Extrinsic Geometry of Convex Surfaces

by
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Volume Thirty-five
Translations of Mathematical Monographs

10.1090/mmono/035

Translations of Mathematical Monographs

Volume 35

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American Mathematical Society
Providence, Rhode Island
1973

ВНЕШНЯЯ ГЕОМЕТРИЯ ВЫПУКЛЫХ ПОВЕРХНОСТЕЙ

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Издательство „Наука"
Главная редакция
Физико-Математической Литературы
Москва 1969

Translated from the Russian by
Israel Program for Scientific Translations

AMS (MOS) subject classifications (1970).
Primary 53C45; Secondary 53A05, 53A35, 35A30.

Library of Congress Cataloging in Publication Data **CIP**

Pogorelov, Alekseĭ Vasil'evich.
Extrinsic geometry of convex surfaces.

(Translations of mathematical monographs, v. 35)
Translation of Vneshni͡ai͡a geometrii͡a vypuklykh poverkhnosteĭ.
1. Convex surfaces. I. Title. II. Series.
QA649.P6313 516'.36 72-11851
ISBN 0-8218-1585-7

Printed in the United States of America

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APPENDIX

UNSOLVED PROBLEMS

Young geometers often experience difficulties in choosing problems for research. We therefore present a few unsolved problems in this Appendix, most of them with hints for solution.

1. Prove the following assertion: The spherical image of a geodesic on a convex surface is a rectifiable curve, i.e. every interior point of the geodesic has a neighborhood whose spherical image has finite length.

We suggest the following approach. First note that it is sufficient to prove rectifiability for the spherical image of a small arc of the geodesic. We may therefore assume that the convex surface is closed. It can always be completed to a complete surface, in such a way that a small arc of a segment (shortest join) remains a segment. Moreover, we may assume that the segment under consideration can be continued as a segment beyond at least one of its endpoints. This is true for any arc of a segment. Thus we may confine attention to a segment γ on a closed convex surface F, which can be continued as a segment beyond one of its endpoints.

Construct a sequence of polyhedra P_n converging to the surface F. Without loss of generality we may assume that the endpoints A and B of the segment lie on all polyhedra P_n. Let γ_n be the segment on P_n connecting A and B. By the inclusion property for segments (Chapter I, §3) the sequence γ_n converges to γ. The spherical images γ_n^* of the segments γ_n converge to the spherical image γ^* of γ. To prove that γ^* is rectifiable, it now suffices to show that the lengths of the curves γ_n^* are uniformly bounded.

Let $\alpha_1, \alpha_2, \cdots$ be the faces of the polyhedron P_n through which the segment γ_n passes, $E_1, E_2, \cdots$ the halfspaces defined by the planes of the faces $\alpha_1, \alpha_2, \cdots$. The intersection of the halfspaces E_n is a solid polyhedron P_n'. Let T be a cube containing all the polyhedra P_n. The intersection of the cube T and the polyhedron P_n' is a certain polyhedron Q_n contained in T. The segment γ_n of P_n lies on the polyhedron Q_n. Since P_n is contained in Q_n, it follows that γ_n is also a segment on Q_n (by Busemann's theorem).

Let $k_1, k_2, \cdots$ be the edges of the polyhedron Q_n which cut the segment γ_n, $\delta_1, \delta_2, \cdots$ their lengths and $\vartheta_1, \vartheta_2, \cdots$ the exterior angles at the edges $k_1, k_2, \cdots$. Then the length of the spherical image of γ_n, i.e. the length of γ_n^*, is equal to $s_n = \vartheta_1 + \vartheta_2 + \cdots$. The quantity $\vartheta_1\delta_1 + \vartheta_2\delta_2 + \cdots$ can be estimated in terms of the integral mean curvature of the polyhedron Q_n, hence in terms of the edge of the cube T containing Q_n. It follows that if $\delta_k > c_0 > 0$ for all k, then $s_n \leqq C(T)/c_0$.

Now assume that the polyhedron Q_n has edges k_s of length less than ϵ_n, and the sum of exterior angles ϑ_s over these edges is greater than σ_n, where $\epsilon_n \to 0$ and $\sigma_n \to \infty$ as $n \to \infty$. One proves that for sufficiently large n the curve γ_n cannot be a segment on Q_n. The reason is that for small ϵ_n and large σ_n a "large amount of curvature" is concentrated near γ_n. Therefore the length of γ_n cannot be the absolute minimum of the lengths of curves connecting A and B on the polyhedron Q_n.

2. Prove the following theorem.

A convex surface, homeomorphic to a disk, with nonnegative (nonpositive) integral geodesic curvature (i.g.c.) along the boundary, can be applied to any isometric surface (i.e. continuously bent into it).

We indicate an approach to the proof. First consider a polyhedron. Let P_1 be a convex polyhedron homeomorphic to a disk, whose angles at the boundary vertices are $\geqq \pi$ (the i.g.c. along the boundary is nonpositive). Let P_2 be a convex polyhedron isometric to P_1. We must show that P_1 can be applied to P_2. Let Q_1 and Q_2 be the convex hulls of P_1 and P_2. They are closed polyhedra. The polyhedron Q_i is the union of P_i and some polyhedron P_i' isometric to a convex plane polygon. Let A and B be two vertices of the polyhedron Q_i on the boundary of the polyhedron P_i, α and β the curvature at these vertices. Connect A and B by a segment γ within the domain P_i'. (This is possible, since P_i' is a convex domain.) Now take two plane triangles with base l equal to the length of γ, and angles $\alpha' \leqq \alpha$, $\beta' \leqq \beta$ at the base. Glue these triangles together along their lateral sides, and glue the bases to the polyhedron Q_i cut along the segment γ. By the Gluing Theorem there exists a closed polyhedron Q_i' which realizes the polyhedral metric obtained by gluing the triangles to the cut. Subjecting the angles α' and β' to a continuous variation from zero to α and β, respectively, we get a continuous deformation of Q_i' (because of monotypy). The domain on Q_i' corresponding under isometry to P_i then undergoes a continuous bending.

Now take two other vertices on the boundary of P_i, or one vertex on the boundary and a new vertex generated by the above gluing procedure.

Connect these vertices by a segment, cut the polyhedron along the segment, and glue two new triangles to the cut. A finite number of repetitions of this procedure transforms the original polyhedron Q_i into a polyhedron $\overline{Q}_i$ which is the union of a polyhedron isometric to P_i and a polyhedron $\widetilde{P}_i$ isometric to a cone. The boundaries of $\widetilde{P}_1$ and $\widetilde{P}_2$ have equal i.g.c. It follows easily that they are isometric. The monotypy theorem for polyhedra then implies that $\overline{Q}_1$ and $\overline{Q}_2$ are congruent, and hence so are the domains on them isometric to P_1 and P_2. We have thus transformed the original polyhedra P_1 and P_2 by a continuous bending into congruent polyhedra. Hence each can be applied to the other.

In order to proceed now from polyhedra to general convex surfaces, one uses simultaneous approximation of isometric convex surfaces by isometric polyhedra and the monotopy theorem for general convex surfaces.

Now assume that the angles at the boundary vertices of the polyhedra P_i do not exceed π. We again consider the convex hulls Q_i. Let $\overline{P}_i$ be the polyhedron completing P_i to the closed polyhedron Q_i. To prove that P_1 can be applied to P_2, it suffices to show that $\overline{P}_1$ can be applied to $\overline{P}_2$, with the conditions governing its contact with P_1 observed at each stage of the deformation. This is the purpose of the following constructions.

By cutting and gluing triangles, one transforms the polyhedron Q_i into a polyhedron $\widetilde{Q}_i$ containing a domain $\widetilde{P}_i$ isometric to $\overline{P}_i$; the remainder of $\widetilde{Q}_i$ is a surface V_i isometric to a cone (Figure 35). Now flatten out the "leaves" of the polyhedron V_i, preserving their convexity. This transforms the domain $\widetilde{P}_i$ completing the polyhedron V_i to the convex hull into some domain $\widetilde{P}_i'$, while the polyhedron V_i becomes a domain on some polyhedral angle V_i'. Now apply the angle V_1' to the angle V_2'. When this is done, the domain $\widetilde{P}_1'$ is applied to P_2'. The result is a continuous deformation of $\overline{P}_1$ into $\overline{P}_2$, with the conditions governing its contact with P_1 observed at each stage of the deformation. One now proceeds as before to general convex surfaces.

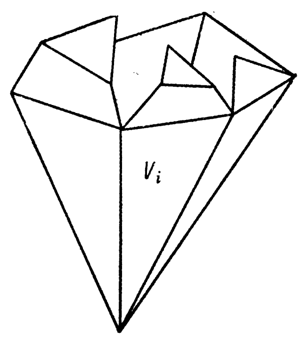

FIGURE 35

3. In §4 of Chapter IV we derived formulas associating with any

pair of isometric surfaces in elliptic space a pair of isometric surfaces in euclidean space. Conversely, each pair of isometric surfaces in euclidean space goes into a pair of isometric surfaces in elliptic space. These formulas involve a vector parameter e_0. Let F_1 and F_2 be two isometric surfaces in euclidean space. Determine the corresponding surfaces Φ_1 and Φ_2 in elliptic space with parameter $e_0 = e_0'$. Now proceed from the surfaces Φ_1 and Φ_2 to a pair of isometric surfaces F_1' and F_2' in euclidean space, using the formulas with parameter $e_0 = e_0''$. Derive formulas setting up the correspondence between the isometric surfaces F_1 and F_2 and the surfaces F_1' and F_2'. Find conditions under which the convexity of the surfaces F_1 and F_2 implies that of F_1' and F_2'. Varying the parameters e_0' and e_0'', and also the relative position of the surfaces F_1 and F_2, consider the problem of transforming a pair of unbounded isometric convex surfaces into a pair of bounded isometric surfaces. In particular, determine whether the monotypy problem for unbounded convex surfaces can be reduced in this way to the monotypy problem for closed convex surfaces or for convex surfaces with fixed boundary.

4. As in the elliptic case, studied in §4 of Chapter IV, formulas can be determined which associate with each pair of isometric surfaces in hyperbolic space a pair of isometric surfaces in euclidean space. Study this correspondence. In particular, find conditions under which convexity of the surfaces in hyperbolic space implies convexity of the corresponding surfaces in euclidean space. Can the monotypy problem for surfaces in hyperbolic space be reduced to the monotypy problem for euclidean space? Some partial results in this direction have been obtained by Gajubov [**34**], but they are far from complete.

5. An incomplete convex metric defined in a domain G is in general not realizable as a convex surface, for the simple reason that the total (integral) curvature of the manifold G with this metric may exceed 4π, while the curvature of a convex surface is always $\leqq 4\pi$. However, there are grounds for the assertion that, under very broad assumptions, this metric is realizable as a *locally* convex surface, i.e. a surface each point of which has a neighborhood which is a convex surface.

Here are some considerations on this problem. Let G be a doubly connected domain (homeomorphic to a circular annulus). Assume that the contours γ_1 and γ_2 bounding the domain are geodesic polygons in the given metric. Divide each side δ_1 of the polygon γ_1 into two by its midpoint P_1, and identify points on γ_1 equidistant from P_1. Do the same for the sides δ_2 of γ_2. The result is a closed manifold R whose curvature is nonnegative everywhere except for two points A_1 and A_2, at which the vertices of the polygons γ_1 and γ_2 are identified.

The manifold R is isometrically embeddable into the locally euclidean space considered in §11 of Chapter VI, with the points A_1 and A_2 on the z-axis. The required realization of R as a locally convex surface is now obtained by mapping the locally euclidean space into euclidean space, identifying geometrically identical points of these spaces.

6. A convex metric M defined in a domain G, homeomorphic to a disk, with boundary γ of nonnegative i.g.c. in the metric M, is realizable as a certain convex cap F. Consider the problem of realizability of a convex metric M defined in a domain G homeomorphic to a disk, whose boundary γ lies on a given surface Φ.

One attack on this problem is as follows. To simplify matters, assume that Φ is an unbounded surface which can be projected in one-to-one fashion onto the entire xy-plane. Let E^+ denote the region of space lying above the surface Φ. Construct a Riemannian space R from two mirror-symmetric copies of the euclidean region E^+ and a regular intermediate layer δ, such that when the thickness of the layer δ tends to zero the space R becomes a metric space R_0 consisting of the two copies of E^+ glued together along the boundary surface Φ. The constructed space R must be symmetric with respect to some totally geodesic surface σ within the layer δ, and the regions E^+ must be symmetric to each other with respect to σ.

Now form a closed manifold M' homeomorphic to a sphere, from two oppositely oriented copies of the manifold M and a regular layer h separating them in such a way that M' admits an inner symmetry with respect to a closed geodesic γ within the layer h, and moreover the two copies of M correspond by symmetry.

The manifold M' is now realized in the space R as a closed surface F' (it is assumed that this can be done). In view of the symmetry of the manifold M' and the space R, this realization can be so constructed that the geodesic γ lies in the surface σ and the surface F' is symmetric with respect to σ. Now letting the thickness of the layers δ and h tend to zero, we get a closed surface F_0 in the space R_0. The domain on this surface lying in the region E^+ furnishes the required realization of the manifold M as a convex surface with boundary on the surface Φ.

7. Complete the proof of the rigidity of multiply-connected locally convex surfaces in a Riemannian space, as indicated in §12 of Chapter VI.

8. Let F_1 and F_2 be two isometric, identically oriented convex surfaces. Assume that corresponding unit vectors on the surfaces satisfy the relation $\tau_1 + \tau_2 \neq 0$. Then the vector valued function $r = \frac{1}{2}(r_1 + r_2)$, where r_1 and r_2 are the radius vectors of corresponding points on F_1 and F_2, defines a convex surface F. Under certain additional assumptions this was proved in Chapters I and II.

The vector field $z = r_1 - r_2$ is a bending field for the surface F. Using this relation between a pair of isometric surfaces and the infinitesimal bendings of the mean surface F, many monotypy problems for convex surfaces can be reduced to the problem of the rigidity of the mean surface.

The following question is natural in this context. Can the surfaces F_1 and F_2 always be so placed that $\tau_1 + \tau_2 \neq 0$ for directions corresponding under the isometry? This is apparently the case for almost all (in measure) relative positions. Prove this assertion.

9. Consider the problem of the existence of a closed convex surface satisfying the equation $f(R_1, R_2, n) = \varphi(n)$, where R_1, R_2 are the principal radii of curvature and n the unit normal to the surface.

This problem can be attacked by the methods of §5 of Chapter VII.

10. Consider the existence problem for a convex surface F whose spherical image coincides with a given convex domain ω on the unit sphere, whose supporting function $H(n)$ coincides with a given continuous function on the boundary of the spherical image, and whose principal radii of curvature at each interior point of the surface satisfy the equation $f(R_1, R_2) = \varphi(n)$, where $f(R_1, R_2) \equiv g(R_1 R_2, R_1 + R_2)$ is strictly monotone in R_1 and R_2, i.e. $\partial f/\partial R_1 > 0$ and $\partial f / \partial R_2 > 0$.

Suppose that the domain ω is in the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$. Set $h(x,y) = H(x,y,1)$, where H is the supporting function of the required surface. The function h satisfies an elliptic equation

$$\Phi(h_{11}, h_{12}, h_{22}, x, y) = 0$$

(see §4, Chapter VII). The existence problem for F reduces to the solvability of the equation $\Phi = 0$. This can be treated on the basis of Bernšteĭn's theorem, first deriving a priori estimates for the posited solution and its first and second derivatives. One first considers the case of analytic functions f, φ and a domain ω bounded by an analytic contour.

For considerations relating to the derivation of a priori estimates, see §§3 and 5 of Chapter VII. We remark that if h_1 and h_2 are solutions of the equations $f = \varphi_1$ and $f = \varphi_2$, then $h_1 - h_2$ cannot assume a maximum in the interior of ω if $\varphi_1 \leqq \varphi_2$. To proceed from analytic to regular data, it suffices to establish a priori estimates for the second derivatives at interior points (see §3 of Chapter VII).

11. It is well known that metric duality can be defined in elliptic space. Let Φ be a convex surface in elliptic space and Φ' the surface polar to Φ (the surface Φ' is the envelope of the polars of the points of Φ). There is a natural correspondence between the points of Φ and Φ': any point P on Φ is associated with the point at which the surface Φ' is tangent to the polar of P. If the curvature K of the space is unity, the extrinsic curvature of Φ on an arbitrary set M is equal to the area of the corresponding set M' on the surface Φ'.

Delete some plane from the elliptic space, and interpret the remaining region on the three-dimensional hemisphere

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1, \quad x_0 > 0.$$

Let Φ be a closed convex surface in the spherical zone $0 < x_0 < \epsilon$. The polar surface Φ' lies in the ϵ-neighborhood of the pole $P(0,0,0,1)$.

The line element of the surface Φ can be expressed as

$$ds^2 = ds_0^2 + \lambda d\sigma^2,$$

where ds_0^2 is the line element of the unit sphere and $\lambda \to 0$ as $\epsilon \to 0$. The extrinsic curvature of the surface Φ is $K_e = \lambda\varphi_\sigma + O(\lambda^2)$, where φ_σ depends on the quadratic form σ.

Project a neighborhood of the pole P of the hemisphere onto the tangent hyperplane E of the hemisphere at P. Let $\overline{\Phi}'$ be the projection of the surface Φ' onto the euclidean space E. Subject the surface $\overline{\Phi}'$ to a similarity mapping with ratio of similitude $1/\lambda$ and let $\epsilon \to 0$. The curvature of the limit surface is $1/\varphi_\sigma$.

Using this construction, derive a new solution of Minkowski's problem, based on the theorem which states that a given metric ds^2 can be realized on a convex surface in elliptic space.

12. In §7 of Chapter VIII we considered the existence of a closed convex surface with given generalized curvature. Analytic interpretation of the result leads to a theorem on the solvability of a certain equation of a very general type defined on the sphere.

Consider the one-dimensional analog of this problem, relaxing the requirement that the curvature be positive. This yields a certain theorem on the existence of a closed curve with given generalized curvature. Analytically speaking, this implies the existence of a periodic solution of an equation $y'' = \varphi(x, y, y')$, where the function φ is periodic in x. For what classes of equations, i.e. for what functions φ, does the geometric theorem guarantee the existence of periodic solutions? Generalize the result to systems of equations

$$y_i'' = \varphi(x, y_1, \cdots, y_n, y_1', \cdots, y_n'), \quad i = 1, 2, \cdots, n.$$

The one-dimensional analog of Minkowski's problem is this: Prove that there exists a closed curve with given radius of curvature $R(\vartheta)$, as a function of the angle of rotation ϑ of the tangent. The problem has a solution if

$$\oint_0^{2\pi} e^{i\vartheta} R(\vartheta)\, d\vartheta = 0.$$

This condition always holds if $R(\vartheta + \pi) = R(\vartheta)$. The supporting function $p(\vartheta)$ of the required curve has a simple expression:

$$p(\vartheta) = \int_0^{\vartheta} e^{i(\vartheta + \tau)} R(\tau)\, d\tau.$$

Using Schauder's fixed-point principle, as in the existence proof for solutions of strongly elliptic Monge-Ampère equations (§8 of Chapter VIII), prove the most general possible theorem on the existence of a closed curve with given generalized length. Interpret the result in terms of the existence of periodic solutions of the equation $y'' = \varphi(x, y, y')$, where φ is periodic in x. Employing geometric ideas, study the problem analytically, under the broadest possible assumptions on the equation. Consider the case of systems of equations.

BIBLIOGRAPHY

G. M. ADELSON-WELSKY

1. *Généralisation d'un théorème géométrique de M. Serge Bernstein,* C. R. (Dokl.) Acad. Sci. URSS **49** (1945), 391-392. MR **8**, 91.

A. D. ALEKSANDROV

2. *Intrinsic geometry of convex surfaces,* OGIZ, Moscow, 1948; German transl., Akademie Verlag, Berlin, 1955. MR **10**, 619; **17**, 74.
3. *Convex polyhedra,* GITTL, Moscow, 1950; German transl., Math. Lehrbücher und Monographien, Forschungsinstitut für Math. II. Abteilung: Math. Monographien, Band VIII, Akademie-Verlag, Berlin, 1958. MR **12**, 732; **19**, 1192.
4. *On a class of closed surfaces,* Mat. Sb. **4** (**46**) (1938), 69-77. (Russian; German summary)
5. *Smoothness of a convex surface of bounded Gaussian curvature,* C. R. (Dokl.) Acad. Sci. URSS **36** (1942), 195-199. MR **4**, 169.
6. *Existence and uniqueness of a convex surface with a given integral curvature,* C. R. (Dokl.) Acad. Sci. URSS **35** (1942), 131-134. MR **4**, 169.
7. *Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it,* Učen. Zap. Leningrad. Gos. Univ. **37** Mat. **3** (1939), 3-35. (Russian) MR **2**, 155.
8. *Dirichlet's problem for the equation* $\operatorname{Det}\|z_{ij}\| = \Phi(z_1, \cdots, z_n, z, x_1, \cdots, x_n)$. I, Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **13** (1958), no. 1, 5-24. (Russian) MR **20** # 3385.
9. *Some theorems on partial differential equations of the second order,* Vestnik Leningrad. Univ. **9** (1954), no. 8, 3-17. (Russian) MR **17**, 493.
10. *Curvature of convex surfaces,* C. R. (Dokl.) Acad. Sci. URSS **50** (1945), 23-26. MR **14**, 577.
11. *The inner metric of a convex surface in a space of constant curvature,* C. R. (Dokl.) Acad. Sci. URSS **45** (1944), 3-6. MR **7**, 167.
12. *On infinitesimal deformations of irregular surfaces,* Mat. Sb. **1** (**43**) (1936), 307-322. (Russian; German summary)
13. *Sur les théorèmes d'unicité pour les surfaces fermées,* C. R. (Dokl.) Acad. Sci. URSS **22** (1939), 99-102.
14. *On the work of S. E. Cohn-Vossen,* Uspehi Mat. Nauk **2** (1947), no. 3 (19), 107-141. (Russian) MR **9**, 485.
15. *Uniqueness theorems for surfaces in the large.* I, Vestnik Leningrad. Univ. **11** (1956), no. 19, 5-17; English transl., Amer. Math. Soc. Transl. (2) **21** (1962), 341-353.
16. *On the theory of mixed volumes of convex bodies.* III. *Extension of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies,* Mat. Sb. **3** (**45**) (1938), 27-46. (Russian; German summary)

A. D. ALEKSANDROV AND E. P. SEN'KIN

17. *On the rigidity of convex surfaces,* Vestnik Leningrad. Univ. **10** (1955), no. 8, 3-13; supplement, ibid. **11** (1956), no. 1, 104-106. (Russian) MR **17**, 295; 998.

A. D. ALEKSANDROV AND V. A. ZALGALLER

18. *Two-dimensional manifolds of bounded curvature,* Trudy Mat. Inst. Steklov. **63** (1962); English transl., *Intrinsic geometry of surfaces,* Transl. Math. Monographs, vol. 15, Amer. Math. Soc., Providence, R. I., 1967. MR **27** # 1911.

I. JA. BAKEL'MAN

19. *Geometric methods of solution of elliptic equations,* "Nauka", Moscow, 1965. (Russian) MR **33** # 2933.

S. N. BERNŠTEĬN

20. *Renforcement de mon théorème sur les surfaces à courbure négative,* Izv. Akad. Nauk SSSR Ser. Mat. **6** (1942), 285-290. MR **5**, 14.

21. *Investigation and integration of second order partial differential equations of elliptic type,* Soobšč. Har'kov. Mat. Obšč. (2) **11** (1910), 1-164. (Russian)

W. BLASCHKE

22. *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie.* Band I: *Elementare Differentialgeometrie,* Springer, Berlin, 1924; reprint, Dover, New York, 1945; Russian transl., ONTI, Moscow, 1935. MR **7**, 391.

T. BONNESEN AND W. FENCHEL

23. *Theorie der konvexen Körper,* Springer, Berlin, 1934; reprint, Chelsea, New York, 1948.

JU. F. BORISOV

24. *The parallel translation on a smooth surface.* I, II, III, Vestnik Leningrad. Univ. **13** (1958), no. 7, 160-171; ibid. **13** (1958), no. 19, 45-54; ibid. **14** (1959), no. 1, 34-50; errata, ibid. **15** (1960), no. 19, 127-129. (Russian) MR **21** # 3032; # 3033; # 3034; **24** # A1078.

H. BUSEMANN AND W. FELLER

25. *Krümmungseigenschaften konvexer Flächen,* Acta Math. **66** (1936), 1-47.

É. CARTAN

26. *Leçons sur la géométrie des espaces de Riemann,* Gauthier-Villars, Paris, 1928; Russian transl., ONTI, Moscow, 1936.

S. E. COHN-VOSSEN

27. *Bending of surfaces in the large,* Uspehi Mat. Nauk **1** (1936), 33-76; reprinted in [**28**], pp. 19-86. (Russian) MR **22** # 4039.

28. *Some problems of differential geometry in the large,* Fizmatgiz, Moscow, 1959. (Russian) MR **22** # 4039.

I. A. DANELIČ

29. *The uniqueness of certain convex surfaces in Lobačevskiĭ space,* Dokl. Akad. Nauk SSSR **115** (1957), 217-219. (Russian) MR **19**, 979.

A. A. DUBROVIN

30. *Regularity of a convex surface with a regular metric in spaces of constant curvature,* Ukrain. Geometr. Sb. Vyp. **1** (1965), 16-27. (Russian) MR **35** # 6102.

N. V. EFIMOV

31. *Qualitative problems of the theory of deformations of surfaces,* Uspehi Mat. Nauk **3** (1948), no. 2 (24), 47-158; English transl., Amer. Math. Soc. Transl. (1) **6** (1962), 274-423. MR **10**, 324.

32. *Qualitative problems of the theory of deformations of surfaces in the small,* Trudy Mat. Inst. Steklov. **30** (1949). (Russian) MR **12**, 531.

33. *On rigidity in the small,* Dokl. Akad. Nauk SSSR **60** (1948), 761-764. (Russian) MR **10**, 324.

G. N. GAJUBOV

34. *Some questions on the theory of surfaces in hyperbolic space,* Taškent. Gos. Univ. Naučn. Trudy Vyp. 245 (1964), 17-22. (Russian) MR **37** # 2141.

E. HEINZ

35. *Neue a-priori Abschätzungen für den Ortsvektor einer Fläche positiver Gaussscher Krümmung durch ihr Linienelement,* Math. Z. **74** (1960), 129-157. MR **22** # 7089.

36. *Über die Differentialungleichung* $0 < \alpha \leqq rt - s^2 \leqq \beta < \infty$, Math. Z. **72** (1959/60), 107-126. MR **22** # 7088.

A. S. KRONROD

37. *On functions of two variables,* Uspehi Mat. Nauk **5** (1950), no. 1 (35), 24-134. (Russian) MR **11**, 648.

N. KUIPER

38. *On C^1-isometric imbeddings.* I, Nederl. Akad. Wetensch. Proc. Ser. A **58** = Indag. Math. **17** (1955), 545-556; Russian transl., Matematika **1** (1957), no. 2, 17-28. MR **17**, 782.

A. S. LEĬBIN

39. *On the deformability of convex surfaces with a boundary,* Uspehi Mat. Nauk **5** (1950), no. 5 (39), 149-159. (Russian) MR **12**, 733.

H. LEWY

40. *A priori limitations for solutions of Monge-Ampère equations,* Trans. Amer. Math. Soc. **37** (1935), 417-434; Russian transl., Uspehi Mat. Nauk **3** (1948), no. 2 (24), 191-215. MR **10**, 41.

I. M. LIBERMAN

41. *Geodesic lines on convex surfaces,* C. R. (Dokl.) Acad. Sci. URSS **32** (1941), 310-313. MR **6**, 100.

A. I. MEDJANIK

42. *On the central symmetry of a closed strictly convex surface,* Ukrain. Geometr. Sb. Vyp. 2 (1966), 52-58. (Russian) MR **35** # 3612.

T. MINAGAWA

43. *Remarks on the infinitesimal rigidity of closed convex surfaces,* Kōdai Math. Sem. Rep. **8** (1956), 41-48. MR **18**, 503.

H. MINKOWSKI

44. *Volumen und Oberfläche,* Math. Ann. **57** (1903), 447-495.

C. MIRANDA

45. *Equazioni alle derivate parziali di tipo ellittico,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 2, Springer-Verlag, Berlin, 1955; English transl., Springer-Verlag, Berlin, 1970; Russian transl., IL, Moscow, 1957. MR **19**, 421.

46. *Su un problema di Minkowski,* Rend. Sem. Mat. Roma **3** (1939), 96-108. MR **1**, 86.

J. F. NASH, JR.

47. *C^1-isometric imbeddings,* Ann. of Math. (2) **60** (1954), 383-396; Russian transl., Matematika **1** (1957), no. 2, 3-16. MR **16**, 515.

L. NIRENBERG

48. *On nonlinear elliptic partial differential equations and Hölder continuity,* Comm. Pure Appl. Math. **6** (1953), 103-156; Addendum, 395; Russian transl., Matematika **3** (1959), no. 3, 9-55. MR **16**, 367.

49. *The Weyl and Minkowski problems in differential geometry in the large,* Comm. Pure Appl. Math. **6** (1953), 337-394. MR **15**, 347.

S. P. OLOVJANIŠNIKOV

50. *Généralisation du théorème de Cauchy sur les polyèdres convexes,* Mat. Sb. **18** (**60**) (1946), 441-446. (Russian; French summary) MR **8**, 169.

51. *On the bending of infinite convex surfaces,* Mat. Sb. **18** (**60**) (1946), 429-440. (Russian; English summary) MR **8**, 169.

A. V. POGORELOV

52. *Quasi-geodesic lines on a convex surface,* Mat. Sb. **25** (**62**) (1949), 275-306; English transl., Amer. Math. Soc. Transl. (1) **6** (1962), 430-473. MR **11**, 201.

53. *Deformation of convex surfaces,* GITTL, Moscow, 1951. (Russian) MR **14**, 400; 1278.

54. *Some results relating to geometry in the large,* Izdat. Har′kov. Gos. Univ., Kharkov, 1961. (Russian)

55. *The rigidity of convex surfaces,* Trudy Mat. Inst. Steklov. **29** (1949). (Russian) MR **12**, 437.

56. *Uniqueness determination of general convex surfaces,* Monografii Instituta Matematiki, vyp. II, Akad. Nauk Ukrain. SSR, Kiev, 1952. (Russian) MR **16**, 162.

57. *Infinitesimal bending of general convex surfaces,* Izdat. Har′kov. Gos. Univ., Kharkov, 1959. (Russian) MR **22** #2960.

58. *On the regularity of convex surfaces with a regular metric in spaces of constant curvature,* Dokl. Akad. Nauk SSSR **122** (1958), 186-187. (Russian) MR **20** #5508.

59. *Some questions in geometry in the large in a Riemannian space,* Izdat. Har′kov. Gos. Univ., Kharkov, 1957. (Russian) MR **20** #4303.

60. *Extension of a general uniqueness theorem of A. D. Aleksandrov to the case of non-analytic surfaces,* Dokl. Akad. Nauk SSSR **62** (1948), 297-299. (Russian) MR **10**, 325.

61. *Regularity of a convex surface with given Gaussian curvature,* Mat. Sb. **31** **(73)** (1952), 88-103. (Russian) MR **14**, 679.

62. *Monge-Ampère equations of elliptic type,* Izdat. Har′kov. Gos. Univ., Kharkov, 1960; English transl., Noordhoff, Groningen, 1964. MR **23** #A1137; **31** #4993.

63. *Surfaces of bounded outer curvature,* Izdat. Har′kov. Gos. Univ., Kharkov, 1956. (Russian)

64. *Topics in the theory of surfaces in an elliptic space,* Izdat. Har′kov. Gos. Univ., Kharkov, 1960. (Russian) MR **22** #8460.

JU. G. REŠETNJAK

65. *Isothermal coordinates on manifolds of bounded curvature.* I, II, Sibirsk. Mat. Ž. **1** (1960), 88-116. 248-276. (Russian) MR **23** #A4072.

R. SAUER

66. *Infinitesimale Verbiegungen zueinander projektiver Flächen,* Math. Ann. **111** (1935), 71-82.

J. SCHAUDER

67. *Über lineare elliptische Differentialgleichungen zweiter Ordnung,* Math. Z. **38** (1934), 257-282.

I. N. VEKUA

68. *Systems of differential equations of the first order of elliptic type and boundary value problems, with an application to the theory of shells,* Mat. Sb. **31** **(73)** (1952), 217-314. (Russian) MR **15**, 230.

69. *Generalized analytic functions,* Fizmatgiz, Moscow, 1959; English transl., Pergamon Press, London; Addison-Wesley, Reading, Mass., 1962. MR **21** #7288; **27** #321.

JU. A. VOLKOV

70. *Stability of the solution of Minkowski's problem,* Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **18** (1963), no. 1, 33-43. (Russian) MR **26** #4258.

H. WEYL

71. *Über die Bestimmung einer geschlossenen konvexen Fläche durch ihr Linienelement,* Vierteljschr. Naturforsch. Ges. Zürich **61** (1916), 40-72; Russian transl., Uspehi Mat. Nauk **3** (1948), no. 2 (24), 159-190. MR **10**, 60.

V. A. ZALGALLER

72. *On curves with curvature of bounded variation on a convex surface,* Mat. Sb. **26** **(68)** (1950), 205-214. (Russian) MR **11**, 681.

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ISBN 978-0-8218-1585-4
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