Interpolation of Linear Operators
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Interpolation of Linear Operators
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Abstract. The book is devoted to an important direction in functional analysis: interpolation theory for linear operators. The main methods for constructing interpolation spaces are expounded and their properties are studied. These methods allow one to look at a number of theorems and inequalities of classical analysis from a new standpoint. Interpolation theory for operators has numerous applications in Fourier series, approximation theory, partial differential equations, etc. Some of them are developed in the book.

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10 9 8 7 6 5 4 3 2 07 06 05 04 03 02
# TABLE OF CONTENTS

FOREWORD TO THE AMERICAN EDITION ........................................... vii

FOREWORD ....................................................................................... ix

CHAPTER I. IMBEDDED, INTERMEDIATE, AND INTERPOLATION BANACH SPACES ........................................... 1
  1. Imbedding of Banach spaces .................................................... 1
  2. Dual spaces of imbedded Banach spaces ................................. 6
  3. Intermediate Banach spaces .................................................. 9
  4. Interpolation spaces and interpolation triples ......................... 18

CHAPTER II. INTERPOLATION IN SPACES OF MEASURABLE FUNCTIONS ......................................................... 39
  Introduction .................................................................................... 39
  1. Positive functions on a semiaxis and their dilation functions .... 46
  2. Rearrangements of measurable functions ................................. 58
  3. Operators in the Banach couple $L_1(0, \infty), L_\infty(0, \infty)$ .... 77
  4. Symmetric spaces. Interpolation between $L_1$ and $L_\infty$ ......... 90
  5. Lorentz and Marcinkiewicz spaces .......................................... 107
  6. Operators of weakened and weak type .................................... 124
  7. The singular Hilbert operator .................................................. 150
  8. Interpolation theorems for spaces with different measures ....... 156
  9. Applications to the theory of orthogonal series ......................... 173

CHAPTER III. SCALES OF BANACH SPACES ........................................ 187
  1. Scales of Banach spaces. Related spaces ................................. 187
  2. Maximal and minimal normal scales ....................................... 192
  3. The scale of Hölder spaces ..................................................... 200
CONTENTS

CHAPTER IV. INTERPOLATION METHODS 209
   Introduction 209
   1. The complex method of interpolation 214
   2. The methods of constants and means (γ- and ξ-methods) 246

CHAPTER V (BY S. G. KREĬN). INTERPOLATION IN SPACES 283
   OF SMOOTH FUNCTIONS 309
   1. Interpolation spaces constructed from an unbounded operator 283
      and a smoothing approximation process
   2. Trace theory 309
   3. Spaces of smooth functions of n variables 322

NOTES ON THE LITERATURE 349

BIBLIOGRAPHY 355

SUBJECT INDEX 373

NOTATION INDEX 375
FOREWORD TO THE AMERICAN EDITION

This edition includes Chapter V, "Interpolation in spaces of smooth functions", written at the same time as the preceding chapters but not included in the Soviet edition for technical reasons. This chapter expounds the abstract scheme for constructing interpolation spaces by means of an unbounded operator in a Banach space, and the corresponding approximation process. Starting points of this theory were papers by J. L. Lions, Lions and J. Peetre, and P. Grisvard, which then were extended by other authors. As an application we consider only Sobolev and Besov spaces.

The reader can get acquainted with other families of spaces in the books referred to in the foreword to the Soviet edition.

The authors express their sincere gratitude to the editor of the translation, Dr. L. J. Leifman, for his penetrating remarks that helped in eliminating a number of shortcomings.
FOREWORD

The present book is devoted to the systematic exposition of a chapter in functional analysis that has appeared and developed in the past two decades and has found applications in various fields.

The basic objects of classical functional analysis were operators acting from one Banach space (or later from a topological linear space) into another. The spaces themselves were considered as given in advance. The change of this ideology was facilitated to a significant extent by the imbedding theorems of S. L. Sobolev, in which a number of fundamental theorems and inequalities of analysis were interpreted as assertions concerning the imbedding of one Banach space into another. Imbedding theorems arose in connection with problems of the theory of partial differential equations, in which for the study of smoothness of solutions a series of spaces is introduced; for the study of the behavior near the boundary of the domain or near some singular points other types of spaces are introduced, the study of values of solutions on manifolds of smaller dimension is performed in still other spaces, etc. The abundance of various spaces required a detailed study of the interrelations between these spaces. Thus a new level of abstraction appeared, on which the Banach spaces themselves are considered as elements of some category. The interpolation theory for linear operators expounded in the book is to a great extent connected with such an approach.

The first interpolation theorem in operator theory was obtained by M. Riesz in 1926 in the form of an inequality for bilinear forms. A sharpening and operator formulation of it were given by G. O. Thorin. An essential further step was the interpolation theorem of J. Marcinkiewicz (1939), whose proof was published by A. Zygmund in 1956. In the fifties important generalizations of the Riesz-Thorin and Marcinkiewicz theorems were obtained by E. M. Stein and G. Weiss. However, all these and other communications were concerned with $L_p$ spaces or spaces similar to them. The
development of general interpolation theorems for families of abstract Hilbert
and Banach spaces began in 1958 independently in several countries. The first
publications are due to J. L. Lions (1958-1960), E. Gagliardo (1959-1960), A.
P. Calderón (1960), and S. G. Krein (1960). The work of J. A. Peetre played
an essential role in the sequel. Several methods have been created for
obtaining interpolation theorems, which have deep interrelations. Moreover,
it became clear fairly soon that the interpolation properties of spaces inter-
mediate between two Banach spaces are consequences of the functoriality of
the methods of construction. Therefore, the main emphasis has been shifted
to the study of properties of intermediate interpolation spaces obtained by
various methods, and to their realization. Along with this, in the work of W.
Orlicz, A. P. Calderón, G. G. Lorentz, E. M. Semenov, and others deep
results have been obtained concerning the interpolation of linear operators in
spaces of measurable functions.

It is impossible to expound all results of interpolation theory for linear
operators in one book. We have tried to illuminate only some of the main
directions in its development: the real and complex methods of constructing
interpolation spaces, the method of scales of Banach spaces, and interpola-
tion in spaces of measurable functions. Supplementary information is con-
tained in remarks and references.

In the development of interpolation theory for operators many new general
notions of functional analysis have emerged. These notions and their interre-
lations are studied in the first chapter of the book. The exposition is based
essentially on the work of N. Aronszajn and E. Gagliardo. To read this
chapter one needs to know only the basic principles of functional analysis.

The second chapter, devoted to interpolation in spaces of measurable
functions, makes up a significant portion of the book. It can be read
independently of the first chapter, from which only the simplest definitions
are needed. The chapter contains a theorem describing all interpolation
spaces between $L_1$ and $L_\infty$, and a theorem which is a further extension of the
Marcinkiewicz theorem. The exposition is pursued as far as concrete applica-
tions, for example, the theory of orthogonal series: convergence properties of
Fourier series and the basis property of a function system are studied.
Moreover, the chapter contains much auxiliary material from the theory of
functions which is discussed little in the literature. Decreasing rearrangements
of measurable functions are studied in detail, function spaces symmetric in
the sense of E. M. Semenov, and in particular, Lorentz and Marcinkiewicz
spaces, are discussed (in the foreign literature similar spaces are called
invariant with respect to permutations). Sharpenings of classical inequalities
of analysis (the Hardy-Littlewood, Hilbert, and other inequaities) are given.
FOREWORD

In the third chapter the theory of scales of Banach spaces, developed mainly in the publications of S. G. Kreĭn and Ju. I. Petunin, is expounded. The prerequisite material for this is contained in the first chapter. Important properties of the scales, in particular their "almost" interpolation properties are also expounded in the fourth chapter. In the last section of the third chapter properties of the classical scale of Hölder spaces important in applications are studied in detail.

In the fourth chapter two methods of constructing interpolation spaces enjoying the largest number of applications are described in detail: the method of complex interpolation proposed independently by A. P. Calderón and J. L. Lions and extensively developed by Calderón, and the method of constants and averages due to J. L. Lions and J. Peetre. The latter method is expounded in the more general form which it acquired in the work of V. I. Dmitriev (who took the most active part in writing the corresponding section). The fourth chapter can be read independently from the second and third chapters.

The book does not include interpolation theory in spaces of smooth functions and its applications.* This theory developed under the influence of the work on imbedding theorems by S. L. Sobolev, S. M. Nikol'skiĭ and their students and followers. The abstract theory did not rise immediately and easily to the level of concrete imbedding theorems obtained by special means. However, now such a theory has been created. Its exposition apparently needs another book. One can get acquainted with it partly in the book [7] by P. L. Butzer and H. Berens. It is expounded more completely in Hans Triebel’s very recent book Interpolation theory, function spaces, differential operators (published by VEB Deutscher Verlag Wiss., Berlin, 1977, and by North-Holland, 1978). (1) One can get acquainted with the applications of this theory to the study of boundary value problems for partial differential equations in the book of J. L. Lions and E. Magenes [27] and in Triebel’s book mentioned above. At the end of the book there is a bibliography covering, in addition, the indicated part of interpolation theory.

As we noted above, some parts of the book were written by V. I. Dmitriev. I. Ja. Šneĭberg provided us with invaluable help. He participated in writing §1 of Chapter IV and read a significant portion of the book. His critical remarks

*Editor’s note. For this translation a new Chapter V was added by the authors to cover this subject.

(1) The authors are grateful to Professor Triebel for making the manuscript of this book available.
enabled us to remove a number of inaccuracies and improve some proofs. The authors express their gratitude to both of them.

Finally, we thank all participants of the Voronezh seminar on interpolation theory for linear operators, and, in particular, M. Š. Braverman, A. A. Dmitriev, E. A. Pavlov, P. A. Kučment, and A. A. Sedaev, for their constant help in the preparation of the book.

*The authors*
NOTES ON THE LITERATURE

Chapter I

§1. As has been mentioned in the Foreword, the notion of imbedding of Banach spaces first appeared in a paper by Sobolev [355]. The important notion of relative completion has been used implicitly by S. G. Kreĭn and Petunin [184]; it has been introduced explicitly by Gagliardo [139] and studied in detail by Aronszajn and Gagliardo [55].

§2. Lemmas 2.1–2.3 and Theorem 2.3 are contained in [55], Theorem 2.1 in [184], and Theorem 2.2 in Berens' book [4].

§3. Intermediate spaces for a couple of Banach spaces were first considered by Lions [202], Formula (3.6) and its consequences are due to Sedaev [41]. Lemma 3.4 is taken from Calderón [95]. The dual spaces of a sum or an intersection of spaces were studied by Aronszajn and Gagliardo [55] (see also [232]); they also introduced the sum and intersection of a family of Banach spaces.

§4. The notions of interpolation triples and interpolation spaces have been introduced in one form or another in all publications devoted to the abstract theory of interpolation of linear operators. The most important results of §4 are due to Aronszajn and Gagliardo [55]. In connection with subsections 4 and 5, see [37]. The important Theorems 4.9 and 4.10 were obtained in [55].

Chapter II

Introduction. Ideal Banach lattices are also called Banach function spaces; their properties are described in [29] and [49] (see also [95], [188], [25] and [47]). Concerning Lebesgue space, see [305].

§1. Theorem 1.1 was obtained by Peetre [284] (concerning (1.7), see [122]). For Lemma 1.1, see [107]. For more detail on logarithmically convex functions, see the book [39]. Lemma 1.4 was obtained in [59].

§2. Theorem 2.1 was obtained by Kreĭn and Semenov [191]. The books [8], [17], and [50] have much information on rearrangements of measurable functions. Many properties of rearrangements have been established in [87], [88], [95], [172], [317], and [312]. Inequality (2.40) is published here for the first time.

§3. Formulas (3.4) and (3.5) were obtained by Peetre in [262] and [261] (see also Oklander [250]). The important inequality (3.14) was established by Lorentz and Shimogaki [227]. Orbits of the semigroup of contractive operators were studied by Ryff [311].

349
§4. Symmetric spaces with the additional assumption that the norm is semicontinuous were introduced by Lorentz under the name \emph{spaces invariant under permutations} in the book [28] (see also Luxemburg [231]). Without this assumption they were studied by Semenov [331]; the main results of subsection 1 were obtained by him. The main theorem describing all interpolation spaces between \( L_p \) and \( L_\infty \) was proved by Calderón [96] by another method. This theorem has a long history: for integral operators in Orlicz spaces it was first proved by Orlicz [256], for integral operators in spaces invariant under permutations by Lorentz [28], and for arbitrary operators in separable spaces or symmetric spaces dual to separable spaces in [241] (see Theorems 4.9 and 4.10). The theorem has been generalized to the case of nonlinear operators satisfying a Lipschitz condition in [257], [227], and [349]. We note that Browder [85] has obtained a general theorem enabling us to obtain interpolation theorems for Lipschitz operators from interpolation theorems for linear operators.

The hypothesis of Theorem 4.3 can be formulated in the following way: the assumptions that \( y \in E, x \in L_1 + L_\infty \), and \( K(t, x) \leq K(t, y) \) imply that \( x \in E \) and \( \|x\|_E \leq \|y\|_E \). In connection with this there arose the following conjecture. Let \((A_0, A_1)\) and \((B_0, B_1)\) be two Banach couples. For the triple \((A_0, A_1, E)\) to be an interpolation triple relative to the triple \((B_0, B_1, F)\) it is necessary and sufficient that the assumptions \( y \in E \), \( x \in B_0 + B_1 \), and \( K(t, x, B_0, B_1) < K(t, y, A_0, A_1) \) imply that \( x \in F \) and \( \|x\|_F < C\|y\|_E \). This conjecture has been confirmed for the following couples: 1) \((L_\infty, L_\infty)\), \((L_p, L_\infty)\) (Lorentz and T. Shimogaki [228]; Sedaev [324]); 2) \((L^{p_0}, L^{p_0})\), \((L^{p_0}, L^{p_1})\) (a_0 and a_1 are weights) (Sedaev and Semenov [327]); 3) \((L_0^\alpha, L_\infty^\alpha)\), \((L_\infty^\alpha, L_\infty^\alpha)\) (Sedaev [324]); 4) \((A_0, A_1)\) \((L_\infty^\alpha, L_\infty^\alpha)\) (Peetre [37] and Sedaev [41]); 5) \((A_0, A_1): (L_\infty^\alpha, L_\infty^\alpha)\) (V. I. Dmitriev [117]); 6) \((L_\infty^\alpha L_\infty^\alpha)\), \((L_\infty^\alpha L_\infty^\alpha)\) (V. I. Dmitriev [119]); 7) \((L_\infty^\alpha L_\infty^\alpha)\), \((L_\infty^\alpha L_\infty^\alpha)\) (Sparr [361]). In the general case the conjecture could not be confirmed (see [37] and [117]). V. I. Dmitriev [119], [10] singled out a general class of couples of spaces (difference couples) for which the conjecture is true.

The action of dilation operators in symmetric spaces has been studied in [348], [350], and [83]. In terms of the norm of the dilation operator, the problem of interpolation of the property of complete continuity of linear operators in intermediate spaces for the couple \((L_1, L_\infty)\) has been solved in [348]. The upper and lower dilation exponents of symmetric spaces were introduced by Boyd [81] under the name \emph{upper and lower indices}. Lemma 4.7 is due to Semenov and plays an important role in what follows. The notion of fundamental function was introduced in [331]. There are examples of spaces for which inequality (4.29) is strict, and, even more, the left and right sides have different asymptotics at infinity (see [350]).

§5. Lorentz spaces were introduced by Lorentz in [225]; he established that their dual spaces are the Marcinkiewicz' spaces. Some properties of Lorentz spaces were obtained in [191]. The spaces \( M_0^\alpha \) were considered by Semenov [330]. The imbedding theorems were obtained in [331]. Theorem 5.9 is contained in [342] under different assumptions.

The first example of a noninterpolation symmetric space was constructed by Russu [307].

§6. Inequality (6.1') is actually contained in [347]. The main interpolation Theorems 6.1 and 6.1' are extensions of Marcinkiewicz' theorem [234], whose proof was published by Zygmund [407]. Many authors have dealt with the generalization of this theorem; see [64], [74], [94], [96], [106], [148], [161], [163], [190], [214], [250], [262], [342], [371], [406], etc. Here the versions of Krein and Semenov are given from [192]. For the case of the \( L_p \) spaces an analogue of Theorem 6.1 can be found in [334], where the conditions on the space \( \hat{E} \) are given in terms of the fundamental function \( \varphi_{\hat{E}} \). However, the proof contains an error and is true only in the case where the norm \( \|e_\iota\|_E \) of the dilation operator coincides with \( M_{\infty} \) (concerning (4.29), see above). A correction and strengthening of this result is expounded in §6 (see [192]). The first theorems on the optimality of interpolation triples of concrete symmetric spaces were obtained by Dikarev and Macaev [111] and Calderón [96]. Calderón's ideas lie at the base of the proofs of the optimality theorems of [192].
NOTES ON THE LITERATURE

Hardy-Littlewood and Hilbert operators and majorant functions for symmetric spaces have been studied in [332], [347], [255], [144], [78], and [218]. A generalization of the Hardy inequality (6.41) has been obtained by F. A Pavlov.

The spaces $L_{p,v}$ are special cases of the Lorentz spaces $\Lambda_{p,q}$ [225]. Interpolation theorems for them are contained in [96]. Theorems 6.12 and 6.13 are new. Applications of Theorem 6.1 to the convolution operator are indicated in [193]. Theorem 6.17 is a sharpening of a result of O'Neil [253].

§7. In discussing the properties of the Hilbert singular operator we have followed Zygmund's book [51]. The main formula (7.9) is due to Stein and Weiss [371]. Theorem 7.2 was proved by Boyd [78].

§8 has an auxiliary character. The operation of taking the dual in symmetric spaces has been studied in [133], [144], [254] and [333].

§9. For the classical Paley theorem, see [50]. The generalization of it in the present form is published here for the first time. The articles [313], [314], [316] and [333] are devoted to the generalization of the Hardy-Littlewood theorem on series with monotone coefficients by means of interpolation theorems. For properties of the Haar system, see [20]. Theorem 9.5 for the space $L_p$ was proved by F. Riesz (see [2] and [50]), for symmetric spaces it was proved by Semenov [333]. Theorem 9.6 for the $L_p$ space was obtained by Marcinkiewicz [235], for Orlicz spaces by V. F. Gapoškin, and for symmetric spaces by Semenov [336].\(^{1)}\) Concerning Theorem 9.8, see [304].

Chapter III

§1. The properties of scales of Banach spaces are expounded in [186]. The notion of normal scale was introduced and studied by Krein [181]. The problem of related Banach spaces was studied by Krein and Petunin in [184]. The condensation of a normal scale by means of relative completion is considered here for the first time.

§2. Maximal scales have been studied in [181], and minimal and regular scales in [185]; their properties are discussed in detail in [186].

§3. The scale of Hölder spaces was considered in detail in [186]. V. Friedrich has kindly indicated to us that there are inaccuracies in [186], which we correct here. The new exposition of interpolation properties of the Hölder scale is based on the work of Petunin and Pličko [297]. We note that in a more general case a detailed exposition of the properties of the Hölder scale and its dual with application to the transportation problem is contained in Friedrichs' book [14].

Chapter IV

Introduction. Theorem 1 was obtained by V. I. Dmitriev [116].

§1. The complex method of interpolation in the form expounded here was suggested independently by Calderón [95] and Lions [204]. The main results of subsections 3 and 4 are contained in Calderón [95]; the useful Remark 2 was made by Stafney [363]. Theorem 1.4 is also due to Calderón [95]; its proof has somewhat been simplified by I. Ja. Šneiberg. In [95], besides the spaces $[A_0, A_1]_v$, the interpolation family of spaces $[A_0, A_1]^{**}$ is also introduced, and the space $\overline{\Lambda}(A_0, A_1)$ of functions in the strip $\Pi$ having the following properties is considered: 1) $\|f(x)\|_{A_0 + A_1} \leq C(1 + |x|); 2) f(x)$ is continuous in the norm of $A_0 + A_1$ in $\Pi; 3) f(x)$ is analytic in $\Pi; 4) f(1$ $+ it_2) - f(1 + it_1)$ belongs to $A_1$ and the difference

\(^{1)}\) Concerning [333] and [336], the remarks to §6 must be taken into account.
f(it_2) - f(it_1) \text{ belongs to } A_0; \text{ moreover,}

\max \left\{ \sup_{A_0} \left| \frac{f(it_2) - f(it_1)}{t_2 - t_1} \right|, \sup_{A_1} \left| \frac{f(1 + it_2) - f(1 + it_1)}{t_2 - t_1} \right| \right\} = \|f\|_{A_0} < \infty.

The space \([A_0, A_1]^\star\) consists of all \(x \in A_0 + A_1\) for which \(x = f(\alpha), f \in \mathcal{B}(A_0, A_1)\), and

\[ \|x\|_{[A_0, A_1]^\star} = \inf_{f(\alpha) = x} \|f\|_{A_0} \] .

It turns out [95] that \([(A_0, A_1)_a] \gamma\) is isometrically isomorphic to the space \([A_0', A_1']^\star\) (if \(A_0 \cap A_1\) is dense in \(A_0 \) and \(A_1\) ). In subsection 6 the dual of \([A_0, A_1]_a\) is described in terms of relative completion (Theorem 1.6, 1. Ja. Weisz). From what has been said we obtain the equality \([A_0, A_1]^\star = [A_0', A_1']_a\). Changes have also been made in the proof of Calderón's reiteration Theorem 1.7. We note that in [320] it is indicated that the condition that \(A_0 \cap A_1\) is dense in \(A_a \cap A_\delta\) is superfluous; however, the proof of this fact contains an error.

The notion of an analytic scale of spaces was introduced by Krein [180]; he established the connection of this notion with the complex method of interpolation [186].

The theory of Hilbert spaces of spaces was constructed independently (in different terms) by Lions [200] and Krein [180]. An important role is played by families of topological linear spaces obtained from a Hilbert space by means of projective or inductive limits. By means of them a number of delicate properties of spaces of analytic functions has been studied.

Theorems 1.11 and 1.13 were obtained from other considerations by E. Heinz, and were sharpened by T. Kato (see [23] and [24]). Heinz has proved an inequality more general than (1.49), in which the fractional power of the operators \(j\) and \(j^*\) is replaced by more general functions. Let us consider the class of functions positive on the semiaxis \([0, \infty)\), admitting analytic continuation to the complex plane with the negative semiaxis removed, which results in a function mapping the upper half-plane into itself. Let the function \(\varphi(t)\) be such that \(\varphi^2(t^{1/2})\) belongs to the indicated class. Under the conditions of Theorem 1.13 we have the inequality

\[ (Tx, y) < \|\varphi(j)x\|_{H_0}\|\varphi(j^*)y\|_{H_0^*} \text{ where } \frac{\varphi(t)}{t} \in [0, \infty]. \]

All interpolation spaces with interpolation constant \(1\) between a couple of Hilbert spaces have been described by means of functions of the indicated class (Fofia and Lions [134], and Donoghue [120]; see also [160]).

The family of spaces \(X_0^{1/2 - \alpha} X_1^{1/2 + \alpha}\) was introduced and studied by Calderón [95]. Here we do not discuss its connection with hyperscales (see [188]). Lozanovskii [230] constructed an example in which the triples \((X_0, X_1, X_0^{1/2 - \alpha} X_1^{1/2 + \alpha})\) and \((Y_0, Y_1, Y_0^{1/2 - \alpha} Y_1^{1/2 + \alpha})\) of ideal lattices are not interpolation triples. Sestakov [340], [341] showed that in the general case \([X_0, X_1]_a\) is the closure of \(X_0 \cap X_1\) in \(X_0^{1/2 - \alpha} X_1^{1/2 + \alpha}\), and consequently is a closed subspace of it. For further study of the family \(X_0^{1/2 - \alpha} X_1^{1/2 + \alpha}\), see [401] and [229].

We note that Schechter [320], [319] constructed a generalization of the complex method of interpolation based on the idea that the intermediate space is constructed not from the values of functions analytic in a strip or their derivatives but rather from the values of some generalized function with compact support defined on these functions.

Interesting results concerning the unambiguously solvability of linear equations and the spectrum of linear operators in the family of spaces \([A_0, A_1]_a\) have been obtained by Sneiberg [353], [354] and Stafney [364].

§2. The methods of constants and averages originate in a paper by Lions and Peetre [214], where they are constructed for the case that \(E_0\) and \(E_1\) are \(L_p\) spaces with power weights. The generalization of these methods to the case of arbitrary ideal lattices discussed here was proposed by Peetre in [35] and [264], and developed by Dmitriev in [114], [116] and [118]. The appearance of spaces of the Calderón scale in the reiteration Theorem 2.8 was unexpected (V. I. Dmitriev [59]).
The $\mathcal{K}$- and $\mathcal{S}$-methods were proposed by Peetre [262] in the case where $E$ is an $L_p$ space with power weight (see subsection 8) and have been the most widely disseminated of all real interpolation methods. These methods were generalized to the case of more general ideal lattices by Peetre [35], Bennett [67], and other authors. For a special case of Theorem 2.10, see [262] and [73].

Theorems 2.11 and 2.12 were proved by Lions and Peetre [214].

Theorem 2.13 on the number of parameters was obtained by Peetre [261]. It has been generalized to the case of arbitrary ideal lattices $E_0$ and $E_1$ by V. I. Dmitriev [118].

Extreme spaces (subsection 9) were studied by Hayakawa [156].

The duality of the methods of constants and averages (in the simplest case when they coincide) was studied by Lions and Peetre [214]. Here we have expounded the results of V. I. Dmitriev [116].

Applications of the methods of constants and averages to quasinormed spaces have been studied in [168] and [157].

The connection between the methods of constants and averages and the theory of scales and the almost interpolation properties of scales were studied by Petunin [186].

Lemma 2.20 and Theorem 2.22 were obtained by Lions and Peetre [214]. The problem of conditions on the commutativity of the functors corresponding to the method of averages and the complex method has been studied by Grisvard [153].

Chapter V

§1. The approach to the construction, by means of an abstract approximation process, of intermediate spaces between a Banach space and the domain of an unbounded operator acting in it is expounded here for the first time. Concrete realizations of it have been studied by many authors. Closely related but different constructions are found in Berens' book [4]. It was apparently there that the connection between the behavior of an approximation process and relative completion (Lemma 1.1) was noticed for the first time. See [24] for more details on the subordinate operators of subsection 4. The approximation process constructed from a power of the resolvent was first studied in Grisvard's fundamental paper [153], the results of which are discussed here only partially. In particular, in it the relation of the intermediate spaces constructed there to the complex interpolation method is studied. Lemma 1.10 is due to Ljubič [220].

For operators satisfying condition (1.15) fractional powers are defined (see the books [23] and [24]), and therefore the theory expounded in subsections 5 and 7 can be carried over to spaces constructed from fractional powers of the operator. The connection between interpolation spaces and domains of fractional powers of operators was studied by Lions [209] for accretive operators in a Hilbert space, and in the general case in a long series of articles by Komatsu [177], and by Sobolevskii in [356]–[359] (see also [72], [246], [392], and [398]).

The construction of interpolation spaces by means of bounded semigroups of operators was first done by Lions and Peetre [213], and since then it has been studied in many publications—see [3], [69]–[71], [91], [92], [151], [153], [222] and [392]. We note that a fairly complete exposition of properties of the spaces constructed from resolvents and semigroups of operators in the case $G = L_p$ is included in the book [7] by Butzer and Berens. Theorem 1.9 was obtained by Grisvard [153]; this theorem and Theorems 1.10 and 1.11 were generalized to the case of fractional powers of operators by Muramatu [246]. The proof of Theorem 1.12 given here is also due to Muramatu. Interesting but apparently not complete research has been carried out in the case where the operators do not commute and are infinitesimal operators of some representation of a Lie group (see [281]).
Grivasdr in [153] began to consider unbounded operators in a couple of Banach spaces. The
imbedding theorem given here, namely Theorem 1.13, is due to Yoshikawa [394]. It has been
developed and applied in [245], [396] and [397].

§2. Spaces of traces were introduced and studied in a series of publications by Lions (see [201],
[202], [205] and [207]). The study has been continued by Grivasdr in [152] and [154]; the
exposition given here is based on his articles.

§3. As we mentioned in the Foreword, Sobolev-Nikol'skii-Slobodeckii-Besov spaces and
imbedding theorems for them served as a guideline for the construction of the corresponding
abstract theory expounded in Chapter V in an incomplete form. In almost all publications
mentioned in §§1 and 2 there are applications to the theory of the indicated spaces. In addition,
we mention the series [379]–[386] of articles by Triebel. We have discussed only the facts which
can be obtained from results of §§1 and 2; besides the abstract theory, we have here used
Mihlin's [30] and P. I. Lizorkin's [217] theorems on multipliers for the Fourier transformation, the
Hestenes-Whitney extension of smooth functions, and other techniques which have become
standard in the theory of partial differential equations. Here we have used the monograph [27] in
an essential way.

We entirely omitted other classes of smooth functions, in particular, Lebesgue spaces or spaces
of Bessel potentials, which are connected with the complex method of interpolation. Their theory
is expounded in Nikol'skii's book [32] without this connection, and in connection with interpola-
tion theory in Triebel's new book Interpolation theory, function spaces, differential operators
mentioned in the Foreword (see also [93], [212] and [218]).

On other classes of functions, see [15], [86], [87], [90], [95], [99]–[101], [129], [171]–[174], [194],
[198], [211], [212], [216], [218], [249], [279], [321], [365], [366], [373] and [374].
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BIBLIOGRAPHY


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### SUBJECT INDEX

<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>absolutely continuous norm</td>
<td>45</td>
</tr>
<tr>
<td>almost imbedded family of Banach spaces</td>
<td>279</td>
</tr>
<tr>
<td>analytic scale of Banach spaces</td>
<td>234</td>
</tr>
<tr>
<td>associated space</td>
<td>45</td>
</tr>
<tr>
<td>averaging operator</td>
<td>80</td>
</tr>
<tr>
<td>B. Levi’s theorem</td>
<td>40</td>
</tr>
<tr>
<td>Banach couple of spaces</td>
<td>9</td>
</tr>
<tr>
<td>Banach function space</td>
<td>40</td>
</tr>
<tr>
<td>basis</td>
<td>178</td>
</tr>
<tr>
<td>unconditional</td>
<td>179</td>
</tr>
<tr>
<td>Besov space</td>
<td>330</td>
</tr>
<tr>
<td>Bochner integrable function</td>
<td>209</td>
</tr>
<tr>
<td>bounded operator in couples of Banach spaces</td>
<td>18</td>
</tr>
<tr>
<td>bounded semigroup</td>
<td>296</td>
</tr>
<tr>
<td>$C^0$-condition</td>
<td>296</td>
</tr>
<tr>
<td>compact scale of Banach spaces</td>
<td>187</td>
</tr>
<tr>
<td>complemented subcouple of a Banach couple</td>
<td>29</td>
</tr>
<tr>
<td>complex method of interpolation</td>
<td>214</td>
</tr>
<tr>
<td>concave function</td>
<td>46</td>
</tr>
<tr>
<td>convergence in measure</td>
<td>39</td>
</tr>
<tr>
<td>conjugation operator</td>
<td>169</td>
</tr>
<tr>
<td>convex function</td>
<td>46</td>
</tr>
<tr>
<td>contraction semigroup</td>
<td>84</td>
</tr>
<tr>
<td>convolution operator</td>
<td>147</td>
</tr>
<tr>
<td>covariant functor</td>
<td>35</td>
</tr>
<tr>
<td>dilation exponents of a function</td>
<td>54</td>
</tr>
<tr>
<td>dilation exponents of a space, upper and lower</td>
<td>99</td>
</tr>
<tr>
<td>dilation functions</td>
<td>53</td>
</tr>
<tr>
<td>dilation operator</td>
<td>96</td>
</tr>
<tr>
<td>dimension of the norm</td>
<td>334</td>
</tr>
<tr>
<td>discretization of an ideal lattice</td>
<td>44</td>
</tr>
<tr>
<td>distribution function</td>
<td>58</td>
</tr>
<tr>
<td>dual family of Banach spaces</td>
<td>195</td>
</tr>
<tr>
<td>dual scale of Banach spaces</td>
<td>196</td>
</tr>
<tr>
<td>Egorov’s theorem</td>
<td>39</td>
</tr>
<tr>
<td>elementary function</td>
<td>92 (Lemma 4.2)</td>
</tr>
<tr>
<td>equimeasurable functions</td>
<td>58</td>
</tr>
<tr>
<td>equivalent functions</td>
<td>48</td>
</tr>
<tr>
<td>Fatou property</td>
<td>44</td>
</tr>
<tr>
<td>Fatou’s lemma</td>
<td>40</td>
</tr>
<tr>
<td>Fourier transformation</td>
<td>325</td>
</tr>
<tr>
<td>fundamental function</td>
<td>101</td>
</tr>
<tr>
<td>generalized function</td>
<td>309</td>
</tr>
<tr>
<td>derivative of a</td>
<td>309</td>
</tr>
<tr>
<td>regular</td>
<td>309</td>
</tr>
<tr>
<td>trace of a</td>
<td>311</td>
</tr>
<tr>
<td>generalized Hardy-Littlewood theorem</td>
<td>224</td>
</tr>
<tr>
<td>generalized Paley theorem</td>
<td>224</td>
</tr>
<tr>
<td>Hardy inequality</td>
<td>167</td>
</tr>
<tr>
<td>Hardy-Littlewood operator</td>
<td>138</td>
</tr>
<tr>
<td>Hilbert operator</td>
<td>140</td>
</tr>
<tr>
<td>singular</td>
<td>150</td>
</tr>
<tr>
<td>Hilbert scale</td>
<td>237</td>
</tr>
<tr>
<td>Hölder scale</td>
<td>201</td>
</tr>
<tr>
<td>ideal Banach lattice</td>
<td>40</td>
</tr>
<tr>
<td>ideal lattice</td>
<td>40</td>
</tr>
<tr>
<td>imbedded family of Banach spaces</td>
<td>279</td>
</tr>
</tbody>
</table>
imbedding constant, 1
imbedding of Banach spaces, 1
  compact, 2
dense, 1
  normalized, 2
incomplete scale of Banach spaces, 188
incomplete scale with base, 188
infinitesimal generator of a semigroup, 297
integral, 210
intermediate space, 15
  of type θ, 257
interpolation constant, 20 (Lemma 4.3)
interpolation functor, 35
  \([A_0, A_1]_α\), 221
  \((A_0, A_1)_θ, p\), 271
  of type α, 36
interpolation property, 195
  almost, 280
  normalized, 195
  strong, 195
interpolation space, 20
  of type α, 22
interpolation theorem, 22
interpolation triples of spaces, 20
  good, 21
interpolation triples of type α, 22
  normalized, 22
intersection of the spaces of a Banach couple, 9
isomorphic Banach couples, 13
isomorphic measure spaces, 46
Lebesgue space, 46
Lebesgue’s theorem, 40
logarithmically convex function, 51
Lorentz space, 107
majorizing normalized scale, 197
Marcinkiewicz space, 112
maximal scale of means, 278
maximal symmetric space, 104
method of constants (\(K\)-method), 246
method of means (\(J\)-method), 248
minimal scale of Banach spaces, 197
Nikol’skii space, 330
normalized interpolation space of type α, 22
normal scale of Banach spaces, 188
  continuous, 189
  maximal, 193
  regular, 196
normative linear manifold, 7
operation **, 124
operator of fractional integration, 149, 150
operator of strong (weakened, weak) type, 130, 131
optimal interpolation triples, 27
orbit of a point, 89
problem of multipliers, 325
quasiconcave function, 49
Rademacher system, 183
rearrangement of a function, 59
reflexive couple of spaces, 9
regular domain, 335
regular ideal lattice, 45
reiteration theorem, 231, 261
related space, 189
relative completion, 3
restriction of an ideal lattice, 43
Riesz-Thorin theorem, 22
right invertible mapping, 33
\(r\)-regular domain, 335
scale of Banach spaces, 187
scale of means, 278
Schwartz space, 325
separable measure, 45
simple function, 39
smallest concave majorant, 47
smoothing approximation process, 283
Sobolev-Slobodeckii space, 331
Sobolev space, 324
space complete with respect to another space, 6
space \(\mathfrak{F}(A_0, A_1)\), 216
space of slowly increasing Schwartz distributions, 325
space of traces, 313
spaces invariant under permutations, 350
strictly simple function, 39
strongly continuous semigroup of operators, 296
strongly measurable function, 209
subadditive function, 51
submultiplicative function, 52
subordinate operator, 290
support of an ideal lattice, 45
symmetric linear subset, 94
symmetric space, 90
total linear manifold, 8
truncation, 102
  right, 102
two-fold, 103
upper and lower indices, 350
weighted ideal lattice, 43
Young inequality, 167
NOTATION INDEX

$C$, imbedding, 2
$\Lambda_\psi$, Lorentz space, 98
$M_\phi$, Marcinkiewicz space, 98
$\chi_A(t)$, characteristic function of a set $A$, 39
$X^N$, truncation, 102
$x_N$, right truncation, 102
$x_M^N$, two-fold truncation, 103
$1^C$, imbedding with imbedding constant 1, 16
$\pi(AB,CD)$, unit ball in $L(AB,CD)$, 23

$L(AB,CD)$, linear space of bounded operators from a Banach couple $A, B$ into a Banach couple $C, D$, 19
$x^*(t)$, rearrangement of $x(t)$, 59
$n_\tau(\tau)$, distribution function of $x$, 58
$M_\delta(s)$, dilation function of $\psi$, 53
$\varphi_E$, fundamental function of the space $E$, 101
$\mathcal{D}(\mathbb{R}^n)$, space of infinitely differentiable functions with compact support, 323
$\mathcal{F}$, Fourier transformation, 325
$E$, closure of $E$ (used occasionally)
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