1. We will first prove that the Volterra operator J_α $(0 < \alpha < \infty)$ is unicellular. To do this we shall show that a vector $f \in L_2(0,1)$ will be a generating vector of the operator J_α[43] if and only if $l_f = 0$.

In fact, it is obvious that $l_{J_\alpha f} \geqq l_f$, and so for $l_f > 0$ the vector f cannot be a generating vector.

Now let $l_f = 0$. We shall show that any vector $g \in L_2(0,1)$ which is orthogonal to all of the vectors $(J_\alpha)^n f$ $(n = 0, 1, \cdots)$ is equal to zero. Indeed, for such a vector

$$\int_0^1 \overline{g(t)} \int_0^t (t-s)^{n\alpha - 1} f(s)\, ds\, dt = 0 \quad (n = 1, 2, \cdots).$$

Replacing $t - s$ by s, we obtain

$$\int_0^1 \overline{g(t)} \int_0^t s^{n\alpha - 1} f(t-s)\, ds\, dt = 0 \quad (n = 1, 2, \cdots)$$

or

$$\int_0^1 \left[\int_s^1 \overline{g(t)} f(t-s)\, dt \right] s^{n\alpha - 1} ds = 0 \quad (n = 1, 2, \cdots).$$

Let n_0 be a positive integer for which $n_0\alpha - 1 > 0$. Denoting the function

$$\left[\int_s^1 \overline{g(t)} f(t-s)\, dt \right] s^{n_0\alpha - 1}$$

by $h(s)$, we obtain

$$\int_0^1 h(s) s^{n\alpha} ds = 0 \quad (n = 1, 2, \cdots). \tag{6.1}$$

Since the system of functions $\{t^{n\alpha}\}_{n=1}^{\infty}$ is complete in $L_2(0,1)$, it follows from (6.1) that $h(t) \equiv 0$, and so

$$\int_t^1 \overline{g(s)} f(s-t)\, ds = 0 \quad (0 \leqq t \leqq 1),$$

or

$$\int_0^t \overline{g(1-s)} f(t-s)\, ds = 0 \quad (0 \leqq t \leqq 1).$$

---

[43] That is, the linear hull of the system of vectors $(J_\alpha)^n f$ $(n = 0, 1, \cdots)$ is dense in $\mathfrak{H}$.

Since $l_f = 0$, it follows from Titchmarsh's theorem (Theorem I.9.3) that $g(t) \equiv 0$ $(0 \leqq t \leqq 1)$. Thus if $l_f = 0$, then f is a generating vector.

Let us now denote by $\mathfrak{H}_l$ $(0 \leqq l \leqq 1)$ the subspace of $L_2(0,1)$, consisting of all functions $f \in L_2(0,1)$ for which $l_f \geqq l$. It is easily seen that each of the subspaces $\mathfrak{H}_l$ $(0 < l < 1)$ is invariant with respect to the operators J_α $(0 < \alpha < \infty)$.

We shall show that any proper invariant subspace $\mathfrak{L}$ of an operator J_α coincides with one of the subspaces $\mathfrak{H}_l$ $(0 < l < 1)$. In fact, no vector f from $\mathfrak{L}$ is a generating vector for an operator J_α; hence $l_f > 0$. By Lemma 6.1

$$l = \min_{f \in \mathfrak{L}} l_f > 0,$$

so that $\mathfrak{L} \subseteq \mathfrak{H}_l$. Let us denote by $f_0 \in \mathfrak{L}$ a vector for which $l_{f_0} = l$.[44] From what was proved above, it is easy to deduce that the sequence $f_1, J_\alpha f_1, \cdots, J_\alpha^n f_1, \cdots$, where $f_1(t) = f_0(t + l)$, is complete in the space $L_2(0, 1 - l)$.

It follows at once that the linear hull of the sequence $\{J_\alpha^n f_0\}_{n=0}^\infty$ is dense in $\mathfrak{H}_l$, and so $\mathfrak{L} = \mathfrak{H}_l$.

Thus, all the operators J_α $(0 < \alpha < \infty)$ are unicellular.

To complete the proof it remains to show that the operators J_α are pairwise nonsimilar. We make a general remark: if two operators A_1 and A_2 $(\in \mathfrak{S}_\infty)$ are similar, then

$$\sup_n \frac{s_n(A_1)}{s_n(A_2)} < \infty \quad \text{and} \quad \sup_n \frac{s_n(A_2)}{s_n(A_1)} < \infty .$$

Indeed, if $A_2 = BA_1B^{-1}$, and hence $A_1 = B^{-1}A_2B$, then

$$s_n(A_2) \leqq |B| \, |B^{-1}| \, s_n(A_1) \quad (n = 1, 2, \cdots),$$
$$s_n(A_1) \leqq |B| \, |B^{-1}| \, s_n(A_2) \quad (n = 1, 2, \cdots).$$

Let us study the asymptotic behavior of the s-numbers of the operator J_α. The numbers $\sigma_n = s_n^{-2}(J_\alpha)$ $(n = 1, 2, \cdots)$ are the characteristic numbers of the integral equation $\phi = \sigma J_\alpha J_\alpha^* \phi$ which, written out in detail, has the form

$$\phi(t) = \frac{\sigma}{\Gamma^2(\alpha)} \int_0^t (t-u)^{\alpha-1} \int_u^1 (s-u)^{\alpha-1} \phi(s) \, ds du. \tag{6.2}$$

Let us consider the case of integer α. In this case equation (6.2) is equivalent to the boundary value problem

[44] It is easily seen that such a vector exists (see the proof of Lemma 6.1).

$$(6.3)\qquad \phi^{(2\alpha)} - \sigma\phi = 0, \quad \phi^{(k)}(0) = \phi^{(\alpha+k)}(1) = 0 \quad (k = 0,1,\cdots,\alpha - 1).$$

The following boundary value problem is simpler:

$$(6.4)\qquad \psi^{(2\alpha)} - \tau\psi = 0, \quad \psi^{(k)}(0) = \psi^{(k)}(1) = 0 \quad (k = 0,1,\cdots,\alpha - 1).$$

Its fundamental functions and characteristic numbers are known:

$$\psi_n(t) = C_n \sin \pi n t; \ \tau_n = (n\pi)^{2\alpha} \quad (n = 1,2,\cdots).$$

Since the changed boundary conditions in the boundary value problem (6.4) cannot change the asymptotic behavior of the eigenvalues (see Theorem VI.1.3), we can assert that $\sigma_n \sim (n\pi)^{2\alpha}$; hence

$$(6.5)\qquad s_n(J_\alpha) \sim 1/(n\pi)^\alpha.$$

We will omit the proof that the asymptotic relation (6.5) also holds for noninteger α. This relation, together with the general remark made earlier regarding the s-numbers of similar operators, shows that the operators J_α are pairwise nonsimilar.

The theorem is proved.

Theorem 6.1 makes possible the following deduction.

In contrast with a finite-dimensional space, where all unicellular operators are similar to each other, *in an infinite-dimensional Hilbert space there exists at least a continuum of unicellular Volterra operators which are pairwise nonsimilar.*

We also note that all the unicellular operators considered in the preceding sections, with the exception of §5, were dissipative Volterra operators with nuclear imaginary components, and consequently their order $p(A)$[45] was always equal to unity. It follows from (6.5) that $p(J_\alpha) = 1/\alpha$. Hence *there exist unicellular Volterra operators of any finite order.*

One can show (see the more general assertion in §6.2) that for $0 < \alpha \leqq 1$ the values of the quadratic form $(J_\alpha f,f)$ $(f \in \mathfrak{H})$ lie in the sector $|\arg\lambda| < \pi\alpha/2$. Thus *the operators* iJ_α $(0 < \alpha \leqq 1)$ *are dissipative Volterra operators of order* $1/\alpha$.

For $1 < \alpha < \infty$ the values of the quadratic form $(J_\alpha f,f)$ $(f \in \mathfrak{H})$ fill up the entire complex plane. The point is that if the order of a Volterra operator is < 1, then this operator cannot be dissipative (see Theorem III.2.4).

[45] We recall that the *order* $p(A)$ of an operator $A \in \mathfrak{S}_\infty$ is the infimum of all numbers p for which $A \in \mathfrak{S}_p$.

2. The unicellularity of the operators J_α for the narrower range $0 < \alpha < 1$ can be obtained as a corollary of Theorem I.9.1 and a general result. To formulate the latter we need the concept of the fractional powers of a dissipative operator (see V. I. Macaev and Ju. A. Palant [1], T. Kato [1], B. Sz.-Nagy and C. Foiaş [4]).

If A ($\in \mathfrak{R}$) is a dissipative operator, then the operator A^ν $(0 < \nu < 1)$ is defined by

$$A^\nu = -\frac{\sin \pi\nu}{\pi} e^{\pi i\nu/2} \int_0^\infty \lambda^{\nu-1} A(A + i\lambda I)^{-1} d\lambda, \tag{6.6}$$

where the integral converges in norm.[46] Putting $A^{n+\nu} = A^n A^\nu$ $(n = 0, 1, \cdots;$ $0 < \nu < 1)$, we obtain the semigroup A^α $(0 < \alpha < \infty)$ of *powers* of the operator A.

1. (JU. A. PALANT).[47] *If A is a dissipative unicellular Volterra operator, then all of its powers A^α $(0 < \alpha < 1)$ are also unicellular Volterra operators.*

The operator IJ_1 is a dissipative unicellular Volterra operator. Substituting $A = iJ_1$ in (6.6) enables us to deduce that $(iJ_1)^\alpha = \exp(\pi i\alpha/2) J_\alpha$ (i.e. $J_1^\alpha = J_\alpha$). Consequently, the operators J_α $(0 < \alpha < 1)$ are Volterra and unicellular.

3. A *formal* analog of a finite-dimensional unicellular operator in an infinite-dimensional space is the operator S, whose matrix relative to some orthogonal basis $\{e_j\}_0^\infty$ of the space $\mathfrak{H}$ has the form

$$\begin{Vmatrix} 0 & 0 & 0 \dots \\ 1 & 0 & 0 \dots \\ 0 & 1 & 0 \dots \\ \cdot & \cdot & \cdot \ \cdot \ \cdot \end{Vmatrix}$$

Obviously S is defined by

$$Se_j = e_{j+1} \quad (j = 0, 1, 2, \cdots),$$

---

[46] For such a definition of the powers A^ν $(0 < \nu \leqq 1)$ of a dissipative operator A, all the values of the quadratic form $(A_\nu f, f)$ $(f \in \mathfrak{H})$ lie in the sector $0 \leqq \arg \lambda \leqq \pi\nu$ (see V. I. Macaev and Ju. A. Palant [1]).

It turns out that if the operator A is dissipative and invertible, then the operator $B = A^{1/n}$, where n is any integer, is the unique operator for which

$$B^n = A \quad \text{and} \quad 0 \leqq \arg(Bf, f) \leqq \pi/n.$$

A generalization of this result to the case of a maximal dissipative unbounded operator was obtained by H. Langer [1].

[47] Oral communication.

and is therefore called the (*right*) *shift* operator.

However, S is not only not a Volterra operator; it is not even completely continuous. Indeed, S is isometric and maps the space $\mathfrak{H}$ onto the subspace orthogonal to the line $\{\lambda e_0\}$.

This operator has nothing in common with a unicellular operator. Besides the obvious invariant subspaces (the closed linear hulls of the vectors $\{e_j\}_n^\infty$, $n = 1, 2, \cdots$), it has a "vast" set of other invariant subspaces which form a complicated structure. Nevertheless, this set admits a complete description, which we shall now give.

It is more convenient to describe all the invariant subspaces of the operator S after realizing the space $\mathfrak{H}$ in the form of the Hardy space $\boldsymbol{H}^2$. The space $\boldsymbol{H}^2$ consists of all analytic functions

$$f(z) = \sum_{j=0}^{\infty} c_j(f)\, z^j$$

for which

$$\sum_{j=0}^{\infty} |c_j|^2 < \infty .$$

Every such function automatically turns out to be holomorphic in the unit disc $|z| < 1$. The scalar product in $\boldsymbol{H}^2$ is defined in a natural way by

$$(f,g) = \sum_{j=0}^{\infty} c_j(f)\overline{c_j(g)}.$$

To map the space $\mathfrak{H}$ unitarily onto the space $\boldsymbol{H}^2$, we associate with every vector

$$f = \sum_{j=0}^{\infty} f_j\, e_j$$

the function

$$f(z) = \sum_{j=0}^{\infty} f_j z^j.$$

Clearly, under this mapping the operator S goes over into the operator of multiplication by the independent variable z.

With every inner function[48] $m(z)$ we associate the subspace $\mathfrak{L}_m$ ($\subset \boldsymbol{H}^2$), consisting of all functions of the form $m(z)f(z)$ ($f(z) \in \boldsymbol{H}^2$).

We shall say that an inner function is normalized, if the first nonzero coefficient in its Maclaurin series expansion is positive.

[48] For the definition of an inner function, see footnote 18), above.

THEOREM 6.2 (A. BEURLING [1]). *The subspaces $\mathfrak{L}_m$, where $m(z)$ is an arbitrary inner function, exhaust all the invariant subspaces of the operator of multiplication by z in H^2. To distinct inner functions there correspond distinct subspaces.*

It follows from Theorem 6.2 that the restriction of the (right) shift operator to any one of its invariant subspaces is again a (right) shift operator which is unitarily equivalent to the original one.

4. Let us consider the operator S_Λ, to which there corresponds in some orthonormal basis $\{e_j\}_0^\infty$ the matrix

$$\left\| \begin{matrix} 0 & 0 & 0 & \dots \\ \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \right\|, \tag{6.7}$$

where $\{\lambda_j\}$ is a sequence of complex numbers.

A finite $(n \times n)$ matrix of the form (6.7) generates a unicellular operator in the space E_n if and only if all of the λ_j are different from zero. This condition is necessary for the unicellularity of the operator S_Λ in the infinite-dimensional case. Indeed, if $\lambda_k = 0$ for some k, then S_Λ has two nonintersecting invariant subspaces: one having basis $\{e_j\}_{j=0}^k$, and the other with basis $\{e_j\}_{j=k+1}^\infty$.

If $\lim \lambda_j = 0$, then, as is easily seen, S_Λ is a Volterra operator. However, the two conditions

$$\lambda_j \neq 0 \ (j = 1, 2, \cdots) \quad \text{and} \quad \lim_{j\to\infty} \lambda_j = 0$$

taken together still turn out to be insufficient for the unicellularity of S_Λ. N. K. Nikol'skiĭ [1] has shown that the operator S_Λ with

$$\lambda_j = \begin{cases} 2^{-j} & \text{for } j \neq 2^p + 1, p = 1, 2, \cdots, \\ 2^{-2(p+1)^2} & \text{for } j = 2^p + 1, p = 1, 2, \cdots \end{cases}$$

is not unicellular. However we have the following result (which is contained in more general results by N. K. Nikol'skiĭ [1]).

2. (N. K. NIKOL'SKIĬ [1]). *Let $\Lambda = \{\lambda_j\}$ be an arbitrary nonincreasing sequence of positive numbers for which the series $\sum_{j=1}^\infty \lambda_j^p$ converges for some p $(0 < p < \infty)$. Then the operator S_Λ is unicellular.*

We note that the condition $\sum_{j=1}^\infty \lambda_j^p < \infty$ means that S_Λ belongs to the class $\mathfrak{S}_p$.

For the special case $\lambda_j = 2^{-j}$ $(j = 1, 2, \cdots)$ the unicellularity of the corresponding operator S_Λ was established earlier by W. F. Donoghue [1].

BIBLIOGRAPHY

N. I. AHIEZER

1. *Lectures in the theory of approximation,* 2nd rev. ed., "Nauka", Moscow, 1965; English transl., Ungar, New York, 1956. MR **20** # 1872; MR **32** # 6108.
2. *The classical moment problem and some related questions in analysis,* Fizmatgiz, Moscow, 1961; English transl., Hafner, New York, 1965. MR **27** # 4028; MR **32** # 1518.

N. I. AHIEZER AND I. M. GLAZMAN

1. *The theory of linear operators in Hilbert space,* GITTL, Moscow, 1950; 2nd rev. ed., "Nauka", Moscow, 1966; English transl., Ungar, New York, 1961. MR **13**, 358; MR **34** # 6527.

V. M. ALEKSANDROV

1. *To the solution of certain contact problems of elasticity theory,* Prikl. Mat. Meh. **27** (1963), 970-972 = J. Appl. Math. Mech. **27** (1963), 1490-1494.

N. ARONSZAJN AND K. T. SMITH

1. *Invariant subspaces of completely continuous operators,* Ann. of Math. (2) **60** (1954), 345-350. MR **16**, 488.

N. H. ARUTJUNJAN

1. *A planar contact problem of the theory of creep,* Prikl. Mat. Meh. **23** (1959), 901-924 = J. Appl. Math. Mech. **23** (1959), 1283-1313.

N. H. ARUTJUNJAN AND M. M. MANUKJAN

1. *A contact problem in the theory of creep taking into account frictional force,* Prikl. Mat. Meh. **27** (1963), 813-820 = J. Appl. Math. Mech. **27** (1963), 1244-1254.

F. V. ATKINSON

1. *Some aspects of Baxter's functional equation,* J. Math. Anal. Appl. **7** (1963), 1-30. MR **27** # 5135.
2. *Discrete and continuous boundary problems,* Academic Press, New York, 1964. MR **31** # 416.

N. K. BARI

1. *Trigonometric series,* Fizmatgiz, Moscow, 1961; English transl., Macmillan, New York and Pergamon Press, Oxford, 1964. MR **23** # A3411; MR **30** # 1347.

M. A. BARKAR′ AND I. C. GOHBERG

1. *Factorization of operators with respect to a discrete chain of projectors in a Banach space,* Mat. Issled. **1** (1966), no. 1, 32-54; English transl., Amer. Math. Soc. Transl. (2) **90** (1970) (to appear). MR **34** # 6529.
2. *Factorization of operators in a Banach space,* Mat. Issled. **1** (1966), no. 2, 98-129; English transl., Amer. Math. Soc. Transl. (2) **90** (1970) (to appear). MR **35** # 780.

G. BAXTER

1. *An operator identity,* Pacific J. Math. **8** (1958), 649-663. MR **21** # 2298.

JU. M. BEREZANSKIĬ

1. *Expansions in eigenfunctions of selfadjoint operators,* "Naukova Dumka", Kiev, 1965; English transl., Transl. Math. Monographs, vol. 17, Amer. Math. Soc., Providence, R. I., 1968. MR **36** # 5768.

A. BEURLING

1. *On two problems concerning linear transformations in Hilbert space,* Acta Math. **81** (1948), 239-255. MR **10**, 381.

M. Š. Birman and M. Z. Solomjak

1. *On double Stieltjes operator integrals*, Dokl. Akad. Nauk SSSR **165** (1965), 1223-1226 = Soviet Math. Dokl. **6** (1965), 1567-1571. MR **32** # 6234.
2. *Double Stieltjes operator integrals*, Probl. Math. Phys., no. 1, Leningrad. Univ., Leningrad, 1966, pp. 33-67 = Topics in Math. Phys., Vol. I, Consultants Bureau, New York, 1967, pp. 25-54. MR **35** # 767b.
3. *Double Stieltjes operator integrals and problems on multipliers*, Dokl. Akad. Nauk SSSR **171** (1966), 1251-1254 = Soviet Math. Dokl. **7** (1966), 1618-1621. MR **35** # 767a.
4. *Double Stieltjes operator integrals*. II, Probl. Math. Phys., no. 2, Leningrad. Univ., Leningrad, 1967, pp. 26-60. (Russian) MR **38**, 2621.

E. Bishop

1. *A duality theorem for an arbitrary operator*, Pacific J. Math. **9** (1959), 379-397. MR **22** # 8339.

M. S. Brodskiĭ

1. *On Jordan cells of infinite-dimensional operators*, Dokl. Akad. Nauk SSSR **111** (1956), 926-929. (Russian) MR **19**, 48.
2. *On a problem of I. M. Gel'fand*, Uspehi Mat. Nauk **12** (1957), no. 2 (74), 129-132. (Russian) MR **20** # 1229.
3. *Integral representations of bounded non-selfadjoint operators with a real spectrum*, Dokl. Akad. Nauk SSSR **126** (1959), 1166-1169. (Russian) MR **21** # 7438.
4. *On the integral representation of dissipative completely continuous operators*, Naučn. Zap. Kafedr Mat. Fiz. Estest. Odessk. Gos. Ped. Inst. **24** (1959), 3-7. (Russian)
5. *On the triangular representation of completely continuous operators with one-point spectra*, Uspehi Mat. Nauk **16** (1961), no. 1 (97), 135-141: English transl., Amer. Math. Soc. Transl. (2) **47** (1965), 59-65. MR **24** # A426.
6. *Unicellularity criteria for Volterra operators*, Dokl. Akad. Nauk SSSR **138** (1961), 512-514 = Soviet Math. Dokl. **2** (1961), 637-639. MR **24** # A1015.
7. *A multiplicative representation of certain analytic operator-functions*, Dokl. Akad. Nauk SSSR **138** (1961), 751-754 = Soviet Math. Dokl. **2** (1961), 695-698. MR **24** # A1031.
8. *Unicellularity of real Volterra operators*, Dokl. Akad. Nauk SSSR **147** (1962), 1010-1012 = Soviet Math. Dokl. **3** (1962), 1749-1751. MR **26** # 6779.
9. *On the unicellularity of the integration operator and a theorem of Titchmarsh*, Uspehi Mat. Nauk **20** (1965), no. 5 (125), 189-192. (Russian) MR **32** # 8055.

M. S. Brodskiĭ and G. È. Kisilevskiĭ

1. *Criterion for the unicellularity of dissipative Volterra operators with nuclear imaginary components*, Izv. Akad. Nauk SSSR. Ser. Mat. **30** (1966), 1213-1228; English transl., Amer. Math. Soc. Transl. (2) **65** (1967), 282-296. MR **34** # 3310.

M. S. Brodskiĭ and M. S. Livšic

1. *Spectral analysis of non-selfadjoint operators and intermediate systems*, Uspehi Mat. Nauk **13** (1958), no. 1 (79), 3-85; English transl., Amer. Math. Soc. Transl. (2) **13** (1960), 265-346. MR **20** # 7221; MR **22** # 3982.

M. S. Brodskiĭ and Ju. L. Šmul'jan

1. *Invariant subspaces of a linear operator and divisors of its characteristic function*, Uspehi Mat. Nauk **19** (1964), no. 1 (115), 143-149. (Russian) MR **29** # 2645.

M. S. Brodskiĭ, I. C. Gohberg, M. G. Kreĭn and V. I. Macaev

1. *Some investigations in the theory of non-selfadjoint operators*, Proc. Fourth All-Union Math. Congress (Leningrad, 1961), vol. II, Sectional Lectures, "Nauka", Leningrad, 1964, pp. 261-271; English transl., Amer. Math. Soc. Transl. (2) **65** (1967), 237-251.

V. M. BRODSKIĬ

1. *On the integral with respect to a chain of orthogonal projectors,* Bul. Akad. Štiince RSS Moldoven, **1966**, no. 4, 59-64. (Russian) MR **36** # 1997.
2. *Multiplicative representation of the characteristic functions of contraction operators,* Dokl. Akad. Nauk SSSR **173** (1967), 256-259 = Soviet Math. Dokl. **8** (1967), 362-366. MR **36** # 3154.

M. S. BUDJANU

1. *A certain theorem on the factorization of operator functions,* Bul. Akad. Štiince RRS Moldoven, **1965**, no. 7, 22-32. (Russian) MR **33** # 7872.

M. S. BUDJANU AND I. C. GOHBERG

1. *A general theorem on factorization of matrix functions,* Studies in Algebra and Math. Anal., Izdat. "Kaetja Moldovenjaske", Kishinev, 1965, pp. 116-121. (Russian) MR **36** # 726.

A. P. CALDERON AND A. ZYGMUND

1. *A note on the interpolation of sublinear operations,* Amer. Math. J. **78** (1956), 282-288. MR **18**, 586.

T. CARLEMAN

1. *Zur Theorie der linearen Integralgleichungen,* Math. Z. **9** (1921), 196-217.
2. *Über die Abelsche Integralgleichung mit konstanten Integrationsgrenzen,* Math. Z. **15** (1922), 111-120.

H. CARTAN

1. *Sur les matrices holomorphes de n variables complexes,* J. Math. Pures Appl. **19** (1940), 1-26. MR **1**, 312.

Ė. R. CEKANOVSKIĬ

1. *The real and imaginary parts of an unbounded operator,* Dokl. Akad. Nauk SSSR **139** (1961), 48-51 = Soviet Math. Dokl. **2** (1961), 881-884. MR **24** # A1619.
2. *Model elements of non-selfadjoint operators,* Dokl. Akad. Nauk SSSR **142** (1962), 1043-1046 = Soviet Math. Dokl. **3** (1962), 256-260. MR **25** # 455.
3. *On the description of the invariant subspaces and of the unicellularity of the operator of integration in the space $W_2^{(r)}$,* Uspehi Mat. Nauk **20** (1965), no. 6 (126), 169-172. (Russian) MR **32** # 8156.

M. COTLAR

1. *A unified theory of Hilbert transforms and ergodic theorems,* Revista Mat. Cuyana **1** (1955), 105-167. MR **18**, 893.

R. COURANT AND D. HILBERT

1. *Methoden der mathematischen Physik.* I, II, Interscience, New York, 1943, 1953; Russian transl., GITTL, Moscow, 1951. MR **5**, 97; MR **13**, 800; MR **16**, 426.

JU. L. DALECKIĬ

1. *On continuous rotation of subspaces in a Banach space,* Uspehi Mat. Nauk **12** (1957), no. 3 (75), 147-154. (Russian) MR **20** # 1199.

JU. L. DALECKIĬ AND S. G. KREĬN

1. *Properties of operators depending on a parameter,* Dokl. Akad. Nauk USSR (1960), 433-436. (Ukrainian)

L. DE BRANGES

1. *Some Hilbert spaces of entire functions,* Proc. Amer. Math. Soc. **10** (1959), 840-846. MR **22** # 4833.
2. *Some Hilbert spaces of entire functions,* Trans. Amer. Math. Soc. **96** (1960), 259-295. MR **24** # A3289a.
3. *Some Hilbert spaces of entire functions.* II, Trans. Amer. Math. Soc. **99** (1961), 118-152. MR **24** # A3289b.

4. *Some Hilbert spaces of entire functions.* III, Trans. Amer. Math. Soc. **100** (1961), 73-115. MR **24** # A3289c.
5. *Some Hilbert spaces of entire functions.* IV, Trans. Amer. Math. Soc. **105** (1962), 43-83. MR **26** # 583.
6. *Some applications of spaces of entire functions,* Canad. J. Math. **15** (1963), 563-583. MR **27** # 3801.
7. *Invariant subspaces of nonselfadjoint transformations,* Bull. Amer. Math. Soc. **69** (1963), 587-590. MR **26** # 6780.
8. *Some Hilbert spaces of analytic functions.* I, Trans. Amer. Math. Soc. **106** (1963), 445-468. MR **26** # 2866.
9. *New and old problems for entire functions,* Bull. Amer. Math. Soc. **70** (1964), 214-223. MR **28** # 1305.
10. *Some Hilbert spaces of analytic functions.* II, J. Math. Anal. Appl. **11** (1965), 44-72. MR **35** # 778.
11. *Some Hilbert spaces of analytic functions.* III, J. Math. Anal. Appl. **12** (1965), 149-186. MR **35** # 779.

W. F. Donoghue

1. *The lattice of invariant subspaces of a completely continuous quasi-nilpotent transformation,* Pacific J. Math. **7** (1957), 1031-1035. MR **19**, 1066.

N. Dunford

1. *A survey of the theory of spectral operators,* Bull Amer. Math. Soc. **64** (1958), 217-274; Russian transl., Matematika **4** (1960), no. 1, 53. MR **21** # 3616.

N. Dunford and J. T. Schwartz

1. *Linear operators.* Part I: *General theory,* Pure and Appl. Math., vol. 7, Interscience, New York and London, 1958; Russian transl., IL, Moscow, 1962. MR **22** # 8302.
2. *Linear operators.* Part II: *Spectral theory, Selfadjoint operators in Hilbert space,* Interscience, New York, 1963; Russian transl., "Mir", Moscow, 1966. MR **32** # 6181.

C. L. Dolph

1. *Recent developments in some non-self-adjoint problems of mathematical physics,* Bull. Amer. Math. Soc. **67** (1961), 1-69. MR **25** # 5612.

V. M. Eni

1. *On the multiplicity of an eigenvector and the multiplicity of a characteristic value of a matrix pencil,* Bul. Akad. Štiince RSS Moldoven. 1965, no. 7, 94-97. (Russian) MR **33** # 5642.

D. K. Faddeev and V. N. Faddeeva

1. *Computational methods of linear algebra,* Fizmatgiz, Moscow, 1960; 2nd ed., 1963; English transl., Freeman, San Francisco, Calif., 1963. MR **28** # 4659; MR **28** # 1742.

F. R. Gantmaher and M. G. Kreĭn

1. *Oscillation matrices and kernels and small oscillations of mechanical systems,* 2nd ed., GITTL, Moscow, 1950; German transl., Akademie-Verlag, Berlin, 1960; English transl., U. S. Atomic Energy Commission, AEC-tr-4481, 1961. MR **14**, 178; MR **22** # 5161.

I. M. Gel'fand and V. B. Lidskiĭ

1. *On the structure of the regions of stability of linear canonical systems of differential equations with periodic coefficients,* Uspehi Mat. Nauk **10** (1955), no. 1 (63), 3-40; English transl., Amer. Math. Soc. Transl. (2) **8** (1958), 143-181. MR **17**, 482; MR **19**, 960.

I. M. Gel'fand, D. A. Raĭkov and G. E. Šilov

1. *Commutative normed rings,* Fizmatgiz, Moscow, 1960; English transl., Chelsea, New York, 1964. MR **34** # 4940.

JU. P. GINZBURG

1. *On J-contractive operator-functions,* Dokl. Akad. Nauk SSSR **117** (1957), 171-173. (Russian) MR **20** # 1203.
2. *The factorization of analytic matrix functions,* Dokl. Akad. Nauk SSSR **159** (1964), 489-492 = Soviet Math. Dokl. **5** (1964), 1510-1514. MR **30** # 3228.
3. *Multiplicative representations of bounded analytic operator-functions,* Dokl. Akad. Nauk SSSR **170** (1966), 23-26 = Soviet Math. Dokl. **7** (1966), 1125-1128. MR **34** # 611.

I. C. GOHBERG

1. *On linear operators depending analytically upon a parameter,* Dokl. Akad. Nauk SSSR **78** (1951), 629-632. (Russian) MR **13**, 46.
2. *Relations between the spectra of the Hermitian components of nilpotent matrices and on the integral of triangular truncation,* Bul. Akad. Štiince RSS Moldoven. **1963**, no. 1, 27-37. (Russian) MR **35** # 2168.
3. *The factorization problem in normed rings, functions of isometric and symmetric operators and singular integral equations,* Uspehi Mat. Nauk **19** (1964) no. 1 (115), 71-124 = Russian Math. Surveys **19** (1964), no. 1, 63-114. MR **29** # 487.

I. C. GOHBERG AND M. G. KREĬN

1. *The basic propositions on defect numbers, root numbers and indices of linear operators,* Uspehi Mat. Nauk **12** (1957), no. 2 (74), 43-118; English transl., Amer. Math. Soc. Transl. (2) **13** (1960), 185-264. MR **20** # 3459; MR **22** # 3984.
2. *Systems of integral equations on a half line with kernels depending on the difference of arguments,* Uspehi Mat. Nauk **13** (1958), no. 2 (80), 3-72; English transl., Amer. Math. Soc. Transl. (2) **14** (1960), 217-287. MR **21** # 1506; MR **22** # 3954.
3. *Completely continuous operators with a spectrum concentrated at zero,* Dokl. Akad. Nauk SSSR **128** (1959), 227-230. (Russian) MR **24** # A1022.
4. *On the theory of triangular representations of nonselfadjoint operators,* Dokl. Akad. Nauk SSSR **137** (1961), 1034-1037 = Soviet Math. Dokl. **2** (1961), 392-396. MR **25** # 3370.
5. *Volterra operators with imaginary component in one class or another,* Dokl. Akad. Nauk SSSR **139** (1961), 779-782 = Soviet Math. Dokl. **2** (1961), 938-985. MR **25** # 3371.
6. *The effect of some transformations of kernels of integral equations upon the equations' spectra,* Ukrain. Mat. Ž. **13** (1961), no. 3, 12-38; English transl., Amer. Math. Soc. Transl. (2) **35** (1964), 263-295. MR **27** # 1788.
7. *On the problem of factoring operators in a Hilbert space,* Dokl. Akad. Nauk SSSR **147** (1962), 279-282 = Soviet Math. Dokl. **3** (1962), 1578-1582. MR **26** # 6777.
8. *Factorization of operators in Hilbert space,* Acta Sci. Math. Szeged **25** (1964), 90-123; English transl., Amer. Math. Soc. Transl. (2) **51** (1966), 155-188. MR **29** # 6313.
9. *Criteria for completeness of the system of root vectors of a contraction,* Ukrain. Mat. Ž. **16** (1964), 78-82; English transl., Amer. Math. Soc. Transl. (2) **54** (1966), 119-124. MR **29** # 2651.
10. *Introduction to the theory of linear nonselfadjoint operators,* "Nauka", Moscow, 1965; English transl., Transl. Math. Monographs, vol. 18, Amer. Math. Soc., Providence, R.I., 1969. MR **36** # 3137.
11. *On the multiplicative representation of the characteristic functions of operators close to unitary ones,* Dokl. Akad. Nauk SSSR **164** (1965), 732-735 = Soviet Math. Dokl. **6** (1965), 1279-1283. MR **33** # 571.

I. C. GOHBERG AND N. JA. KRUPNIK

1. *On the norm of the Hilbert transform in L_p spaces,* Funkcional. Anal. i Priložen. **2** (1968), no. 2, 91-92 = Functional Anal. Appl **2** (1968), 180-181.

V. P. GURARIĬ

1. *The Fourier transform in* $L^2(-\infty, \infty)$ *with a weight,* Mat. Sb. **58(100)** (1962), 439-452. (Russian) MR **26** # 2810.

H. HELSON

1. *Lectures on invariant subspaces,* Academic Press, New York and London, 1964. MR **30** # 1409.

E. HILLE AND R. S. PHILLIPS

1. *Functional analysis and semi-groups,* Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R.I., 1957. MR **19**, 664.

K. HOFFMAN

1. *Banach spaces of analytic functions,* Prentice-Hall, Englewood Cliffs, N. J., 1962; Russian transl., IL, Moscow, 1963. MR **24** # A2844.

I. S. IOHVIDOV AND M. G. KREĬN

1. *Spectral theory of operators in a space with an indefinite metric.* I, Trudy Moskov. Mat. Obšč. **5** (1956), 367-432; English transl., Amer. Math. Soc. Transl. (2) **13** (1960), 105-175. MR **18**, 320; MR **22** # 3983.

V. A. JAKUBOVIČ

1. *Construction of the group of symplectic matrices and the structure of the set of nonstable canonical systems of differential equations with periodic coefficients,* Mat. Sb. **44 (86)** (1958), 313-352. (Russian) MR **20**, 3325.
2. *Arguments on the group of symplectic matrices,* Mat. Sb. **55 (97)** (1961), 255-280. (Russian) MR **26** # 5072.
3. *The structure of the functional space of complex canonical equations with periodic coefficients,* Dokl. Akad. Nauk SSSR **139** (1961), 54-57 = Soviet Math. Dokl. **2** (1961), 887-890. MR **25** # 5232.

P. D. KALAFATI

1. *On a new orthogonal system of functions,* Dokl. Akad. Nauk SSSR **105** (1955), 631-633. (Russian) MR **18**, 310.
2. *Integral equations with homogeneous skewsymmetric kernels,* Trudy Nikolaevskogo Korablestroitel. Inst. **8** (1956), 246-255. (Russian)

G. K. KALISCH

1. *A functional analysis proof of Titchmarsh's theorem on convolution,* J. Math. Anal. Appl. **5** (1962), 176-183. MR **25** # 4307.

T. M. KARASEVA

1. *On a test for the boundedness of the solutions of Hill's differential equations,* Prikl. Mat. Meh. **20** (1956), 549-551. (Russian) MR **18**, 306.
2. *On the rate of growth and boundedness of solutions of second order differential equations with periodic coefficients,* Izv. Akad. Nauk SSSR. Ser. Mat. **29** (1965), 41-70; English transl., Amer. Math. Soc. Transl. (2) **79** (1969), 181-214. MR **31** # 3664.

T. KATO

1. *Fractional powers of dissipative operators,* J. Math. Soc. Japan **13** (1961), 246-274. MR **25** # 1453.

M. V. KELDYŠ AND V. B. LIDSKIĬ

1. *On the spectral theory of non-selfadjoint operators,* Proc. Fourth All-Union Math. Congress (Leningrad, 1961), vol. I, Izdat. Akad. Nauk SSSR, Leningrad, 1963, pp. 101-120. (Russian) MR **30** # 1414.

G. È. KISILEVSKIĬ

1. *On the unicellularity of dissipative Volterra operators,* Ukrain. Mat. Ž. **16** (1964), 690-696. (Russian) MR **30** # 419.

2. *Conditions for unicellularity of dissipative Volterra operators with finite-dimensional imaginary component,* Dokl. Akad. Nauk SSSR **159** (1964), 505-508 = Soviet Math. Dokl. **5** (1964), 1527-1531. MR **30** # 5262.
3. *On the reduction of a certain class of infinite-dimensional operators to Jordan normal form,* First Republ. Math. Conf. of Young Researchers, Part I, Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, 1965, pp. 332-341. (Russian) MR **33** # 3101.
4. *On the order of the characteristic matrix functions of dissipative Volterra operators,* Dokl. Akad. Nauk SSSR **159** (1964), 730-733 = Soviet Math. Dokl. **5** (1964), 1579-1582. MR **30** # 5163.
5. *Invariant subspaces of dissipative Volterra operators with nuclear imaginary components,* Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 3-23 = Math. USSR Izv. **2** (1968), 1-20. MR **36** # 4375.

A. N. Kolmogorov
1. *Sur les fonctions harmoniques conjuguées et les séries de Fourier,* Fund. Math. **7** (1925), 23-28.

M. G. Kreĭn
1. *On the theory of entire matrix functions of exponential type,* Ukrain. Mat. Ž. **3** (1951), 164-173. (Russian) MR **14**, 981.
2. *On certain problems on the maximum and minimum of characteristic values and on the Ljapunov zones of stability,* Prikl. Mat. Meh. **15** (1951), 323-348; English transl., Amer. Math. Soc. Transl. (2) **1** (1955), 163-187. MR **13**, 348; MR **17**, 484.
3. *On inverse problems for a nonhomogeneous cord,* Dokl. Akad. Nauk SSSR **82** (1952), 669-672. (Russian) MR **14**, 649.
4. *On a generalization of investigations of Stieltjes,* Dokl. Akad. Nauk SSSR **87** (1952), 881-884. (Russian) MR **14**, 868.
5. *On integral equations generating differential equations of second-order,* Dokl. Akad. Nauk SSSR **97** (1954), 21-24. (Russian) MR **16**, 372; MR **17**, 1436.
6. *On a new method of solving linear integral equations of the first and second kind,* Dokl. Akad. Nauk SSSR **100** (1955), 413-416. (Russian) MR **17**, 375.
7. *On criteria of stable boundedness of solutions of periodic canonical systems,* Prikl. Mat. Meh. **19** (1955), 641-680. (Russian) MR **17**, 1088.
8. "The basic propositions of the theory of λ-zone of stability of a canonical system of linear differential equations with periodic coefficients", in *Memorial volume to A. A. Andronov,* Izdat. Akad. Nauk SSSR, Moscow, 1955, pp. 413-498. (Russian) MR **17**, 738.
9. *On the theory of the accelerants and S-matrices of canonical differential systems,* Dokl. Akad. Nauk SSSR **111** (1956), 1167-1170. (Russian) MR **19**, 277.
10. *Integral equations on the half-line with a kernel depending on the difference of the arguments,* Uspehi Mat. Nauk **13** (1958), no. 5 (83), 3-120; English transl., Amer. Math. Soc. Transl. (2) **22** (1962), 163-288. MR **21** # 1507.
11. *A contribution to the theory of linear nonselfadjoint operators,* Dokl. Akad. Nauk SSSR **130** (1960), 254-256 = Soviet Math. Dokl. **1** (1960), 38-40. MR **24** # A1024.
12. *Sur le problème du prolongement des fonctions Hermitiennes, positives et continues,* Dokl. Akad. Nauk SSSR **26** (1940), 17-22. MR **2**, 361.
13. *On a remarkable class of Hermitian operators,* C. R. (Dokl.) Acad. Sci. URSS **44** (1944), 175-179. MR **6**, 269.
14. *On the logarithm of an infinitely decomposable Hermite-positive function,* C. R. (Dokl.) Acad. Sci. URSS **45** (1944), 91-94. MR **6**, 269.
15. *On the problem of continuation of helical arcs in Hilbert space,* C. R. (Dokl.) Acad. Sci. URSS **45** (1944), 139-142. MR **6**, 269.

16. *The fundamental propositions of the theory of representations of Hermitian operators with deficiency index (m, m)*, Ukrain. Mat. Ž. **1** (1949), no. 2, 3-66. (Russian) MR **14**, 56.
17. *Solution of the inverse Stürm-Liouville problem*, Dokl. Akad. Nauk SSSR **76** (1951), 21-24. (Russian) MR **12**, 613.
18. *Determination of the density of a nonhomogeneous symmetric cord by its frequency spectrum*, Dokl. Akad. Nauk SSSR **76** (1951), 345-348. (Russian) MR **13**, 43.
19. *On the indeterminate case of the Stürm-Liouville boundary problem in the interval* (0, ∞), Izv. Akad. Nauk SSSR Ser. Mat. **16** (1952), 293-324. (Russian) MR **14**, 558.
20. *On some cases of effective determination of the density of an inhomogeneous cord from its spectral function*, Dokl. Akad. Nauk SSSR **93** (1953), 617-620. (Russian) MR **15**, 796.
21. *On a basic approximation problem of the theory of extrapolation and filtration of stationary random processes*, Dokl. Akad. Nauk SSSR **94** (1954), 13-16. (Russian) MR **16**, 53.
22. *On a method of effective solution of an inverse boundary problem*, Dokl. Akad. Nauk SSSR **94** (1954), 987-990. (Russian) MR **16**, 38.
23. *Continuous analogs of propositions on polynomials orthogonal on the unit circle*, Dokl. Akad. Nauk SSSR **105** (1955), 637-640. (Russian) MR **18**, 291.
24. *On some new problems of the theory of oscillations of Stürmian systems*, Prikl. Mat. Meh. **16** (1952), 555-568. (Russian) MR **14**, 650.
25. *Analytic problems and results in the theory of linear operators in Hilbert space*, Internat. Congr. Math. (Moscow, 1966), pp. 189-216; English transl., Amer. Math. Soc. Transl. (2) **90** (1970) (to appear).

M. G. Kreĭn and V. A. Jakubovič

1. *Hamiltonian systems of linear differential equations with periodic coefficients*, Proc. Internat. Sympos. Nonlinear Vibrations, vol. I, 1961, Izdat. Akad. Nauk. Ukrain. SSR, Kiev, 1963. (Russian) MR **28** # 312.

M. G. Kreĭn and G. Ja. Ljubarskiĭ

1. *Analytic properties of the multipliers of periodic canonical differential systems of positive type*, Izv. Akad. Nauk SSSR. Ser. Mat. **26** (1962), 549-572; English transl., Amer. Math. Soc. Transl. (2) **89** (1970) (to appear). MR **26** # 400.

M. G. Kreĭn and Ju. L. Šmul'jan

1. *On fractional-linear transformations with operator coefficients*, Mat. Issled. **2** (1967), no. 3, 64-96. (Russian)

S. G. Kreĭn and Ju. I. Petunin

1. *Scales of Banach spaces*, Uspehi Mat. Nauk **21** (1966), no. 2 (128), 89-168 = Russian Math. Surveys **21** (1966), no. 2, 85-159. MR **33** # 1719.

H. K. Langer

1. *Über die Wurzeln eines maximalen dissipativen Operators*, Acta Math. Acad. Sci. Hungar. **13** (1962), 415-424. MR **26** # 1757.
2. *Ein Zerspaltungssatz für Operatoren in Hilbertraum*, Acta Math. Acad. Sci. Hungar. **12** (1961), 441-445. MR **25** # 3378.

P. D. Lax and R. S. Phillips

1. *Scattering theory*, Pure and Appl. Math., vol. 26, Academic Press, New York, 1967. MR **36** # 530.

B. Ja. Levin

1. *Distribution of zeros of entire functions*, GITTL, Moscow, 1956; English transl., Transl. Math. Monographs, vol. 5, Amer. Math. Soc., Providence, R. I., 1964. MR **19**, 402; MR **28** # 217.

V. B. LIDSKIĬ

1. *Conditions for completeness of the system of root subspaces for non-selfadjoint operators with discrete spectra,* Trudy Moskov. Mat. Obšč. **8** (1959), 83-120; English transl., Amer. Math. Soc. Transl. (2) **34** (1963), 241-281. MR **21** # 6539.

O. N. LITVINENKO

1. *A method for the synthesis of non-homogeneous lines,* Radiotehn. i Èlektron. **11** (1966), 1869-1870. (Russian)

M. S. LIVŠIC

1. *On a class of linear operators in a Hilbert space,* Mat. Sb. **19 (61)** (1946), 239-262; English transl., Amer. Math. Soc. Transl. (2) **13** (1960), 61-83. MR **8**, 588; MR **22** # 3981a.
2. *On the spectral resolution of linear non-selfadjoint operators,* Mat. Sb. **34 (76)** (1954), 145-199; English transl., Amer. Math. Soc. Transl. (2) **5** (1957), 67-114. MR **16**, 48; MR **18**, 748.
3. *Theory of non-selfadjoint operators and its applications,* Trudy Third All-Union Math. Congress, vol. 3, 1956, pp. 269-276. (Russian)
4. *Operators, oscillations, waves,* "Nauka", Moscow, 1966. (Russian)

M. S. LIVŠIC AND V. P. POTAPOV

1. *A theorem on the multiplication of characteristic matrix functions,* Dokl. Akad. Nauk SSSR **72** (1950), 625-628. (Russian) MR **11**, 669.

A. M. LJAPUNOV

1. *General problem of stability of motion,* GITTL, Moscow, 1950; French transl., Ann. of Math. Studies no. 17, Princeton Univ. Press, Princeton, N. J., and Oxford Univ. Press, London, 1947. MR **9**, 34; MR **12**, 612; MR **15**, 1139.
2. *On a series, related to the theory of linear differential equations with periodic coefficients,* Collected works, Vol. II, Izdat. Akad. Nauk SSSR, 1956, pp. 387-390.

JU. I. LJUBIČ AND V. I. MACAEV

1. *To the spectral theory of linear operators in Banach space,* Dokl. Akad. Nauk SSSR **131** (1960), 21-23 = Soviet Math. Dokl. **1** (1960), 184-186.
2. *Operators with separable spectrum,* Mat. Sb. **56 (98)** (1962), 433-468; English transl., Amer. Math. Soc. Transl. (2) **47** (1965), 89-129. MR **25** # 2450.

V. I. MACAEV

1. *On the growth of entire functions that admit a certain estimate from below,* Dokl. Akad. Nauk SSSR **132** (1960), 283-286 = Soviet Math. Dokl. **1** (1960), 548-552. MR **23** # A328.
2. *A class of completely continuous operators,* Dokl. Akad. Nauk SSSR **139** (1961), 548-551 = Soviet Math. Dokl. **2** (1961), 972-975. MR **24** # A1617.
3. *Volterra operators obtained from self-adjoint operators by perturbation,* Dokl. Akad. Nauk SSSR **139** (1961), 810-813 = Soviet Math. Dokl. **2** (1961), 1013-1016. MR **25** # 457.
4. *A method of estimation for the resolvents of non-selfadjoint operators,* Dokl. Akad. Nauk SSSR **154** (1964), 1034-1037 = Soviet Math. Dokl. **5** (1964), 236-241. MR **28** # 1495.
5. *Several theorems on completeness of root subspaces of completely continuous operators,* Dokl. Akad. Nauk SSSR **155** (1964), 273-276 = Soviet Math. Dokl. **5** (1964), 396-399. MR **28** # 2444.

V. I. MACAEV AND JU. A. PALANT

1. *On the powers of a bounded dissipative operator,* Ukrain. Mat. Ž. **14** (1962), 329-337. (Russian) MR **26** # 4184.

E. MAGENES

1. *Spazi di interpolazione ed equazioni a derivate parziali,* Atti del VII Congresso dell'Unione Matematica Italiana, Genova, 1963, Edizioni Cremonese, Rome, 1965, pp. 134-197; Russian transl., Uspehi Mat. Nauk **21** (1966), no. 2 (128), 169-218. MR **35** # 5925; **34** # 3074.

G. F. MANDŽAVIDZE

1. *Approximate solution of boundary problems of the theory of analytic functions,* Investigations on modern problems of the theory of functions of a complex variable, Fizmatgiz, Moscow, 1960, pp. 365-370. (Russian) MR **22** # 6890.

G. F. MANDŽAVIDZE AND B. V. HVEDELIDZE

1. *On the Riemann-Privalov problem with continuous coefficients,* Dokl. Akad. Nauk SSSR **123** (1958), 791-794. (Russian) MR **22** # 3806.

P. MASANI

1. *The Laurent factorization of operator-valued functions,* Proc. London Math. Soc. **6** (1956), 59-69. MR **18**, 138.

J. MIKUSINSKI

1. *Une simple démonstration du théorème de Titchmarsh sur la convolution,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 715-717. MR **22** # 2854.

B. S. MITJAGIN

1. *An interpolation theorem for modular spaces,* Mat. Sb. **66** (**108**) (1965), 473-482. (Russian) MR **31** # 1562.

G. M. MORDASOVA

1. *Asymptotic behavior of the eigenvalues of a linear integral equation with discontinuous kernel,* Dokl. Akad. Nauk SSSR **142** (1962), 1022-1025 = Soviet Math. Dokl. **3** (1962), 234-238. MR **26** # 4134.

B. SZ.-NAGY

1. *Sur les contractions de l'espace de Hilbert,* Acta Sci. Math. Szeged **15** (1953), 87-92. MR **15**, 326.
2. *Sur les contractions de l'espaces de Hilbert.* II, Acta Sci. Math. Szeged **18** (1957), 1-14. MR **19**, 873.

B. SZ.-NAGY AND C. FOIAŞ

1. *Sur les contractions de l'espace de Hilbert.* III, Acta Sci. Math. Szeged **19** (1958), 25-45. MR **21** # 2188.
2. *Sur les contractions de l'espace de Hilbert.* IV, Acta Sci. Math. Szeged **21** (1960), 251-259. MR **23** # A3445.
3. *Sur les contractions de l'espace de Hilbert.* V: *Translations bilatérales,* Acta Sci. Math. Szeged **23** (1962), 106-129. MR **26** # 2890.
4. *Sur les contractions de l'espace de Hilbert.* VI: *Calcul fonctionnel,* Acta Sci. Math. Szeged **23** (1962), 130-167. MR **26** # 2891.
5. *Sur les contractions de l'espace de Hilbert.* VII: *Triangulations canoniques. Fonction minimum,* Acta Sci. Math. Szeged **25** (1964), 12-37. MR **30** # 455.
6. *Sur les contractions de l'espace de Hilbert.* VIII: *Fonctions caractéristiques. Modèles fonctionnels,* Acta Sci. Math. Szeged **25** (1964), 38-71. MR **30** # 2348.
7. *Sur les contractions de l'espace de Hilbert.* IX: *Factorisations de la fonction caractéristique. Sous-espaces invariants,* Acta Sci. Math. Szeged **25** (1964), 283-316. MR **31** # 5101.
8. *Sur les contractions de l'espace de Hilbert.* X: *Contractions similaires à des transformations unitaires,* Acta Sci. Math. Szeged **26** (1965), 79-91. MR **34** # 1856.
9. *Corrections et compléments aux Contractions.* IX, Acta Sci. Math. Szeged **26** (1965), 193-196. MR **33** # 4689.

10. *Sur les contractions de l'espace de Hilbert.* XI: *Transformations unicellulaires,* Acta Sci. Math. Szeged **26** (1965), 301-324. MR **32** # 8176.
11. *Décomposition spectrale des contractions presque unitaires,* C. R. Acad. Sci. Paris Ser. A-B **262** (1966), A440-A442. MR **33** # 3111.
12. *Sur les contractions de l'espace de Hilbert.* XII: *Fonctions intérieures admettant des facteurs extérieurs,* Acta Sci. Math. Szeged **27** (1966), 27-33. MR **33** # 6414.

M. A. NAĬMARK
1. *Linear differential operators,* GITTL, Moscow, 1954; English transl., Ungar, New York, 1968; German transl., Akademie-Verlag, Berlin, 1960. MR **16**, 702.

N. K. NIKOL'SKIĬ
1. *Unicellularity and non-unicellularity of weighted shift operators,* Dokl. Akad. Nauk SSSR **172** (1967), 287-290 = Soviet Math Dokl. **8** (1967), 91-94. MR **36** # 728.

R. S. PHILLIPS
1. *Dissipative operators and hyperbolic systems of partial differential equations,* Trans. Amer. Math. Soc. **90** (1959), 193-254; Russian transl., Matematika **6** (1962), no. 4, 11-70. MR **21** # 3669.

A. I. PLESNER
1. *Spectral theory of linear operators,* "Nauka", Moscow, 1965. (Russian) MR **33** # 3106.

A. I. PLESNER AND V. A. ROHLIN
1. *Spectral theory of linear operators,* Uspehi Mat. Nauk **1** (1946), no. 1 (11), 71-191; English transl., Amer. Math. Soc. Transl. (2) **62** (1967), 29-175. MR **9**, 43.

V. T. POLJACKIĬ
1. *On the reduction of quasi-unitary operators to a triangular form,* Dokl. Akad. Nauk SSSR **113** (1957), 756-759. (Russian) MR **19**, 873.

G. JA. POPOV
1. *On a method of solution of the axisymmetric contact problem of the theory of elasticity,* Prikl. Mat. Meh. **25** (1961), 76-85 = J. Appl. Math. Mech. **25** (1961), 105-118. MR **27** # 3124.
2. *On an approximate method for solving certain planar contact problems of elasticity theory,* Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **14** (1961), no. 3, 81-96. (Russian)
3. *Certain properties of classical polynomials and their application to contact problems,* Prikl. Mat. Meh. **28** (1964), 442-451 = J. Appl. Math. Mech. **28** (1964), 544-555. MR **31** # 3647.
4. *Concerning a planar contact problem for an elastic half-strip,* Izv. Akad. Nauk Armjan. SSR Ser. Meh. **4** (1965), 153-157. (Russian)

V. P. POTAPOV
1. *The multiplicative structure of J-contractive matrix functions,* Trudy Moskov. Mat. Obšč. **4** (1955), 125-236; English transl., Amer. Math. Soc. Transl. (2) **15** (1960), 131-243. MR **17**, 958; MR **22** # 5733.
2. *On holomorphic matrix functions bounded in the unit circle,* Dokl. Akad. Nauk SSSR **72** (1950), 849-852. (Russian) MR **13**, 736.

I. I. PRIVALOV
1. *Boundary properties of analytic functions,* GITTL, Moscow, 1950; German transl., Hochschulbücher für Math., Bd. 25, VEB Deutscher Verlag, Berlin, 1956. MR **13**, 926; MR **18**, 727.

I. E. PROKOPOVIČ
1. *On the solution of the plane contact problem taking account of creep,* Prikl. Mat. Meh. **20** (1956), 680-687. (Russian) MR **19**, 342.

I. M. Rapoport

1. *On a variational problem in the theory of ordinary differential equations with boundary conditions,* Dokl. Akad. Nauk SSSR **73** (1950), 889-890. (Russian) MR **12**, 183.
2. *On asymptotic methods in the theory of differential equations,* Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1954. (Russian) MR **17**, 734.

F. Riesz and B. Sz.-Nagy

1. *Leçons d'analyse fonctionelle,* Akad. Kiadó, Budapest, 1952; 2nd ed., 1953; 3rd ed., Akad. Kiadó, Budapest and Gauthier-Villars, Paris, 1955; English transl., Ungar, New York, 1955. MR **14**, 286; MR **15**, 132; MR **16**, 837; MR **17**, 175.

M. Riesz

1. *Sur les maxima des formes bilinéaires et sur les fonctionelles linéaires,* Acta Math. **49** (1926), 465-497.
2. *Sur les fonctions conjuguées,* Math. Z. **27** (1937), 213-244.

N. A. Rostovcev

1. *On the solution of a plane contact problem,* Prikl. Mat. Meh. **17** (1953), 99-106. (Russian) MR **14**, 1146.

L. A. Sahnovič

1. *On the reduction of Volterra operators to the simplest form and on inverse problems,* Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 235-262. (Russian) MR **19**, 970.
2. *Spectral analysis of Volterra operators and some inverse problems,* Dokl. Akad. Nauk SSSR **115** (1957), 666-669. (Russian) MR **19**, 866.
3. *Spectral analysis of operators of the form* $Kf = \int_0^x f(t)k(x-t)\,dt$, Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 299-308. (Russian) MR **20** # 5409.
4. *The reduction of non-selfadjoint operators to triangular form,* Izv. Vysš. Učebn. Zaved. Matematika **1959**, no. 1 (8), 180-186. (Russian) MR **25** # 460.
5. *A study of the "triangular form" of non-selfadjoint operators,* Izv. Vysš. Učebn. Zaved. Matematika **1959**, no. 4 (11), 141-149; English transl., Amer. Math. Soc. Transl. (2) **54** (1966), 75-84. MR **25** # 461.

S. O. Šatunovskiĭ

1. *Introduction to analysis,* "Mathesis", Odessa, 1923.

R. Schatten

1. *Norm ideals of completely continuous operators,* Ergebnisse der Math. und ihrer Grenzgebiete, Heft 27, Springer-Verlag, Berlin, 1960. MR **22** # 9878.

I. Schur

1. *Über die characteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen,* Math. Ann. **66** (1909), 488-510.

E. M. Semenov

1. *Interpolation of linear operators in symmetric spaces,* Dokl. Akad. Nauk SSSR **164** (1965), 746-749 = Soviet Math. Dokl. **6** (1965), 1294-1298. MR **32** # 8139.

N. P. Sergeev

1. *On a method for solving linear integral equations,* Leningrad. Gos. Ped. Inst. Učen. Zap. **33** (1956), 149-153. (Russian)

I. B. Simonenko

1. *Riemann's boundary value problem with a continuous coefficient,* Dokl. Akad. Nauk SSSR **124** (1959), 278-281. (Russian) MR **22** # 3805.

Ju. L. Šmul'jan

1. *Isometric operators with infinite deficiency indices and their orthogonal extensions,* Dokl. Akad. Nauk SSSR **87** (1952), 11-14. (Russian) MR **14**, 882.
2. *Operators with degenerate characteristic functions,* Dokl. Akad. Nauk SSSR **93** (1953), 985-988. (Russian) MR **15**, 803.

S. L. SOBOLEV

1. *Remarks on the numerical solution of integral equations*, Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 431; English transl., Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **11** (1957), no. 4 (23), 5. MR **18**, 322; MR **21** # 439.

I. JA. ŠTAERMAN

1. *The contact problem of the theory of elasticity*, GITTL, Moscow, 1949. (Russian) MR **11**, 756.

A. V. ŠTRAUS

1. *Characteristic functions of linear operators*, Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 43-74; English transl., Amer. Math. Soc. Transl. (2) **40** (1964), 1-37. MR **25** # 4363.

E. C. TITCHMARSH

1. *The theory of functions*, 2nd ed., Oxford Univ. Press, Oxford, 1939; Russian transl., GITTL, Moscow, 1951. MR **33** # 5850.
2. *Reciprocal formulae involving series and integrals*, Math. Z. **25** (1926), 321-381.

J. WERMER

1. *The existence of invariant subspaces*, Duke Math. J. **19** (1952), 615-622. MR **14**, 384.

N. WIENER AND P. MASANI

1. *The prediction theory of multivariate stochastic processes*. II: *The linear predictor*. Acta Math. **99** (1958), 93-137. MR **20** # 4325.

F. WOLF

1. *Operators in Banach space which admit a generalized spectral decomposition*, Nederl. Akad. Wetensch. Proc. Ser. A **60** = Indag. Math. **19** (1957), 302-311. MR **19**, 869.

N. E. ŽUKOVSKIĬ

1. *Conditions for the finiteness of the integrals of the equation* $d^2y/dx^2 + py = 0$, Selected papers, Vol. 1, OGIZ, Moscow, 1948, pp. 246-253. (Russian) MR **11**, 574.

A. ZYGMUND

1. *Trigonometric series*, Vols. I, II, 2nd rev. ed., Cambridge Univ. Press, New York, 1959; Russian transl., "Mir", Moscow, 1965. MR **21** # 6498; MR **31** # 2554.

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An abstract Volterra operator is, roughly speaking, a compact operator in a Hilbert space whose spectrum consists of a single point $\lambda = 0$. The theory of abstract Volterra operators, significantly developed by the authors of the book and their collaborators, represents an important part of the general theory of non-self-adjoint operators in Hilbert spaces.

The book, intended for all mathematicians interested in functional analysis and its applications, discusses the main ideas and results of the theory of abstract Volterra operators. Of particular interest to analysts and specialists in differential equations are the results about analytic models of abstract Volterra operators and applications to boundary value problems for ordinary differential equations.

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