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The Lore of Large Numbers

Philip J. Davis

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THE LORE OF LARGE NUMBERS

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THE LORE OF LARGE NUMBERS

by

Philip J. Davis

National Bureau of Standards

He telleth the number of the stars;
He calleth them all by their names.
Psalm 147

6

THE MATHEMATICAL ASSOCIATION OF AMERICA

Illustrations by the author and Carl Bass

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Other titles in preparation.

CONTENTS

LIST OF ILLUSTRATIONS AND TABLES

Figure

Figure

Preface

The numbers have always been a source of wonder, and the sequence of integers whose unending procession is contrasted with the finiteness of human experience is surely the first place where mathematics appears as the product of soaring imagination. Children as well as adults raise many questions about large numbers. It has been my idea to answer these questions and at the same time to reveal a bit of the modern mathematical horizon via such questioning.

I have used the idea of large numbers as a binding theme for this book and, employing only the simplest materials, have tried to produce a feeling for numbers, their magnitude, and their growth. By means of approximate computation and estimation, I have suggested that numbers may be handled lightly and efficiently as friends with whom one discourses rather than as enemies against whom one struggles. In the later sections, large (and small) numbers take their place in mathematics and science, and are presented in a way which anticipates certain developments in abstract algebra and analysis and numerical analysis. I have tried to intimate—and this also has been a major aim—that mathematics is a living thing that grows and changes with the generations.

The problems are an integral part of the book. This does not mean that they must all be worked or be handled in the same manner. Some are meant to be read for they drive home a point or contain additional lore. Some are meant to be grappled with. Some require proofs, a word that should be interpreted flexibly. Some require additional information which the student must seek out for himself. For the most part, the problems can be worked with arithmetic alone, but this is not a text book and there are no "boxed methods" to serve as a complete guide. Every book should contain something for the reader to grow on; I have therefore included material in Part II which is more advanced, but which should not be allowed to impede the flow of the narrative.

To a certain extent this book is a family document. It was conceived the day I was asked by my children "What comes after millions?" It was begun after the question had been revised to "What comes after billions?" I hope with them, as with all who may be attracted to this book, that they will first grow into it and then grow out of it, and through it to the fertile acres of mathematical thought.

I should like to express my thanks to the School Mathematics Study Group for including this book in its monograph series, the *New Mathematical Library*, and to the Editorial Panel for many fine suggestions. Several discussions with Dr. Karl Goldberg during the initial writing were very helpful. Thanks are due also to Mrs. Molly F. Hevenor for helping with the secretarial work. I am grateful also to the editorial staff of the *New Mathematical Library* for its assistance, and particularly to Mrs. Jacqueline Lewis for her preparation of "Answers to Selected Problems."

Philip J. Davis

Chevy Chase, Md.
April 1961

Answers to Selected Problems

Set 2 (page 7)

1. The man on the street hardly ever does arithmetic with temperatures. He merely compares them: $52°$ is colder than $83°$ etc. He is using them as ordinals. The weather bureau also uses them as cardinals as, for instance, when they are added and divided to obtain average temperatures.

Set 3 (page 10)

2. He travels 11 blocks north and 6 blocks south, 9 blocks east and 9 west. At the end of his walk he is $11 - 6 = 5$ blocks north and $9 - 9 = 0$ blocks east. (What we have done to solve this problem is to say that 6 blocks south $= -6$ blocks north etc.) The Drunken Sailor was actually suggested by a very important problem in applied mathematics known as "the problem of the random walk."
3. $1.414 \times 1.414 = 1.999396 \qquad 2 - 1.999396 = 0.000604$
   $1.4142 \times 1.4142 = 1.99996164 \qquad 2 - 1.99996164 = 0.00003836$

Set 4 (page 17)

1. Sixteen quadrillion, three hundred seventy-five trillion, two hundred eighty-nine billion, one hundred eighty-two million, three hundred forty-three thousand, eight hundred ninety-two has 153 letters, 5 hyphens and 5 commas, altogether 163 symbols.

3. There are 9 positive one digit numbers $(1, \ldots, 9)$. There are 90 positive two digit numbers $(10, \ldots, 19, 20, \ldots, 29, \ldots, 90, \ldots, 99)$.
In general there are $9\overbrace{0\ldots00}^{n-1 \text{ zeros}}$ positive n digit numbers.

4. 9, 171.

5. If one of the whole numbers has more digits than the other, this number is the larger. Otherwise, starting at the left, compare the digits in the same position in each number until they are unequal. The larger digit belongs to the larger number. That is, let $a = a_n a_{n-1} \ldots a_1$ and $b = b_n b_{n-1} \ldots b_1$ be the two numbers. If
$$a_n = b_n\,,\ a_{n-1} = b_{n-1}\,,\quad \ldots,\quad a_{k+1} = b_{k+1}\,,\ a_k < b_k\,,\ \text{then } a < b\,.$$
Thus $6 = 6$, $0 = 0$, $2 < 6$ and so $60291 < 60621$.
The eye and the mind can go through this rule in a flash. If the numbers are large and unpunctuated, say
$$10069000701000 \quad \text{and} \quad 1006090000700100\,,$$
more time is required.

6. 9,964,411, 1,144,699.

7. The largest is 322,110, the smallest is $011223 = 11{,}223$. But this is not a conventional way of writing numbers. The zero must occur inside. So we place it where it causes the least damage: 101,223.

8. Eight, eighteen, eleven, fifteen, five, four, fourteen, nine, nineteen, one, seven, seventeen, six, sixteen, ten, thirteen, three, twelve, twenty, two.

9. LXXXVIII = 88, C = 100. The first occupies more space than the second, but is smaller. With Arabic numerals, the integer that occupies the most space is always the largest!

10.
$$\begin{aligned}(\mathrm{X}+\mathrm{X}+\mathrm{I}+\mathrm{I})(\mathrm{L}+\mathrm{X}+\mathrm{V}-\mathrm{I}) &= \mathrm{D}+\mathrm{C}+\mathrm{L}-\mathrm{X}+\\ &\ \mathrm{D}+\mathrm{C}+\mathrm{L}-\mathrm{X}+\\ &\ \qquad\quad \mathrm{L}+\mathrm{X}+\mathrm{V}-\mathrm{I}\\ &\ \qquad\quad \mathrm{L}+\mathrm{X}+\mathrm{V}-\mathrm{I}\\ \hline &= \mathrm{DDCCLLLLVV}-\mathrm{II}\\ &= \mathrm{DDCCCCVIII}\\ &= \mathrm{MCDVIII}\end{aligned}$$

11. If we use the duodecimal system, i.e., base 12, we have
$$\begin{aligned}\text{one dozen} &= (1\times 12)+(0\times 1) = (\text{in duodecimal})\ 10\\ \text{one gross} &= (1\times12\times12)+(0\times12)+(0\times1) = (\text{in duodecimal})\ 100\\ \text{one great gross} &= (1\times12\times12\times12)+(0\times12\times12)+(0\times12)+(0\times1)\\ &= (\text{in duodecimal})\ 1000\end{aligned}$$

12. Folio: The page size of a book made up of printer's sheets folded into 2 leaves.

    Quarto: The page size of a book made up of printer's sheets folded into 4 leaves.

    Octavo: The page size of a book made up of printer's sheets folded into 8 leaves.

    Sixteenmo: The page size of a book made up of printer's sheets folded into 16 leaves.

    Thirty-twomo: The page size of a book made up of printer's sheets folded into 32 leaves.

    This suggests the binary system.

13. Suppose we allow plates with the numbers 0, 01, 0063, etc. To make one million plates we need†

Number of Plates with	Using Numbers		Using Letters	
1 symbol	10		26	
2 symbols	10×10		26×26	
3 symbols	$10 \times 10 \times 10$		$26 \times 26 \times 26$	
4 symbols	$10 \times 10 \times 10 \times 10$		$26 \times 26 \times 26 \times 26$	475,254
5 symbols	$10 \times 10 \times 10 \times 10 \times 10$	111,110		524,746
6 symbols		888,890		
Total		1,000,000		1,000,000

If we use numbers, we need a plate large enough to carry 6 symbols. Using letters, we need only 5 symbols. Provided that letters and numbers are the same size and that all plates are made the same width, the lettered plates use only 5/6 the amount of metal.

14. Add the pence column (74d), divide by 12, write the remainder (2d) and carry the quotient (6s) to the shilling column. Then add this column (136s), divide by 20, write 16s and carry £6 to the pound column whose sum is then £11; and Jefferson is correct.

15. The number N will be of the binary form $n_k n_{k-1} \dots n_1 0$, where the digits n_i are either 0 or 1. In the decimal system‡

$$N = n_k \times 2^k + n_{k-1} \times 2^{k-1} + \dots + n \times 2 + 0 .$$

Since every term in this sum is divisible by 2, so is the sum; i.e. so is N.

† Suppose we have 10 symbols and we wish to make different plates using, say, 3 of these symbols. For the first symbol we may use any one of the original 10 symbols, and likewise for the second and third symbols. Therefore the number of different plates would be $10 \times 10 \times 10 = 1000$. (This subject matter is covered in discussions on permutations and combinations.)

‡ Powers are defined in the next section.

If M ends in 2 zeros then

$$M = m_r \times 2^r + \ldots + m_2 \times 2^2 + 0 \times 2 + 0,$$

and M is divisible by $2^2 = 4$. In general if the number ends in t zeros it will be divisible by 2^t. Note that in the decimal system (base 10), if a number ends in t zeros it is divisible by 10^t.

Set 5 (page 20)

1. $5^1 = 5$, $5^2 = 5 \times 5 = 25$, $5^3 = 5 \times 5^2 = 125$, $5^4 = 5 \times 5^3 = 625$, $5^5 = 5 \times 5^4 = 3125$.
2. Since $1 \times 1 \times \ldots \times 1 = 1$, one to any power is one.
3. $(1.5)^1 = 1.5$, $(1.5)^2 = 2.25$, $(1.5)^3 = 3.375$, $(1.5)^4 = 5.0625$, $(1.5)^5 = 7.59375$.
4. Since $0 \times 0 \times \ldots \times 0 = 0$, zero to any power is zero.
5. $2^7 = 128$, $7^2 = 49$; $4^3 = 64$, $3^4 = 81$.
6. 10^{100} written out has 100 zeros and 1 one, i.e., 101 symbols. In exponent form it has 5 symbols, a saving of 96 symbols.
7. $5^6 = 15{,}625$, $6^5 = 7{,}776$.
8. $2^{3+5} = 2^8 = 256$, $(3 + 5)^2 = 8^2 = 64$.
9. $2^7 =$ 128 has 3 digits
   $2^{10} =$ 1,024 has 4 digits
   $2^{14} =$ 16,384 has 5 digits
   $2^{17} =$ 131,072 has 6 digits
   $2^{20} =$ 1,048,576 has 7 digits

   By inspection, we see that the number of digits is about one third the exponent. By using logarithms this can be made more precise. The idea is to express the number 2^n as a number 10^k. If k is between 1 and 2, 10^k has 2 digits. If k is between 2 and 3, 10^k has 3 digits. In general if k is between $m - 1$ and m then 10^k has m digits. Set $10^k = 2^n$. Then $k \log_{10} 10 = n \log_{10} 2$. Since $\log_{10} 10 = 1$ and $\log_{10} 2 = .3010\ldots$, we have $k = n \log_{10} 2 \approx .3010n$. Therefore the number of digits in 2^n equals the first integer greater than $n \log_{10} 2 \approx .3010n$. Example: $n = 17$, $.3010 \times 17 = 5.117$. The first integer greater than 5.117 is 6, and 2^{17} has six digits.

Set 6 (page 26)

1. $10{,}000{,}000{,}000 = 10^{10}$.
2. A million billion = a billion million = a quadrillion = 10^{15}.
3. $(3 \times 100) + 3 = 303$, so a centillion would be 10^{303}.

4. The number consists of 66 nines.
5. Reading from the left to the right we have a one, nine zeros, a one, nine zeros and a one.
7. Twelve quadrillion, three hundred forty-five trillion, six hundred seventy-eight billion, nine hundred million.
8. 6,500,000,000,000,000,000,000.

Set 7 (page 28)

1. a) $6^8 \times 6^{10} = 6^{8+10} = 6^{18}$.
   b) $6^8 \times 6^8 = 6^{8+8} = 6^{16}$ or $6^8 \times 6^8 = (6^8)^2 = 6^{16}$.
   c) $(6^8)^3 = 6^{24}$.
   d) $2^4 \times 4^2 = 2^4 \times (2^2)^2 = 2^4 \times 2^4 = 2^8$.
2. $100^{10} = (10^2)^{10} = 10^{20}$; $\therefore 10^{100} > 100^{10}$.
   $8^{16} = (2^3)^{16} = 2^{48}$, $16^8 = (2^4)^8 = 2^{32}$, $2^{48} > 2^{32}$.
3. $(2 + 5)^4 = 7^4 = 2401$, $2^4 + 5^4 = 16 + 625 = 641$.
4. $(1 + 2 + 3)^4 = 6^4 = 1296$, $1^4 + 2^4 + 3^4 = 1 + 16 + 81 = 98$.
5. Using the first law of exponents: $a^m \times (a^n \times a^p) = a^m \times a^{n+p}$. Let $n + p = r$ and use the first law again: $a^m \times a^r = a^{m+r}$. Substitute $n + p$ for r.
6. $(ab)^m = ab \times ab \times \ldots \times ab$ (m ab's multiplied together). Since we may change the order of the terms (we say that multiplication is commutative; i.e., $3 \times 2 = 2 \times 3$) we can write all the a's then all the b's:
   $(ab)^m = a \times \ldots \times a \times b \times \ldots \times b$ (m a's multiplied together followed by m b's multiplied together). But m a's multiplied together is a^m, m b's is b^m, and so we have $a^m b^m$.
7. Using the second law of exponents, $(a^m)^m = a^{m^2}$; using it again,
   $$[(a^m)^m]^m = (a^{m^2})^m = a^{m \cdot m^2} = a^{m^3}.$$
8. $2^{2^2} = 2^4 = 16$, $2^{2^3} = 2^8 = 256$, $3^{2^2} = 3^4 = 81$.
9. $3^{(2^3)} = 3^8$, $(3^2)^3 = 3^6$, $3^8 \neq 3^6$.
10. $2^{2^{2^2}} = 2^{16} = 65{,}536$.

Set 8 (page 30)

1. a) 3.65×10^2, b) 1.0078×10^4, c) 6.329480×10^6.
2. 2.5×10^9, 6.5×10^{12}.

Set 10 (page 33)

1. 372,950, 372,900, 373,000, 400,000.
2. 300,000.

5. The number has been rounded.

6. This number rounded to one significant figure is 3,000,000 which is its order of magnitude.

7. a) $2^{15} = 32{,}768 \approx 32{,}800 = 3.28 \times 10^4$.
   b) There are $(36)^3$ cubic inches in a cubic yard.
   $$36^3 = 46656 \approx 46700 = 4.67 \times 10^4.$$

Set 11 (page 36)

1. a) $1.80 + $10.80 + $0.00 + $9.20 = $21.80 (nearest 10 cents).
   $2.00 + $11.00 + $0.00 + $9.00 = $22.00 (nearest dollar).
   b) $70 \times 90 \times 40 = 7 \times 9 \times 4 \times 10^3 = 252 \times 10^3 \approx 3 \times 10^5$.

2. 5280 ft/mi $\times$ 12 in/ft $\approx 5300 \times 12 = 63{,}600 \approx 6.4 \times 10^4$ in/mi. (Rounded to 2 significant figures.)

3. 1 rod = 16.5 feet, 1 sq rod $\approx (16 \text{ ft})^2 = 256 \approx 2.6 \times 10^2$ sq ft.
   1 acre $= 1.6 \times 10^2$ sq rods $\approx 1.6 \times 2.6 \times 10^4$ sq ft.
   $= 4.16 \times 10^4 \approx 4.2 \times 10^4$ sq ft. (Rounded to 2 significant figures.)

4. $\pi = 3.14\ldots \approx 3.1$.
   Area $= \pi r^2 \approx 3.1 \times 120 \times 120 = 3.1 \times 1.2 \times 1.2 \times 10^4$
   $\approx 3.1 \times 1.4 \times 10^4 = 4.34 \times 10^4$
   $\approx 4.3 \times 10^4$ sq in. (Rounded to 2 significant figures.)

5. $\pi \approx 3$, radius $= 18$ in ≈ 20 in,
   Volume $= \frac{4}{3}\pi r^3 \approx \frac{4}{3} \times 3 \times 20 \times 20 \times 20 = 32 \times 10^3$
   $\approx 3 \times 10^4$ cu in. (Rounded to 1 significant figure.)

6. Approximately 2,000 years, approximately 50 Sundays to a year, therefore approximately 1×10^5 Sundays. (Rounded to one significant figure.)

7. To the nearest cent. Rounding occurs, for instance, when interest is computed.

8. Round all numbers to 1 significant figure.
   No. of seconds in a year
   $\approx 6 \times 10$ sec/min $\times 6 \times 10$ min/hr $\times 2 \times 10$ hr/day $\times 4 \times 10^2$ days/yr
   $\approx 288 \times 10^5 \approx 3 \times 10^7$ sec/yr.
   Thus the sun travels approximately
   $$3 \times 10 \text{ mi/sec} \times 3 \times 10^7 \text{ sec/yr} = 9 \times 10^8 \text{ mi/yr}.$$

9. Radius $\approx 4 \times 10^3$ miles, $(4 \times 10^3)^2 = 4^2 \times (10^3)^2 = 16 \times 10^6$,
   Area $= 4\pi r^2 \approx 12 \times 16 \times 10^6 = 192 \times 10^6 \approx 2 \times 10^8$ sq mi.

10. From Problem 4, 1 acre $\approx 4.2 \times 10^4$ sq ft, 1 sq ft $\approx 1.4 \times 10^2$ sq in.
    Therefore
    1 acre $\approx 1.4 \times 10^2 \times 4.2 \times 10^4 \approx 5.88 \times 10^6 \approx 5.9 \times 10^6$ sq in.
    Income/acre $\approx .25 \times 5.9 \times 10^6 \approx 1.475 \times 10^6 \approx \$1.5 \times 10^6$.

Set 12 (page 39)

6. A recent Washington, D.C. phone book lists 52 columns of Smiths out of a total of about 5400 columns. The ratio is therefore $\frac{52}{5400} \approx$ (say) $\frac{1}{100}$. (What is this ratio for your city?) Of course, the phone book lists businesses, offices, etc., and Washington may not be a typical city, but these figures would indicate that about 1 person in 100 is a Smith. At this rate there should be around 2,000,000 Smiths in the U.S.A.

8. The actual count (in the King James version) is supposed to be 2,728,800 for the Old Testament, 838,380 for the New Testament. Total: 3,567,180.

9. The Holland-America line estimates 36,855 pounds of meat, 9,250 pounds of fowl, and 11,560 pounds of fish.

Set 13 (page 43)

1. 4.0082×10^{-3}, 2.223×10^{-8}.

2. .0000000229, .000030077.

3. 1 picogram = 10^{-12} grams, 1 centigram = 10^{-2} grams. Therefore 1 gram = 10^{12} picograms = 10^{2} centigrams, or, dividing by 10^{2}, 10^{10} picograms = 1 centigram.

4. $\frac{3}{127} = .02362\ldots$, $\frac{4}{169} = .02366\ldots$, so $\frac{4}{169} > \frac{3}{127}$.

5. Find a common denominator for the fractions and compare the numerators:

$$\frac{3 \times 169}{127 \times 169}, \quad \frac{4 \times 127}{127 \times 169}$$

$$3 \times 169 = 507, \quad 4 \times 127 = 508, \quad 508 > 507.$$

6. Multiply numerator and denominator by the appropriate number so that all the denominators equal 64: $\frac{7}{64}''$, $\frac{8}{64}''$, $\frac{10}{64}''$, $\frac{16}{64}''$, $\frac{11}{64}''$, $\frac{12}{64}''$.
Arrange in order of size: $\frac{7}{64}''$, $\frac{8}{64}''$, $\frac{10}{64}''$, $\frac{11}{64}''$, $\frac{12}{64}''$, $\frac{16}{64}''$.

7. a) Since

$$\frac{1}{30}+\frac{1}{30}+\frac{1}{30}+\frac{1}{30}+\frac{1}{30}+\frac{1}{30} < \frac{1}{25}+\frac{1}{26}+\frac{1}{27}+\frac{1}{28}+\frac{1}{29}+\frac{1}{30}$$
$$< \frac{1}{25}+\frac{1}{25}+\frac{1}{25}+\frac{1}{25}+\frac{1}{25}+\frac{1}{25},$$

the answer is surely greater than $\frac{6}{30} = .20$ and less than $\frac{6}{25} = .24$. A good guess might be the average of these values: .22.

b) Exact answer: $\dfrac{1,560,647}{7,125,300} = .219028\ldots.$

Note the large numbers that arise from this "simple" addition.

Set 14 (page 46)

1. $2.72 \times 3.03 \times 10^{-5} \times 10^{8} = 2.72 \times 3.03 \times 10^{3} = 8.2416 \times 10^{3}$.

2. $\frac{1}{7^8}$ is larger than $\frac{1}{7^{16}}$.

3. $2^{-(2^3)} = 2^{-8} = \frac{1}{2^8} = \frac{1}{256}, \quad 2^{(2^3)} = 2^8 = 256,$

$(-2)^{2^3} = (-2)^8 = (-1 \times 2)^8 = (-1)^8 \times (2)^8 = 256,$

$(-2)^{-2^3} = \frac{1}{(-2)^{2^3}} = \frac{1}{256}.$

5. $$\left(\frac{a}{b}\right)^m = \overbrace{\frac{a}{b} \times \ldots \times \frac{a}{b}}^{m \text{ factors}} = a \times \frac{1}{b} \times \ldots \times a \times \frac{1}{b}, \text{ or}$$

$$\overbrace{a \times \ldots \times a}^{m \text{ factors}} \times \overbrace{\frac{1}{b} \times \ldots \times \frac{1}{b}}^{m \text{ factors}} = a^m \times \left(\frac{1}{b}\right)^m.$$

6. One square fermi $= 10^{-13} \times 10^{-13} = 10^{-26}$ square centimeters $= 10^{-2} \times 10^{-24}$ sq. cm. $= 10^{-2}$ barns,

or 1 barn $= 10^2$ sq. fermis.

Set 15 (page 50)

4. a) Choose the unit of length to be less than the length of the smaller object.
   b) Choose the unit length greater than the length of the larger object.
   c) Choose the unit length between the lengths of the two objects.

5. $\frac{L}{L} = I, \quad I \times L = I. \quad$ Therefore $\left(\frac{L}{L}\right) \times L = I$.

6. $\frac{L}{S} = L, \left(\frac{L}{S}\right) \times S = L \times S = I; \left(\frac{S}{S}\right) = I, \ L \times \left(\frac{S}{S}\right) = L \times I = I$.

7. $S^2 = S \times S = S, \quad L^2 = L \times L = L, \quad I^2 = I \times I = I$.

8. $$S^n = \overbrace{S \times S \times \ldots \times S}^{n \text{ factors}} = S, \quad L^n = \overbrace{L \times L \times \ldots \times L}^{n \text{ factors}} = L,$$

$$I^n = \overbrace{I \times \ldots \times I}^{n \text{ factors}} = I.$$

9. This follows immediately from 8, for, e.g., S to any positive power is S; thus $S^m = S$, $S^n = S$, $S^{m+n} = S$, and

$$S^m \times S^n = S \times S = S = S^{m+n}.$$

10. Again, since $S^n = S$, $(S^n)^m = (S)^m = S$ and $S^{nm} = S^k$ (where $k = nm$) $= S$. (The same reasoning holds for L and I.) So the second law holds.

11. $\sqrt{S}$ designates the collection of numbers which is obtained by taking square roots of numbers in S. Since the square root of a number between 0 and 1 is itself between 0 and 1, and since, conversely, every number between 0 and 1 is the square root of some number between 0 and 1, we have $\sqrt{S} = S$. $\sqrt{L}$ and $\sqrt{I}$ can be defined in a similar way and we have $\sqrt{L} = L$, $\sqrt{I} = I$. We might also observe that
$$S \times S = S^2 = S\,,$$
and so "extracting the square root" of both sides yields $S = \sqrt{S}$. Similar observations hold for L and I.

12. $a = S$, $b = L$.
$$S \times L = I\,, \qquad \sqrt{S \times L} = \sqrt{I} = I\,,$$
$$\sqrt{S} = S\,, \qquad \sqrt{L} = L\,, \qquad \sqrt{S} \times \sqrt{L} = S \times L = I\,.$$
Therefore
$$\sqrt{S \times L} = \sqrt{S} \times \sqrt{L}.$$
The other possibilities are established in the same way. The rules hold.

Set 16 (page 54)

4. $9{,}999{,}999{,}999 \approx 10^{10}$. $\dfrac{10^{10}\text{ subtractions}}{1300\text{ subtractions/minute}} = \dfrac{10^8}{13}\text{ min} \approx 15\text{ years.}$

5. On some machines the motor will run indefinitely until it has been turned off by pressing a stop key.

Set 17 (page 65)

1. $3\frac{10}{71} = 3.1408\ldots$, $3\frac{1}{7} = 3.1428\ldots$; since $\pi = 3.14159\ldots$, it lies between $3\frac{10}{71}$ and $3\frac{1}{7}$, and agrees with these numbers to 2 decimal places.

2. By long division, $355/113 = 3.1415929\ldots$, but $\pi = 3.1415926\ldots$; the numbers agree to six places.

3. The average of $3\frac{1}{7}$ and $3\frac{10}{71}$ is (see Problem 1),
$$\tfrac{1}{2}(3.1408\ldots + 3.1428\ldots) = \tfrac{1}{2}(6.2836\ldots) = 3.1418\ldots.$$
π is closer to this average than to either number.

4. Radius of earth $\approx 4{,}000$ mi $\approx 2.53 \times 10^6$ in.,
circumference $\approx 1.58 \times 10^9$ in. $\quad 2^{30} \approx 1.07 \times 10^9$,
$$\frac{\text{circumference}}{2^{30}} \approx \frac{1.58 \times 10^9}{1.07 \times 10^9} \approx 1.5\text{ in.}$$

5. $(3.14159265\ldots)^2 - 10 \approx -.13$
$22(3.14159265\ldots)^4 - 2143 \approx .0000027$
$9(3.14159265\ldots)^4 - 240(3.14159265\ldots)^2 + 1492 \approx -.023.$

6. $a = \frac{13}{64} \times 31{,}680 = 13 \times 495 \text{ in.} \approx 13 \times 5 \times 10^2 \text{ in.}$
$b = \frac{11}{64} \times 31{,}680 = 11 \times 495 \text{ in.} \approx 11 \times 5 \times 10^2 \text{ in.}$

$$\text{area} = \pi ab \approx 3.1 \times 1.3 \times 1.1 \times 2.5 \times 10^7 \text{ sq. in.}$$
$$= \frac{3.1 \times 1.3 \times 1.1 \times 2.5 \times 10^7}{1.4 \times 10^2} \text{ sq. ft.}$$
$$\frac{3.1 \times 1.3 \times 1.1 \times 2.5 \times 10^7}{1.4 \times 4.4 \times 10^6} \text{ acres} \approx 18 \text{ acres.}$$

Set 18 (page 74)

1. We first define $n!$ (read n factorial).

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 1,$$

that is, $1! = 1$, $2! = 2 \times 1 = 2$, $3! = 3 \times 2 \times 1 = 6$.

Then Liouville's number has a 1 in the $n!$ decimal place for $n = 1, 2, 3, \ldots$ and 0 elsewhere. Since the ones do not appear periodically the number is *not* a repeating decimal and so it is not a fraction. Since the only digits which appear are 0 and 1 the number is not normal.

2. 362 0's, 428 1's, 409 2's, 369 3's, 405 4's,
418 5's, 398 6's, 376 7's, 405 8's, 430 9's.

If the digits were divided absolutely evenly there would be 400 of each.

3.

	Observed Frequency	Expected Frequency
Busts	115	$.2952 \times 400 = 118.08$
One Pair	198	$.5040 \times 400 = 201.6$
Two Pairs	53	$.1080 \times 400 = 43.2$
Three of a kind	26	$.0720 \times 400 = 28.8$
Full house	5	$.0090 \times 400 = 3.6$
Straight	2	$.0072 \times 400 = 2.88$
Four of a kind	1	$.0045 \times 400 = 1.8$
Five of a kind	0	$.0001 \times 400 = .04$

Set 19 (page 82)

1. $\frac{1}{13} = 0.076923\ldots$, $\frac{1}{51} = 0.0196078431372549\ldots$,
$\frac{1}{61} = 0.01639344262295081967213114754098360$
$655737704918032786885245901\ldots.$

2. $$\begin{array}{rl} 10{,}000r &= 2468.2468\ldots \\ r &= .2468\ldots \\ \hline 9{,}999r &= 2468, \end{array} \qquad r = \frac{2468}{9999}.$$

3. $$\begin{array}{rl} 10{,}000r &= 1961.1961\ldots \\ r &= .1961\ldots \\ \hline 9{,}999r &= 1961, \end{array} \qquad r = \frac{1961}{9999}.$$

Set 20 (page 90)

1. The sum of the digits is 41, $4 + 1 = 5$; the number is not a square.
2. The sum of the digits is 36, $3 + 6 = 9$; this answer is admissible for a square, but this test does not suffice. By taking square roots, we obtain 1164 exactly.
3. Adding the digits as above the final answer is 9; this answer is admissible for a cube, but this test does not suffice. Taking cube roots, we obtain 1776 exactly.
4. Let s_1 and s_2 be two squares. Then $s_1 \times s_2$ is the square of the number $\sqrt{s_1} \times \sqrt{s_2}$. For example, let $s_1 = 4$, $s_2 = 9$, then $\sqrt{s_1} = 2$, $\sqrt{s_2} = 3$, $s_1 \times s_2 = 36$, $\sqrt{s_1} \times \sqrt{s_2} = 6$, $6^2 = 36$. In general, the product of n squares $s_1 \times \ldots \times s_n$ is the square of the number $\sqrt{s_1} \times \ldots \times \sqrt{s_n}$.
5. Let c_1 and c_2 be two cubes. Then $c_1 \times c_2$ is the cube of the number $\sqrt[3]{c_1} \times \sqrt[3]{c_2}$. The proof for the general case is the same.
6. We observe first that, if the number n^2 ends in an even digit, then n^2 is even. Moreover, n must be even because, if n were odd, say $n = 2m + 1$, then $n^2 = 4m^2 + 4m + 1 = 2[2m^2 + 2m] + 1$ would be odd and could not end in an even digit. Since n is divisible by 2, it is twice another number, say $n = 2k$. Hence $n^2 = 4k^2$ is divisible by 4.
7. If n^2 ends in a zero, then it is a multiple of 10, i.e. it has the factors 2 and 5. But the square n^2 must have all its prime factors repeated an even number of times (half of them come from the first n, the other half from the second n in n^2). So n^2 must have the factors $2^2 \times 5^2 = 100$, or $2^4 \times 5^4 = 10{,}000$, etc. and hence ends in an even number of zeros.
8. The smallest number divisible by $1, 2, \ldots, 9, 10$ is 2520 and the smallest number which we can raise to this power is 2, i.e., 2^{2520} is the number.

$$\begin{aligned} 2^{2520} &= (2^{1260})^2 = (2^{840})^3 = (2^{630})^4 = (2^{504})^5 = (2^{420})^6 = (2^{360})^7 \\ &= (2^{315})^8 = (2^{280})^9 = (2^{252})^{10}. \end{aligned}$$

9. $1936 = 44^2$. The next year that is a square is $45^2 = 2025$.

10. $1728 = 12^3$. The next year that is a cube is $13^3 = 2197$.

11. 360,360 is divisible

by:	since:
2	0 is divisible by 2
3	$3+6+0+3+6+0 = 18$ is divisible by 3
4	$12 + 0 = 12$ is divisible by 4
5	0 is divisible by 5
6	it is divisible by 2 and 3
7	$(3 \times 0) + (2 \times 6) - (1 \times 3) - (3 \times 0) - (2 \times 6) + (1 \times 3) = 0$ is divisible by 7
8	$0 + (2 \times 6) + (4 \times 3) = 24$ is divisible by 8
9	$3 + 6 + 3 + 6 = 18$ is divisible by 9
10	0 is the last digit
11	$0 - 6 + 3 - 0 + 6 - 3 = 0$ is divisible by 11
12	it is divisible by 4 and 3
13	$(10 \times 0) - (4 \times 6) - (1 \times 3) + (3 \times 0) + (4 \times 6) + (1 \times 3) = 0$ is divisible by 13.

12.

1	2	3	4	5	6	7	8	9	10	11	12	13
1	2	3	2^2	5	2×3	7	2^3	3^2	2×5	11	$2^2\times3$	13

Hence the least common multiple of the first 13 integers is

$$2^3 \times 3^2 \times 5 \times 7 \times 11 \times 13 = 360{,}360\,.$$

13. For the number $10\ldots01$ to be divisible by 7, the sum

$$(3 \times 1) + (2 \times 0) - (1 \times 0) - (3 \times 0) - (2 \times 0) + (1 \times 0) + (3 \times 0) + \ldots + (n \times 1)\,,$$

where n is one of the numbers $3, 2, -1, -3, -2, 1,$ must be divisible by 7. But this sum is equal to $(3 \times 1) + (n \times 1)$ which is divisible by 7 only if $n = -3$. Thus the sequence must end with -3 so we can have 2 zeros, or we can add 6 more zeros until we come to -3 the second time, etc. So the possible number of zeros is given by $2 + 6m$ where m may be $1, 2, 3, \ldots$. The same type of reasoning proves the statement for divisibility by 13.

14. To test the given number consisting of n digits, all ones, for divisibility by 7, we cut off the sum

$$(3 \times 1) + (2 \times 1) - (1 \times 1) - (3 \times 1) - (2 \times 1) + \ldots$$

(where the dots indicate repetition of these terms) after n terms and find that, when $n = 1, 2, 3, 4, 5,$ or 6, the corresponding sum is $3, 5, 4, 1, -1,$ or 0. For $n = 7, 8, \ldots,$ the values $3, 5, \ldots$ of the corresponding sums just repeat. The only one of these sums divisible by 7 is 0. Each time we have six 1's in a row the sum will be 0 so the number of 1's must be a multiple of 6. The same type of argument proves that this is also true for divisibility by 13.

15. $1 \times 2 \times 3 \times 4 \times \ldots \times 10 + 1 = 3{,}628{,}801$,

$$1 - 0 + 8 - 8 + 2 - 6 + 3 = 0,$$

and 0 is divisible by 11.

16. Let p be the prime, let 6 go into p n times with a remainder r where r may be 1, 2, 3, 4, or 5, i.e.,

$$\frac{p}{6} = n + \frac{r}{6} \quad \text{or} \quad p = 6n + r\,.$$

Now if r is a digit other than 1 or 5, the right side may be factored:

$$p = 6n + 2 = 2(3n + 1), \qquad p = 6n + 3 = 3(2n + 1),$$
$$p = 6n + 4 = 2(3n + 2),$$

and p will be divisible by 2, 3, or 4 and thus cannot be a prime.

17. We shall verify that 1951 and 1973 are primes by showing that they are not divisible by any of the primes 2, 3, 5, 7, ... The question is how far must we go to be sure? It will suffice to test all primes smaller than 45 because 45^2 is greater than both 1951 and 1973, and if 1951 had a factor greater than 45, then its other factor must be smaller than 45 and hence it lies among the primes we are testing. The same is true for 1973. Now we just verify that neither 1951 nor 1973 is divisible by 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43.

19. a) $1729 = 1728 + 1 \quad = 12^3 + 1^3$
$\qquad = 1000 + 729 = 10^3 + 9^3$.

b) 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728 are the cubes less than 1729; try all possibilities.

20. Let T be a triangular number. Then

$$T = 1 + 2 + \ldots + n = \frac{n}{2}(1 + n)$$

$$8T + 1 = \frac{8n}{2}(1 + n) + 1 \quad = 4n^2 + 4n + 1 = (2n + 1)^2\,.$$

So $8T + 1$ is the square of the number $2n + 1$. Conversely, if $8N + 1$ is a square, it must be the square of an odd number: $8N + 1 = (2n + 1)^2$. Consequently $N = \frac{1}{2}n(1 + n)$ and hence must be triangular.

Set 21 (page 101)

1. Find the digit sums of the summands:

$$\begin{array}{lcl}
1 + 2 + 9 + 5 = 17, & & 1 + 7 = 8; \\
2 + 8 + 7 + 2 = 19, & 1 + 9 = 10, & 1 + 0 = 1; \\
9 + 0 + 0 + 1 = 10, & & 1 + 0 = 1; \\
6 + 2 + 1 + 4 = 13, & & 1 + 3 = 4; \\
7 + 7 + 2 + 8 = 24, & & 2 + 4 = 6.
\end{array}$$

Add these digit sums: $8 + 1 + 1 + 4 + 6 = 20$, $2 + 0 = 2$.
Find the digit sum of the reputed sum: $6 + 4 + 2 = 12$, $1 + 2 = 3$.
Since $3 \neq 2$, the sum is not correct; in fact the correct sum is 63920.

2. $1 + 6 + 9 + 2 + 0 + 2 + 0 + 3 + 0 + 2 + 0 + 2 + 0 + 1 = 28$, $2 + 8 = 10$, $1 + 0 = 1$ and the number is not a multiple of 9.
3. The sums of the individual digits in the two numbers are the same, and this sum is the remainder when the number is divided by 9 (provided that, if the sum exceeds 8, we replace it by its residue). The residue of the difference of two numbers is equal to the difference of their residues. (This may be proved in the same way as for the sum.) The difference of the residues of these two numbers is zero (since their residues are the same) and so this difference is divisible by 9.
4. To be divisible by 9 the sum of the digits must be a multiple of 9 and thus must contain at least 9 ones. The smallest such number is 111,111,111.
5. The $9 \times m$th term, $m = 1, 2, 3, \ldots$, has $9m$ sevens. The sum of the digits of this term is $9 \times m \times 7$ which is certainly divisible by 9. There are no other multiples of 9 in the series of digit sums.
6. Every 3rd term in the series of 6's is divisible by 9 since each time we add 3 sixes we add 18 to the sum of the digits and 18 is divisible by 9.

   In the second series, consider separately the 1st, 3rd, 5th, ... terms

$$2, \quad 252, \quad 25252, \quad \ldots,$$

   and the 2nd, 4th, 6th, ... terms

$$25, \quad 2525, \quad 252525, \quad \ldots.$$

   The $(2m + 1)$st term consists of $(m + 1)$ 2's and m 5's. The $(2m)$th term consists of m 2's and m 5's. The digit sum of the $(2m + 1)$st term is therefore $2(m + 1) + 5m = 7m + 2$. The digit sum of the $2m$th term is $2m + 5m = 7m$. These must be multiples of 9. Now $7m$ is a multiple of 9 only when $m = 9, 18, \ldots$, i.e., a multiple of 9: $m = 9k$. $7m + 2$ is a multiple of 9 only if $m = 1, 10, 19, \ldots$, i.e., one more than a multiple of 9: $m = 9k + 1$. Thus, finally, the $2m = 2(9k) = 18k$th terms and the

$$2m + 1 = 2(9k + 1) + 1 = (18k + 3)\text{rd}$$

   terms are the only multiples of 9 in the series. These are the 3rd, 18th, 21st, 36th, 39th, ... terms.
7. $r(79{,}532{,}853) = r(42) = 6$, $r(93{,}758{,}479) = r(52) = 7$, $r(6 \times 7) = r(42) = 6$, $r(7{,}456{,}879{,}327{,}810{,}587) = r(87) = r(15) = 6$, and the test is satisfied.
8. $r(123{,}456) = r(21) = 3$, $r(789{,}123) = r(30) = 3$, $r(3 \times 3) = 0$. $r(97{,}42d{,}969{,}088) = r(62 + d)$. We must have $r(62 + d) = 0$, i.e., $62 + d$ must be a multiple of 9, so $d = 1$.
9. $r(162{,}5d0{,}792) = r(32 + d)$. $32 + d$ must be a multiple of 9, so $d = 4$.
10. $r(6d{,}d23) = r(11 + 2d)$. $11 + 2d$ must be a multiple of 9, so $d = 8$.

Set 22 (page 114)

1. $$\begin{aligned} 2a + b &= 99 \\ a + 2b &= 108 \end{aligned} \qquad \begin{aligned} 2a + b &= 99 \\ 2a + 4b &= 216 \\ \hline -3b &= -117 \quad \text{1st reduced system} \\ b &= 39 \end{aligned}$$

$$2a = 99 - 39 = 60, \qquad a = 30\,.$$

2. $$\begin{aligned} 6x - y + z &= 8 \quad \text{(1st equation)} \\ 6x + 30y - 6z &= 42 \quad \text{(2nd equation)} \\ \hline -31y + 7z &= -34 \end{aligned}$$

$$\begin{aligned} 3x + 15y - 3z &= 21 \quad \text{(2nd equation)} \\ 3x - 8y + 4z &= 3 \quad \text{(3rd equation)} \\ \hline 23y - 7z &= 18 \end{aligned}$$

$$\left.\begin{aligned} -31y + 7z &= -34 \\ 23y - 7z &= 18 \end{aligned}\right. \quad \text{1st reduced system}$$

$$\begin{aligned} \hline -8y &= -16 \quad \text{2nd reduced system} \\ y &= 2 \end{aligned}$$

$$\begin{aligned} -62 + 7z &= -34 \\ 7z &= -34 + 62 = 28 \\ z &= 4 \end{aligned} \qquad \begin{aligned} 6x - 2 + 4 &= 8 \\ 6x &= 8 + 2 - 4 = 6 \\ x &= 1\,. \end{aligned}$$

3. As above, $-31y + 7z = -34$

$$\begin{aligned} 8x + 40y - 8z &= 56 \quad \text{(2nd equation)} \\ 8x + 9y - z &= 10 \quad \text{(3rd equation)} \\ \hline 31y - 7z &= 46, \text{ or } -31y + 7z = 46 \end{aligned}$$

$$\left.\begin{aligned} -31y + 7z &= -34 \\ -31y + 7z &= 46 \end{aligned}\right. \quad \text{1st reduced system}$$

$$\begin{aligned} \hline 0 &= -80\,. \end{aligned}$$

But this last equation is impossible and the given system of equations has no solution. In this case the system is called *inconsistent.* Inconsistent systems place contradictory requirements upon the unknowns.

4. We note that if we rewrite the 9 equations replacing G by I, I by G, etc. (as indicated in the problem) we obtain the same set of equations, and so $G = I$, etc. Physically we should expect that the average temperature at points which occupy positions symmetric with respect to the center line (CW) of the square will be equal. We have eliminated the 3rd, 6th and 9th equations since they are equal respectively to the 1st, 4th and 7th.

$$\begin{array}{ll}
(1) & 4I = H + N + 200 \\
(2) & 4H = 2I + M + 100 \\
(3) & 4N = I + M + S + 100 \\
(4) & 4M = 2N + H + R \\
(5) & 4S = N + R + 140 \\
(6) & 4R = 2S + M + 40
\end{array}$$

$$\begin{array}{ll}
(1) & 4I - H - N = 200 \\
(2) & \underline{-4I + 8H - 2M = 200} \\
(7) & 7H - 2M - N = 400
\end{array}
\qquad
\begin{array}{ll}
(1) & 4I - H - N = 200 \\
(3) & \underline{-4I - 4M - 4S + 16N = 400} \\
(8) & -H - 4M - 4S + 15N = 600
\end{array}$$

Equations (4), (5), (6), (7), (8) are the 1st reduced system.

$$\begin{array}{ll}
(4) & -7H + 28M - 14N - 7R = 0 \\
(7) & \underline{7H - 2M - N = 400} \\
(9) & 26M - 15N - 7R = 400
\end{array}
\qquad
\begin{array}{ll}
(4) & -H + 4M - 2N - R = 0 \\
(8) & \underline{-H - 4M - 4S + 15N = 600} \\
(10) & 8M + 4S - 17N - R = -600
\end{array}$$

Equations (5), (6), (9), (10) are the 2nd reduced system.

$$\begin{array}{ll}
(5) & 4S - N - R = 140 \\
(6) & \underline{-4S + 8R - 2M = 80} \\
(11) & -N + 7R - 2M = 220
\end{array}
\qquad
\begin{array}{ll}
(5) & 4S - N - R = 140 \\
(10) & \underline{4S - 17N - R + 8M = -600} \\
 & 16N - 8M = 740 \\
(12) & 4N - 2M = 185
\end{array}$$

Equations (9), (11), (12) are the 3rd reduced system.

$$\begin{array}{ll}
(9) & -7R + 26M - 15N = 400 \\
(11) & \underline{7R - 2M - N = 220} \\
 & 24M - 16N = 620 \\
(13) & 6M - 4N = 155
\end{array}
\qquad
\begin{array}{ll}
(12) & 4N - 2M = 185 \\
(13) & \underline{-4N + 6M = 155} \\
 & 4M = 340 \\
 & M = 85
\end{array}
\quad \text{4th reduced system}$$

Having determined M, we determine the other unknowns by back substitution:

$$(13)\ 4N = 6 \times 85 - 155 = 355, \qquad N = \frac{355}{4};$$

$$(11)\ 7R = \frac{355}{4} + 170 + 220 = \frac{1915}{4}, \qquad R = \frac{1915}{28};$$

$$\begin{aligned}
(10)\ 4S &= -8 \times 85 + 17 \times \frac{355}{4} + \frac{1915}{28} - 600 \\
&= \frac{8320}{28} = \frac{2080}{7}, \\
S &= \frac{2080}{28} = \frac{520}{7};
\end{aligned}$$

$$(7)\ 7H = 400 + 2 \times 85 + \frac{355}{4} = \frac{2635}{4}, \qquad H = \frac{2635}{28};$$

$$(1)\ 4I = \frac{2635}{28} + \frac{355}{4} + 200 = \frac{2680}{7}, \qquad I = \frac{670}{7}.$$

Therefore

$$G = I = \frac{670}{7} \approx 95.7\,, \quad H = \frac{2635}{28} \approx 94.1\,, \quad L = N = \frac{355}{4} = 88.75,$$

$$M = 85\,, \qquad Q = S = \frac{520}{7} \approx 74.3\,, \qquad R = \frac{1915}{28} \approx 68.4\,.$$

Set 23 (page 125)

1. The general term of sequence (A) is $100 + 1{,}000n$, that of sequence (B), $3^n \times 2$. We want to know when

$$\text{the ratio } \frac{3^n \times 2}{100 + 1{,}000n}$$

of the two terms is at least 1. This ratio will be greater than 1 for n greater than or equal to 8, i.e. beginning with the 9th term.

2. The general term of the ratios is $\frac{n}{\sqrt{n}}$. Multiplying this by $1 = \sqrt{n}/\sqrt{n}$ we have

$$\frac{n}{\sqrt{n}} = \frac{n\sqrt{n}}{n} = \sqrt{n}.$$

As n becomes larger and larger, so does $\sqrt{n}$.

3. The two ratios to be considered are

$$\frac{n\sqrt{n}}{n} = \sqrt{n} \quad \text{and} \quad \frac{n^2}{n\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}\,.$$

As n becomes larger, so does $\sqrt{n}$.

4. The general term is $n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}$ (n n's). There are infinitely many sequences with more rapid growth. Consider for example the sequence with general term $\left(n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}\right)^2$. Then the ratio of the 2nd scale to the 1st is $n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}$ which increases as n increases.

5. The number of ancestors n generations removed is 2^n.

6. 2^{40}. The population wasn't this large ($2^{40} > 10^{12}$ = one trillion). This shows that people intermarry; for the answer to Problem 5 is correct only if the ancestors are all different.

8. 1 2 3 5 8 13 21 34 55 89 144 233 . . .
   1 2 3 5 8 13 21 34 55 89 144 . . . 1st differences

   Taking growth differences again and again always gives us the Fibonacci sequence, i.e., none of the growths ever stops growing. On the other hand the nth difference of the scale of nth power is a constant, and hence this scale grows less rapidly than the Fibonacci sequence. The ratios of successive terms are:

$$\frac{2}{1} = 2\,,\ \frac{3}{2} = 1.5\,,\ \frac{5}{3} = 1.66\ldots,\ \frac{8}{5} = 1.60\,,\ \frac{13}{8} = 1.625\,,$$

$$\frac{21}{13} = 1.615\ldots,\ \frac{34}{21} = 1.619\ldots,\ \frac{55}{34} = 1.617\ldots,\ \frac{89}{55} = 1.618\ldots \text{ etc}$$

   Notice that beginning with the first ratio every other term is greater than

$$\tfrac{1}{2}(1 + \sqrt{5}) = 1.618034\ldots$$

   while the in between terms are less than it; but as we go on in the sequence, the terms come closer and closer to this "limit."

Miscellaneous Problems (page 127)

1. HUNDRED represents the number

$7(26)^6 + 20(26)^5 + 13(26)^4 + 3(26)^3 + 17(26)^2 + 4(26) + 3.$

THOUSAND begins with $19(26)^7$ and is therefore even larger.

MILLION is

$12(26)^6 + 8(26)^5 + 11(26)^4 + 11(26)^3 + 8(26)^2 + 14(26) + 13.$

Hence THOUSAND > MILLION > HUNDRED

$$\text{ONE} + \text{ONE} = 2[14(26)^2 + 13(26) + 4] = 28(26)^2 + (26)^2 + 8$$
$$= (26)^3 + 3(26)^2 + 8 = \text{BDAI}.$$

3. Area (circle) $= \pi r^2 \approx 3.14 \times 10^2$ sq. in. The screen forms small squares; the area of each is $\frac{1}{11} \times \frac{1}{11} = (\frac{1}{11})^2$ sq. in. Hence the circle contains approximately $3.14 \times 10^2 \times 11^2 \approx 38{,}000$ small squares. Each of these has $\frac{2}{11}$ in. of wire in it. (We allow only two adjacent sides so as to prevent counting double from square to square.) Hence there are approximately

$$3.14 \times 10^2 \times 11^2 \times \tfrac{2}{11} = 3.14 \times 10^2 \times 22 \approx 6{,}900 \text{ inches of wire}.$$

7. Number the particles

$$1, 2, 3, 4, \ldots, \quad \tfrac{3}{2} \times 136 \times 2^{256},$$

and let i be the number of the particle we are trying to guess. We first find in which half of this set of numbers i lies by asking, say, "does i lie in the second half of this set?" Next we find in which half of this half, i.e. in which quarter of the original set, i lies. Proceeding in this way, the 264th question will tell us in which $(1/2)^{264}$ part of the original set i lies. Now we notice that

$$\frac{(3/2) \times 136 \times 2^{256}}{2^{264}} = \frac{(3/2) \times 17 \times 2^3 \times 2^{256}}{2^{264}} = \frac{3 \times 17}{2^6} = \frac{51}{64} < 1.$$

So after the 264th guess we have less than one particle left and we have found the particle.

9. We notice, by considering successive powers of 3, that the last digit, d, forms a repeating sequence 1, 3, 9, 7, 1, 3, 9, ... Consider any non-negative power of 3, say 3^m. We may write $m = 4n + r$ where $n = 0, 1, 2, \ldots,$ and $r = 0, 1, 2$ or 3. Then if $r = 0$, $d = 1$; if $r = 1$, $d = 3$; if $r = 2$, $d = 9$; and if $r = 3$, $d = 7$. For the number 3^{1001}, $m = 1001$, $n = 250$, $r = 1$ and so the last digit is 3. To find the 1st digit, we use logarithms.

$$\operatorname{Log}_{10} 3^{1001} = 1001 \log_{10} 3 = 1001 \times .47712 = 477.59 \ldots;$$

using anti-logarithms, we find that $3^{1001} = 303\ldots$, i.e., the first digit is 3.

Bibliography

(S): Contains material appropriate to students.
(T): Contains material appropriate to teachers.
(A): Contains advanced mathematical material.

HISTORY OF MATHEMATICS

Smith, D. E., *History of Mathematics*, Dover reprint, New York, 1958. Though not up to date, this is particularly good for elementary work. Contains many illustrations. (S,T)

van der Waerden, B. L., *Science Awakening*, Nordhoff, Gröningen, 1954. A history of mathematics in antiquity. An exquisite book. (S,T)

WEIGHTS AND MEASURES

Judson, L. V., *Units and Systems of Weights and Measures. Their Origin, Development, and Present Status*, National Bureau of Standards Circular 570, U. S. Government Printing Office, Washington 25, D. C., 1956. (S, T)

MATHEMATICAL FALLACIES

Maxwell, E. A., *Fallacies in Mathematics*, Cambridge University Press, Cambridge, 1959. A collection of fallacious arguments in algebra, geometry, trigonometry, and calculus. (S, T)

HISTORY OF π

Hobson, E., *Squaring the Circle*, Chelsea reprint, New York, 1958. The story to the end of the 19th Century. (S, T)

REPEATING DECIMALS

Rademacher, H. and Toeplitz, O., *The Enjoyment of Mathematics*, Princeton University Press, Princeton, 1947. Section 23. Selections from mathematics for the somewhat advanced amateur. (T, A) While many textbooks on algebra or higher algebra have a chapter on repeating decimals, Rademacher and Toeplitz go beyond the usual treatment.

IRRATIONAL NUMBERS, TRANSCENDENTAL NUMBERS, NORMAL NUMBERS

Niven, I., *Numbers: Rational and Irrational*, The New Mathematical Library, Vol. 1, 1961 (S, T)

Niven, I., *Irrational Numbers*, The Carus Mathematical Monographs, No. 11, 1956. (T, A)

GEOMETRIC CONSTRUCTIONS

Dickson, L. E., *First Course in the Theory of Equations*, Wiley, New York, 1921, Chapter III. (T, A)

Bieberbach, L., *Theorie der Geometrischen Konstruktionen*, Birkhäuser, Basel, 1952. (A)

ELECTRONIC COMPUTING MACHINERY

From among the many books on this topic, we select the following as being most suitable for this bibliography:

McCormick, E. M., *Digital Computer Primer*, McGraw-Hill, New York, 1959. (S, T)

THE THEORY OF NUMBERS

Uspensky, J. V. and Heaslet, M. A., *Elementary Number Theory*, McGraw-Hill, New York, 1939. (T)

Ore, O., *Number Theory and Its History*, McGraw-Hill, New York, 1948 (T)

PRIME NUMBERS

Lehmer, D. N., *List of Prime Numbers from 1 to 10,006,721*, Hafner reprint, New York, 1956. Contains an informative introduction on the history of the prime number problem. (S, T, A)

LARGE NUMBERS IN THE THEORY OF NUMBERS

Those interested will find much material in the back issues of the periodical *Mathematical Tables and Other Aids to Computation*, 1943– . This periodical is currently entitled, *Mathematics of Computation*. (S, T, A)

SYSTEMS OF SIMULTANEOUS LINEAR EQUATIONS

Though all books on algebra contain chapters devoted to this topic, the reader may be interested in seeing this material against the wider backdrop of computational problems.

Milne, W. E., *Numerical Calculus*, Princeton University Press, Princeton, 1949, Chapter I. (T, A)

Kunz, K. S., *Numerical Analysis*, McGraw-Hill, New York, 1957, Chapter 10. (T, A)

GROWTH IN MATHEMATICS

Sawyer, W. W., *What Is Calculus About?* The New Mathematical Library, Vol. 2, 1961. (S, T)

All texts on calculus contain pertinent material in their chapters on sequences, limits, and indeterminate forms. (T) For the ideas of P. Dubois-Reymond and for the "algebra" of growth see:

Hardy, G. H., *Orders of Infinity*, Cambridge University Press, Cambridge, 2nd edition, 1954. (A)

Borel, E., *Leçons sur la théorie de la croissance*, Paris, 1910. (A)

GROWTH AND FORM IN NATURE

Thompson, D. W., *Growth and Form*, Cambridge University Press, Cambridge, 2nd edition, 1954. A classic. The illustrations alone are fascinating and should be highly suggestive. (S, T)

Kovach, L. D., "Life Can Be So Nonlinear," *American Scientist*, June 1960, pp. 218–225. An entertaining and instructive little article. (S, T)

Apropos of the size of Gargantua (Section 12) the reader might like to look at the essay "On Being the Right Size" which appears in Haldane, J. B. S., *Possible Worlds*, New York, 1928. (S, T)

LARGE NUMBERS AND ASTRONOMY

Gamow, G., *1, 2, 3, Infinity*, Viking, New York, 1957. (S, T)

Shapley, H., *Of Stars and Men*, Washington Square Press, New York, 1958. (S, T)

GENERAL

James and James, *Mathematics Dictionary*, Princeton, 1959. Contains many interesting little essays as part of the definitions. (S, T, A)

Kasner, E. and Newman, J., *Mathematics and the Imagination*, Simon & Schuster, New York, 1940, 1945. Popular essays. (S, T)

Littlewood, J. E., *A Mathematician's Miscellany*, Methuen, London, 1953. Mathematical nuggets by one of England's leading mathematicians. (S, T, A)

Littlewood, J. E., "Large Numbers," *The Mathematical Gazette*, vol. 32 (1948), pp. 163–171. (S, T, A)

The Lore of Large Numbers

Large numbers have always been a source of wonder and many questions are raised about them. "What comes after billions?" or " How do the relative sizes of an atom and a man compare to the relative sizes of a man and the sun?" The author has answered some of these questions by explaining the arithmetic and the uses of large numbers in a way which introduces the reader to the horizons of modern mathematics. Using large num bers as a unifying theme and employing only the simplest materials, the author provides the reader with an understanding for numbers, their magnitude, and their growth. The reader is introduced to exponents, computation, number theory, and to the rapidity of growth of sequences. Several historical passages reveal mathematics as a living thing that grows and changes with the generations. Tables listing interesting and useful numbers in the physical universe are appended.

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