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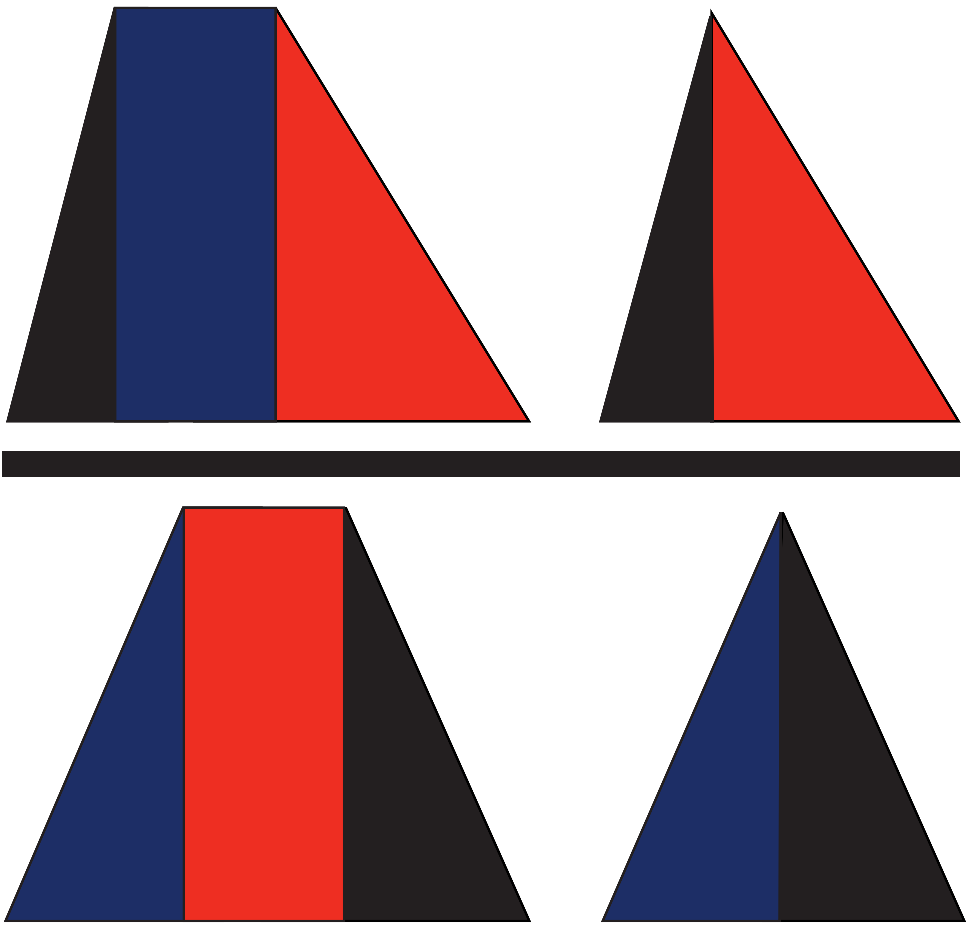
GEOMETRIC INEQUALITIES

NICHOLAS D. KAZARINOFF



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GEOMETRIC INEQUALITIES

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GEOMETRIC INEQUALITIES

by

Nicholas D. Kazarinoff

University of Michigan

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INDEX OF NUMBERED THEOREMS

1. If $a > b$ and $b > c$, then $a > c$. 9
2. If $a > b$ and $c \geq d$, then $a + c > b + d$. 10
3. If $a > b > 0$ and $c \geq d > 0$, then
(1) $ac > bd$ (2) $ac > bc$ and (3) $1/a < 1/b$. 10
4. If $a > b > 0$ and if $p > 0$, then $a^p > b^p$; if $p < 0$, then $a^p < b^p$. 11
5. For every positive integer n
$2\sqrt{n+1} - 2\sqrt{n} < 1/\sqrt{n} < 2\sqrt{n} - 2\sqrt{n-1}$. 14
6. If $a_i > 0$ $(i = 1, \cdots, n)$ and if $a_1 \cdot a_2 \cdot \cdots \cdot a_n = 1$, then $\sum_1^n a_i \geq n$, with equality holding if and only if $a_i = 1$ for each i. 19
7. If $a_i > 0$ $(i = 1, \cdots, n)$ and if $\sum_1^n a_i = nA$, then $a_1 \cdot a_2 \cdot \cdots \cdot a_n \leq A^n$ with equality if and only if $a_1 = a_2 = \cdots = a_n$. 20
8. If $a_i > 0$ $(i = 1, \cdots, n)$, then $\sqrt[n]{a_1 \cdot \cdots \cdot a_n} \leq \sum_1^n a_i/n$ with equality holding if and only if $a_1 = a_2 = \cdots = a_n$. 24
9. The Isoperimetric Theorem
(A) Of all plane figures with a given perimeter, the circle has the greatest area. 30
(B) Of all plane figures with a given area, the circle has the least perimeter. 30
For Three-dimensional Space
(A) Of all solids with a given surface area, the sphere has the greatest volume. 30

(B) Of all solids with a given volume, the sphere has the least surface area. 30

10. (A) Of all triangles with a common base and perimeter, the isosceles triangle has the greatest area. 32
(B) Of all triangles with a common base and area, the isosceles triangle has the smallest perimeter. 32

10A′ If two triangles have the same base and the same perimeter, the one with the smaller difference in the lengths of its legs has the larger area. 32

11A. Of all triangles with a given perimeter, the equilateral triangle has the greatest area. 38

11B. Of all triangles with a given area, the equilateral triangle has the least perimeter. 42

12. Of all n-gons inscribed in a given circle, the regular n-gon has the greatest area. 44

13. Of all quadrilaterals with a given area, the square has the least perimeter. 48

14. A quadrilateral with given sides has the greatest area when it can be inscribed in a circle. 50

15. Of all quadrilateral prisms with a given volume, the cube has the least surface area. 51

16. Given any n-gon which does not have all its sides of equal length, one can construct another n-gon of a larger area, with the same perimeter and with all sides of equal length. 54

17. Given an acute-angled triangle, the vertices of the inscribed triangle with the smallest perimeter are the feet of the altitudes of the given triangle. 77

18. (Erdös-Mordell) If P is any point inside or on the boundary of a triangle ABC, and if p_a, p_b, and p_c are the distances from P to the sides of the triangle, then $\overline{PA} + \overline{PB} + \overline{PC} \geq 2(p_a + p_b + p_c)$, with equality if and only if $\triangle ABC$ is equilateral and the point P is its circumcenter. 78

19. (Pappus) Let ABC be any triangle. Let $AA'C'C$ and $BB''C''C$ be any two parallelograms constructed on AC and BC respectively, so that either both parallelograms are outside the triangle or both are not entirely outside the triangle. Prolong their sides $A'C'$ and $B''C''$ to meet in P. Construct a third parallelogram $ABP''P'$ on AB with AP' parallel to CP and $\overline{AP'} = \overline{CP}$. The area of $ABP''P'$ is equal to the sum of the areas of the parallelograms $AA'C'C$ and $BB''C''C$. 84

Anybody who liked his first geometry course (and some who did not) will enjoy the simply stated geometric problems about maximum and minimum lengths and areas in this book. Many of these already fascinated the Greeks, for example the problem of enclosing the largest possible area by a fence of given length, and some were solved long ago; but other remain unsolved even today. Some of the solutions of the problems posed in this book, for example the problem of inscribing a triangle of smallest perimeter into a given triangle, were supplied by world famous mathematicians, others by high school students.

NICHOLAS D. KAZARINOFF was born in Ann Arbor, Michigan, in 1929. He received his BA and MA in physics from the University of Michigan, where his father, Donat K. Kazarinoff, taught mathematics for thirty-five years. He received his PhD in mathematics in 1954 from the University of Wisconsin, and is now Associate Professor of Mathematics at the University of Michigan. During the academic year 1959–60 he was on leave to do research at the Mathematics Research Center, US Army, in Madison, Wisconsin, and during the academic year 1960–61 he was in residence at the Steklov Mathematics Institute of the Academy of Sciences in Moscow, USSR.

The author's scientific interests lie in the fields of differential equations and geometry, especially the geometry of convex sets. **Geometric Inequalities** is an outcome of an informal evening seminar for high school student in Ann Arbor.

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