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Uses of Infinity

Leo Zippin

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USES OF INFINITY

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USES OF INFINITY

by

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USES OF INFINITY

Preface

Most of this book is designed so as to make little demand on the reader's technical competence in mathematics; he may be a high school student beginning his mathematics now or one who has put away and forgotten much of what he once knew. On the other hand, the book is mathematical except for the first chapter—that is to say, it is a carefully reasoned presentation of somewhat abstract ideas. The reader who finds the material interesting must be prepared, therefore, to work for it a little, usually by thinking things through for himself now and then and occasionally by doing some of the problems listed. Solutions to some of these are given at the end of the book. But it will not pay the reader to stop too long at any one place; many of the ideas are repeated later on, and he may find that a second view of them leads to understanding where a first view was baffling. This style of presentation is imposed upon an author by the nature of mathematics. It is not possible to say at once all of the key remarks which explain a mathematical idea.

Many a reader is perhaps wondering whether it is possible for fellow human beings to communicate upon a topic as remote-sounding as "uses of infinity"; but, as we shall see, any two people who know the whole numbers,

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad \cdots,$$

can talk to each other about "infinities" and have a great deal to say.

I have written this book from a point of view voiced in a remark by David Hilbert when he defined mathematics as "the science of infinity". An interesting theorem of mathematics differs from interesting results in other fields because over and above the surprise and beauty of what it says, it has "an aspect of eternity"; it is always part of an infinite chain of results. The following illustrates what I mean: the fact that $1 + 3 + 5 + 7 + 9$, the sum of the first five odd integers, is equal to 5 times 5 is an interesting oddity; but the theorem that *for all* n the sum of the first n odd integers is n^2 is mathematics.

I hope that the reader will believe me when I say that professional mathematicians do not profess to understand better than anybody else what, from a philosophical point of view, may be called "the meaning of infinity." This is proved, I think, by the fact that most mathematicians do not talk about this kind of question, and that those who do do not agree.

Finally, I wish to express my especial thanks to Mrs. Henrietta Mazen, a teacher of mathematics at the Bronx High School of Science, who selected and edited the material in this monograph from a larger body of material that I had prepared. The reader who enjoys this book should know that in this way a considerable role was played in it by Mrs. Mazen. I am also indebted to Miss Arlys Stritzel who supplied most of the solutions to the problems posed in the book.

Solutions to Problems

CHAPTER TWO

2.1 This is a non-terminating sequence of sets of musical compositions, the first set consisting of compositions for one voice part or instrument, the second set of pieces for two performers, the third of pieces for three performers, and so on.†

2.2 This is a periodic sequence of the four classes of hits in baseball. The iteration dots indicate that we are to repeat the same sequence of classes again and again.

2.3 Collections of siblings born on the same day make up the terms of this sequence. The first term is the collection of all individuals with one such sibling, the second is the set of individuals with two such siblings, etc. The terms which occur beyond a certain point in this infinite sequence are empty sets.

2.4 Here we have a list of the names of the days of the week. In this case the iteration dots represent an abbreviation for the days Saturday and Sunday.

2.5 This is a periodic sequence, the terms of which are the first letters in the names of the days of the week, in the order of the days, beginning with the letter M corresponding to Monday. The first term occurs again after six more terms, and from then on the entire period is repeated over and over.

2.6 The first of these three sets is the collection of all integers n of the form $n = 3q$ where q is an integer. It is easily seen that this set consists of the numbers $0, 3, 6, 9, \cdots$.

The second set is composed of all integers n of the form $n = 1 + 3q$. Since each such number has the remainder 1 when divided by 3, the numbers of the infinite sequence $1, 4, 7, 10, \cdots$ belong to the second set.

† Remark added by the author: I can see "quintet", "sextet", "septet", "octet", but there I get stuck. In the absence of a clear-cut rule as to just how to continue, I would agree with a student who called the question unclear and would count all answers correct.

The numbers in the third set are each of the form $2 + 3q$, where q is an integer, and hence each has remainder 2 when divided by 3. Thus, the third set has the elements $2, \; 5, \; 8, \; 11, \; \cdots$.

Inasmuch as every integer when divided by 3 has one and only one of the remainders $0, \; 1, \; 2,$ we know that these three nonoverlapping infinite sets together comprise the entire set of integers.

2.7 If q divides n, then $n = qb$ and $n + 1 = qb + 1$, where b is an integer. In other words, if n is divisible by q, $n + 1$ has the remainder 1 when divided by q, and hence n and $n + 1$ have no common factor.

2.8 The general principle which is suggested by an examination of these tables is that for every k by k multiplication table, where k is any positive whole number, the sum of the numbers in the table is the square of the sum of the first k positive integers. The sum of the integers in any lower gnomon-figure is the cube of the smallest integer in that gnomon.

2.9 $$\frac{1}{7} = 1\cdot 10^{-1} + 4\cdot 10^{-2} + 2\cdot 10^{-3} + 8\cdot 10^{-4} + 5\cdot 10^{-5} + 7\cdot 10^{-6} + \frac{1}{7}\cdot 10^{-7}$$

$$= .142857 + .000000142857 + \frac{1}{7}\cdot 10^{-14} = .142857142857 \cdots .$$

Similarly, $$\frac{1}{9} = 1\cdot 10^{-1} + \frac{1}{9}\cdot 10^{-2} = .111111 \; \cdots .$$

$$\frac{1}{11} = 0\cdot 10^{-1} + 9\cdot 10^{-2} + \frac{1}{11}\cdot 10^{-3} = .090909 \; \cdots .$$

$$\frac{1}{99} = 0\cdot 10^{-1} + 1\cdot 10^{-2} + \frac{1}{99}\cdot 10^{-3} = .010101 \; \cdots .$$

Every terminating decimal may be written in the form $a/10^k$ where a is an integer; for example,

$$3.572 = \frac{3572}{10^3}.$$

If a contains factors 2^s and/or 5^t with $0 < s, \; t \leq k$, then $a/10^k$ is not in lowest terms and may be reduced as follows:

$$\frac{a}{10^k} = \frac{2^s\cdot 5^t\cdot b}{2^k\cdot 5^k} = \frac{b}{2^{k-s}\cdot 5^{k-t}} = \frac{b}{2^n\cdot 5^m}.$$

2.10 The symbols $0.9090909090 \cdots$ and $0.0909090909 \cdots$ are the non-terminating decimal representations of $\frac{10}{11}$ and $\frac{1}{11}$, respectively. When we "add" these expressions the resulting symbol is $.9999999999 \cdots$, whereas the sum of $\frac{10}{11}$ and $\frac{1}{11}$ is $\frac{11}{11} = 1$.

2.11 Since

$$n \div m = n \times \frac{1}{m},$$

the method is clear. For example, to divide 7 by 9 we look up the reciprocal of 9 in the table of Figure 2.4 and write

$$7 \div 9 = 7 \times \frac{1}{9} = 7 \times .111 \cdots = .777 \cdots.$$

2.12 Construct the circle of radius PQ with center Q; see Figure 2.7. Using P as center, and the same radius, swing the compass to find the points P_1 and Q_1 of intersection of this circle with the first circle.

Next, open the compass to the width P_1Q_1 ; draw circles with center P_1 and Q_1 respectively. One of their intersections (the one to the right of P and Q) is the desired point R.

To see that P, Q, R are collinear and $PQ = QR = d$, observe that P_1PQ and Q_1PQ are two equilateral triangles with common base PQ which is bisected by the segment P_1Q_1 in the point M (see Figure 2.7), and that P_1Q_1R is an equilateral triangle with base P_1Q_1 and whose altitude MR lies on the line through P and Q.

Moreover, if $PQ = d$, then

$$P_1Q_1 = 2P_1M = d\sqrt{3} \quad \text{and} \quad MR = \frac{1}{2}P_1Q_1\sqrt{3} = \frac{3d}{2};$$

$$QR = MR - MQ = \frac{3d}{2} - \frac{d}{2} = d.$$

This method shows us how to construct an infinite sequence of points on a line. Simply pick two points, call them P and Q, get R, and repeat the above construction on Q and R, getting R', etc.

2.13 Suppose P is to the left of Q. Line up the ruler with P and Q so that its right end is at Q and make a mark on the ruler at the place where P falls. This mark divides the ruler into two parts, one of length $PQ = d$ on the right, and one of smaller length d' on the left. After drawing the segment PQ, line up the ruler with the segment PQ so that the division mark on the ruler falls on Q. Then the right end of the ruler will be at a distance d from Q, on PQ extended. Denote the endpoint of this extension by Q_1 . Next move the ruler in the direction from Q to Q_1 along the line until the division mark is on the new point Q_1 . The right endpoint of the ruler will be at a point Q_2 on the line through P and Q, at a distance $2d$ from Q. Continue this process indefinitely in order to extend the line through P and Q indefinitely to the right.

In order to extend the line through P and Q to the left, we just reflect the method just described.

If d' were longer than d, the same method would work but it would be more economical to interchange the roles of d and d'.

2.14 Once the direction of the road and the point at which it is to enter the mountain are determined, it is only necessary to line up every three consecutive guide-posts. This can be checked at each advance.

2.15 The successive midpoints approach the point which is at a distance from A equal to $\frac{2}{3}$ of the length of AB.

2.20 (a) Divide a segment into 5 equal parts, hold one part, and give 3 pieces away. Then one part remains and we hold $\frac{1}{4}$ of the amount which has been distributed. If we repeat the same process over and over, each time dividing the one remaining part into 5 equal pieces, we shall continue to hold $\frac{1}{4}$ of the total amount distributed and a smaller and smaller amount of the original segment will remain to be distributed.

Thus

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots = \frac{1}{4}.$$

Similarly,

(b) $$\frac{1}{8} + \frac{1}{64} + \frac{1}{512} + \cdots = \frac{1}{7};$$

(c) $$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots = \frac{1}{9}.$$

2.21 Divide the segment into n equal parts, hold m of them, leave m of them to work on and give away the remaining $n - 2m$ parts. Thus $m + (n - 2m) = n - m$ parts have been distributed, and we hold m parts; hence we hold $m/(n - m)$ of the distributed amount.

We treat the remaining part in the same way, dividing it into n equal pieces, holding m, giving away $n - 2m$ and keeping m to be worked on. Continuing in this way, we shall always hold $m/(n - m)$ of the distributed part while the part to be worked on gets smaller and smaller.

2.22 It follows from $4 = 2 \cdot 2$ that $\sqrt{4} = 2$, so that $\sqrt{4}$ is rational. To prove that $\sqrt{3}$ and $\sqrt{5}$ are irrational, we need only consider the following:

(a) If we assume that $\sqrt{3} = p/q$, where p/q is that fraction (among all equivalent fractions) which has the smallest denominator, then we have

$$1 < \frac{p}{q} < 2,$$
$$q < p < 2q,$$
$$0 < p - q < q,$$

and

$$3q^2 = p^2,$$
$$3q^2 - pq = p^2 - pq,$$
$$q(3q - p) = p(p - q),$$

which implies, in contradiction to our initial assumption, that

$$\frac{p}{q} = \frac{3q - p}{p - q}.$$

(b) Assume that $p/q = \sqrt{5}$, p/q the fraction with smallest denominator. Then $p = \sqrt{5}q$ and $2 < p/q < 3$ give us

$$2q < p < 3q,$$
$$0 < p - 2q < q,$$

and

$$p^2 = 5q^2,$$
$$p^2 - 2pq = 5q^2 - 2pq,$$
$$p(p - 2q) = q(5q - 2p),$$

so that

$$\frac{p}{q} = \frac{5q - 2p}{p - 2q},$$

where $p - 2q < q$.

2.23 From $p = \sqrt{7}q$ and $2 < p/q < 3$ we get

$$2q < p < 3q,$$
$$0 < p - 2q < q;$$
$$p^2 = 7q^2,$$
$$p^2 - 2pq = 7q^2 - 2pq,$$
$$p(p - 2q) = q(7q - 2p).$$

Then

$$\frac{p}{q} = \frac{7q - 2p}{p - 2q}$$

contradicts the assumption that $\sqrt{7}$ can be represented as a ratio of two integers, p and q, where q is the smallest possible denominator.

2.24 $\sqrt{8} = \sqrt{4\cdot 2} = \sqrt{4}\cdot\sqrt{2} = 2\sqrt{2}$. If $\sqrt{8} = p/q$, then $2\sqrt{2} = p/q$, or $\sqrt{2} = p/2q$ (a contradiction since $\sqrt{2}$ is irrational).

2.25 $p = \sqrt{n}q$ and $k^2 < n < (k+1)^2$ imply that

$$k^2 < \left(\frac{p}{q}\right)^2 < (k + 1)^2,$$
$$k < \frac{p}{q} < k + 1,$$
$$kq < p < (k + 1)q,$$
$$0 < p - kq < q.$$

Since $p^2 = nq^2$,

$$p^2 - kpq = nq^2 - kpq,$$
$$p(p - kq) = q(nq - kp),$$
$$\frac{p}{q} = \frac{nq - kp}{p - kq} \quad \text{where } p - kq < q.$$

If there were a fraction whose square is the integer N, we would write it with as small a denominator as possible, say $p/q = \sqrt{N}$, and $q \neq 1$ by assumption, so $q^2 \neq 1$. Hence the fraction p^2/q^2 would lie between consecutive integers k and $k + 1$ and we can produce the above contradiction.

2.26 If $\sqrt{s}$ is not an integer, then, by problem 2.25, s is not the square of a fraction. But if $2\sqrt{s} = n$, where n is an integer, then s is equal to the fraction $(n/2)^2$. Hence, unless $\sqrt{s}$ is an integer, we are involved in a contradiction.

2.27 To prove that $12.5 \times a = 100a/8$ we need only note that

$$12.5 = 12 + \frac{1}{2} = \frac{24 + 1}{2} = \frac{25}{2} = \frac{100}{8}.$$

2.28 Let a be any integer with digits $a_k a_{k-1} \cdots a_0$. Then we may write

$$\begin{aligned} a &= a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_1 \cdot 10 + a_0 \\ &= a_k(10^k - 1 + 1) + a_{k-1}(10^{k-1} - 1 + 1) + \cdots + a_1(10 - 1 + 1) + a_0 \\ &= a_k 99\cdots9 + a_k + a_{k-1}99\cdots9 + a_{k-1} + \cdots + a_1 \cdot 9 + a_1 + a_0 \\ &= 3[a_k(33\cdots3) + a_{k-1}(33\cdots3) + \cdots + a_1 \cdot 3] \\ &\qquad + a_k + a_{k-1} + a_{k-2} + \cdots + a_1 + a_0 . \end{aligned}$$

Since the first expression on the right is a multiple of 3, a has the same remainder upon division by 3 as the second term on the right, i.e. the sum of the digits of a. Another way of saying this is that a and the sum of its digits belong to the same residue class (mod 3).

If $a_k + a_{k-1} + \cdots + a_0 < 10$, the proof is complete. If not, let

$$a_k + a_{k-1} + \cdots + a_0 = b = b_j 10^j + b_{j-1}10^{j-1} + \cdots + b_1 \cdot 10 + b_0 ,$$

and proceed in the same manner as above; eventually the sum of digits will be less than 10, and we shall have reached the root number $r(a)$. This shows that a and the sum b of its digits and the sum c of the digits of b etc. down to $r(a)$ are all in the same residue class modulo 3.

2.29 (a) and (b) Let us consider the infinite sequence of decimals

$$\begin{aligned} a_1 &= .1111\cdots, \\ a_2 &= .101010\cdots, \\ a_3 &= .100100100\cdots, \\ a_4 &= .1000100010001\cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ a_k &= .1000\cdots1000\cdots1000\cdots, \end{aligned}$$

and note that the period of a_k is k.

(c) .1010010001000010000010000001$\cdots$.

CHAPTER THREE

3.1 The proof of this theorem for the case $AB':B'B = m:n$, where m and n are positive integers, is essentially the same as the proof given in Chapter 6, Section 6.2(a). For $m = \sqrt{2}$, $n = 1$, the theorem can be proved by the methods used in Sections 6.2(b) and 6.2(c) for the incommensurable case.

3.2 Such a rectangle does not exist because it would lead to the equation $1 = 0 \cdot x$, where x is the length of the other side. But this contradicts the rule of arithmetic: $x \cdot 0 = 0$ for all x whatsoever.

3.3 Construct the right triangle ADC (see Figure 3.6) with legs of lengths x and 1. Draw the perpendicular to the hypotenuse AC through C. Extend line AD to meet this perpendicular at B. The segment DB then has the desired length $y = 1/x$ because the length of the altitude CD of right triangle ACB is the mean proportional between the lengths of $AD = x$ and $DB = y$:

$$\frac{y}{1} = \frac{1}{x}.$$

3.4 The parabola $y = x^2$ is the locus of points (x, y) such that, for every abscissa x, the ordinate y satisfies the relation

$$\frac{y}{x} = \frac{x}{1}.$$

This relation suggests again the construction of a right triangle so that the altitude to the hypotenuse has length x and divides the hypotenuse into segments of lengths 1 and y. In Figure 3.7 this construction was carried out for two given values of x, x_1 and x_2, and the corresponding values of y are y_1 and y_2. (The details of this construction are left to the reader.)

The parabola shown in Figure 3.8 may now be plotted either by using values (x, y) obtained from the construction of Figure 3.7 as coordinates for points of the graph, or it may be plotted directly by carrying out the construction in the coordinate plane as indicated: For each abscissa x, find the point Z: $(x, -1)$, connect it to the origin O, draw a perpendicular to the resulting segment at O and locate the point W: (x, y) at which this perpendicular intersects the vertical line x units away from the y-axis. In the right triangle OWZ the altitude to the hypotenuse has length x and divides ZW into segments of length 1 and y so that the relation

$$\frac{1}{x} = \frac{x}{y} \quad \text{or} \quad y = x^2$$

is satisfied for each W so constructed.

3.5 If x is a given length then $\sqrt{x}$ can always be constructed by virtue of the relation

$$\frac{1}{\sqrt{x}} = \frac{\sqrt{x}}{x}.$$

For example, in Figure 3.7, let $AE = 1$ as before, extend AE to B so that EB has the given length x, and draw the semicircle with AB as diameter. The perpendicular to AB at E will intersect this semicircle at a point O, and EO, the altitude of right triangle AOB, will have length $\sqrt{x}$.

The length $\sqrt{x}$ can also be read off the parabola of Figure 3.8: just find a point whose distance from the horizontal axis is x; its distance from the vertical axis is $\sqrt{x}$.

3.6 To find approximately the cube root $\sqrt[3]{a}$ of a number a one determines the point of intersection P of the graph of $y = x^3$ with the horizontal line $y = a$ and measures the abscissa of P. The coordinates of P are $(\sqrt[3]{a}, a)$.

3.8 A careful construction will show that as n gets larger and larger, the quantity $\sqrt{n+1} - \sqrt{n}$ becomes smaller and smaller.

3.9 Let $a = \sqrt{n+1}$, $b = \sqrt{n}$. Then

$$(\sqrt{n+1} + \sqrt{n})\cdot(\sqrt{n+1} - \sqrt{n}) = 1,$$

or

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Clearly, as n gets larger and larger the denominator $\sqrt{n+1} + \sqrt{n}$ increases, and it follows from the above identity that the quantity $\sqrt{n+1} - \sqrt{n}$ becomes smaller and smaller; that is, the difference between $\sqrt{n+1}$ and $\sqrt{n}$ can be made as small as we wish (it is always greater than 0) by taking n large enough.

3.10 We multiply the expression by

$$\frac{\sqrt{2n+1} + \sqrt{2n}}{\sqrt{2n+1} + \sqrt{2n}}$$

and obtain

$$\sqrt{n}\,\frac{2n+1-2n}{\sqrt{2n+1} + \sqrt{2n}} = \frac{\sqrt{n}}{\sqrt{2n+1} + \sqrt{2n}} = \frac{1}{\sqrt{2 + \frac{1}{n}} + \sqrt{2}}.$$

As n gets larger and larger, $1/n$ becomes smaller and smaller so that this expression approaches

$$\frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

3.11 If $0 < y < \frac{1}{2}\pi$, then (see Figure 3.15), $\sin y < y < \tan y$; dividing by $\tan y$, we obtain

$$\cos y < \frac{y}{\tan y} < 1.$$

As y decreases, $\cos y$ approaches 1 so that $y/\tan y$ is squeezed between 1 and a number close to 1.

3.12 (a) PROOF OF THEOREM 3.1. Let S be the sequence $x_1, x_2, x_3, \cdots$ having the limit L, and suppose that $y_1, y_2, y_3, \cdots$ is any subsequence S' of S (i.e., S' is an infinite sequence whose terms are some or all of the terms of S, arranged in the order in which they occur in S).

Now S has the limit L means that the sequence $(x_1 - L)$, $(x_2 - L)$, $(x_3 - L)$, $\cdots$ approaches 0; that is, for every integer n, there are at most a finite number (depending on n) of terms $x_k - L$, $k = 1, 2, 3, \cdots$, which are numerically larger than $1/n$. But if S' is a subsequence of S, then every term of the sequence $(y_1 - L)$, $(y_2 - L)$, $\cdots$ is identical to some term $x_k - L$, so that, for every integer n, there can be at most a finite number of terms $y_{k'} - L$, $k' = 1, 2, 3, \cdots$, numerically larger than $1/n$.

Thus, every subsequence of an infinite sequence with limit L also has the limit L.

(b) PROOF OF THEOREM 3.2. $a + x - (a + L) = x - L$, so that if $(x_1 - L)$, $(x_2 - L)$, $(x_3 - L)$, $\cdots$ approaches zero, then $[a + x_1 - (a + L)]$, $[a + x_2 - (a + L)]$, $\cdots$ also approaches zero; therefore $a + x_1$, $a + x_2$, $a + x_3$, $\cdots$ has the limit $a + L$.

(c) PROOF OF THEOREM 3.3. We wish to show that, for every integer m, all but a finite number of terms kx_i satisfy

$$|kx_i - kL| < \frac{1}{m}$$

provided that the sequence x_i has L as limit. In other words we know that for every n, all but a finite number of x_i satisfy

$$|x_i - L| < \frac{1}{n}.$$

In particular, take n to be the nearest integer greater than or equal to $|k|m$. Then

$$|kx_i - kL| = |k||x_i - L| < |k|\frac{1}{n} \leq |k|\frac{1}{|k|m} = \frac{1}{m}$$

for all but a finite number of terms. This argument holds for every integer m.

(d) We prove first: if $x_1, x_2, x_3, \cdots$ has the limit L and $y_1, y_2, y_3, \cdots$ has the limit M, then $x_1y_1, x_2y_2, x_3y_3, \cdots$ has the limit LM.

Following the hint, we write

$$\begin{aligned} x_i y_i - LM &= x_i y_i - x_i M + x_i M - LM \\ &= x_i(y_i - M) + M(x_i - L). \end{aligned}$$

By assumption, for every integer n,

$$|x_i - L| < \frac{1}{n} \quad \text{and} \quad |y_i - M| < \frac{1}{n}$$

for all but a finite number of x_i, y_i. Moreover, since the x_i have a limit, all but a finite number are certainly bounded by some constant, say C. Thus

$$\begin{aligned} |x_i y_i - LM| &\leq |x_i||y_i - M| + M|x_i - L| \\ &\leq C\frac{1}{n} + M\frac{1}{n} = (C + M)\frac{1}{n}. \end{aligned}$$

Now, given any integer m, we can achieve

$$|x_i y_i - LM| \leq \frac{1}{m}$$

merely by choosing the integer n so that

$$\frac{C + M}{n} < \frac{1}{m},$$

thus establishing, for every integer m,

$$|x_i y_i - LM| < \frac{1}{m}$$

for all but a finite number of terms $x_i y_i$.

To prove Theorem 3.4, that the limit of x_1^2, x_2^2, $\cdots$ is L^2, use the above result with $y_i = x_i$ and $L = M$. To prove that the limit of x_1^k, x_2^k, $\cdots$ is L^k, we simply repeat the argument $k - 1$ times.

3.13 (a) This problem is not at all easy. Take the case $0 < r < 1$. One standard proof runs as follows:

Since $1/r > 1$, $1/r = 1 + h$, $h > 0$. Now

$$\left(\frac{1}{r}\right)^n = (1 + h)^n > 1 + nh$$

for every n, and so $(1 + h)^n$ increases without limit as n increases. Therefore the reciprocal, r^n, goes to zero.

The assertion that $(1 + h)^n$ increases without limit as $n \to \infty$ can also be proved by appeal to the Bolzano-Weierstrass principle described in the next section. For if L is any number such that

$$(1 + h)^n \leq L$$

for all n, then

$$(1 + h)^{n-1} \leq \frac{L}{1 + h} = L' < L$$

for all n; and so, given *any* upper bound L to this sequence, there would exist a *smaller* upper bound L'. Such a sequence is incompatible with the Bolzano-Weierstrass principle.

Let us suppose now that $r = 1$. Then $r^2 = 1$, $r^3 = 1$, $r^4 = 1$, $\cdots$ and each number of the sequence $(r - 1)$, $(r^2 - 1)$, $\cdots$ equals 0. Since no term of this sequence exceeds $1/n$, $n > 0$, the sequence approaches 0 and r, r^2, r^3, $\cdots$ has the limit 1 when $r = 1$.

(b) From the identity

$$a(1 - x)(1 + x + x^2 + \cdots + x^{n-1}) = a(1 - x^n)$$

we get

$$a \cdot 1 + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r};$$

then, since r^n approaches 0 if $|r| < 1$, we have

$$\lim_{n \to \infty} \frac{1 - r^n}{1 - r} = \frac{1}{1 - r}, \qquad -1 < r < 1.$$

Hence, the sum of any infinite geometric series $a + ar + ar^2 + \cdots$ with ratio r numerically smaller than 1 is $a/(1 - r)$.

3.14 The sequence of rational numbers constructed in the text, in decimal form, is

1. 0 000000000···,
.5 0 00000000···,
2.00 0 0000000···,
.333 3 333333···,
3.0000 0 00000···,
.25000 0 0000···,
.666666 6 666···,
1.5000000 0 00···,
4.00000000 0 0···,
.200000000 0 ···,
············.

We shall construct a number whose kth decimal is always one or the other of the digits 2 or 3 but differs from the kth decimal of the kth number in this list.

Let us choose 3 as its first decimal since 3 is different from the first decimal of the first number. Let us choose 3 for the second decimal of the number we are constructing because 3 is different from the second decimal of the second number in the list. Similarly, let us choose $3(\neq 0)$ for the third decimal, $2(\neq 3)$ for the fourth decimal, $3(\neq 0)$ for the fifth, $3(\neq 0)$ for the sixth, $3(\neq 6)$ for the seventh, etc., until we come to a kth number with 3 in the kth place; then we will choose 2.

3.15 Before classifying the letters of the alphabet, we shall consider, for the sake of definiteness, the letter T. We shall simplify the situation by considering an uncountable set of identical (i.e. congruent) letters T (e.g. with a stem of one inch and a top bar of one inch) whose vertices are labelled A, B, C, see Figure 3.18(a).

On any piece of paper of finite dimensions, an uncountable set of such T's must contain an uncountable subset of T's such that the vertices labelled A are within $\frac{1}{8}$, say, inch of each other. In this set, there is an uncountable subset such that the vertices labelled B are within $\frac{1}{8}$ inch of each other. And in this set, there is an uncountable subset such that the vertices labelled C are within $\frac{1}{8}$ inch of each other.

Now then, let ABC and $A'B'C'$ [see Figure 3.18(b)] be two T's of the kind described. They cross. This can be proved by elementary geometry from the fact that a straight line divides the plane into two regions (called the two *sides* of the line), and segments connecting points on opposite sides of the line must cross the line. This establishes the impossibility of writing an uncountable number of congruent non-crossing T's on a page. Note that if we merely required the distances AA' and BB' to be small, then the T's could possibly stand as they do in Figure 3.19, and not cross. We shall not treat the case of T's of varying sizes here, but the same result can be proved.

All letters of the alphabet that contain a configuration such as we encountered in the letter T (i.e. an intersection of two segments or curves where at least one of the segments extends beyond the point of intersection) are in the same class as T. They are A, B, E, F, H, K, P, Q, R, T, X, Y.

For all other letters, it is possible to scribble an uncountable set of them on a page. Figures 3.20(a), (b) illustrate this fact for the letters L, O respectively. In each case, the fact that a line segment contains an uncountable number of points gives the clue.

CHAPTER FOUR

4.1 The phrase "if this limit exists" has been omitted. The statement $L = \lim_{n\to\infty} L_n$ makes sense only if the L_n have a limit, and in this case it asserts that L is the value of this limit.

4.2 The direct computation of the lengths S_1, S_2, $\cdots$ of the sides of equilateral triangles whose bases are on the x-axis and whose vertices lie on the curve $y = x^2$ is somewhat awkward; fortunately the question posed in the problem can easily be answered without such a computation: The length of the resulting zig-zag is again 2 because, as in Example 2, it is twice as long as the distance from (1, 0) to the origin.

4.3 We calculate the distances

$$B_1T = 1, \quad B_3T = \frac{1}{2}, \quad B_5T = \frac{1}{4}, \quad \cdots, \quad B_{2n+1}T = \frac{1}{2^n}, \quad \cdots$$

which approach zero. Hence T is the limit of the sequence

$$B_1, \quad B_3, \quad \cdots \quad B_{2n+1}, \quad \cdots .$$

The distances $B_{2n}B_{2n-1}$ can be represented as sides of equilateral triangles of lengths $1/2^{n-1}$ and so these distances also approach 0.

By virtue of the triangle inequality, we have the following relations between lengths:

$$B_{2n}T \leq B_{2n}B_{2n-1} + B_{2n-1}T.$$

As n increases each term on the right approaches zero (by what we showed above) and hence their sum approaches zero. Therefore T is also the limit of points with even subscripts.

4.4 The point Z: $(\sqrt{2}, \sqrt{2}/3)$ is the limit of the sequence of points D_1, D_2, $\cdots$. The abscissas of D_1, D_2, $\cdots$ are

$$x_1 = \frac{\sqrt{2}}{2}, \quad x_2 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4}, \quad x_3 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{8}, \cdots,$$

$$x_n = \frac{\sqrt{2}}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right), \cdots$$

When these finite geometric progressions are summed, they have the form

$$x_1 = \sqrt{2}\left(1 - \frac{1}{2}\right), \quad x_2 = \sqrt{2}\left(1 - \frac{1}{4}\right),$$

$$x_3 = \sqrt{2}\left(1 - \frac{1}{8}\right), \cdots, \quad x_n = \sqrt{2}\left[1 - \left(\frac{1}{2}\right)^n\right], \cdots;$$

these numbers come arbitrarily close to $\sqrt{2}$ since the sequence $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\cdots$ approaches zero. This shows that $\lim_{n\to\infty} x_n = \sqrt{2}$, and this is the meaning of the phrase "the abscissa of Z is the limit of the abscissas of D_1, D_2, $\cdots$".

To prove that the ordinates

$$y_1 = \frac{\sqrt{2}}{2}, \quad y_2 = \frac{\sqrt{2}}{2}\left(1 - \frac{1}{2}\right), \quad y_3 = \frac{\sqrt{2}}{2}\left(1 - \frac{1}{2} + \frac{1}{4}\right),$$

$$\cdots, \quad y_n = \frac{\sqrt{2}}{2}\left[1 - \frac{1}{2} + \frac{1}{4} - \cdots + \left(-\frac{1}{2}\right)^{n-1}\right], \cdots$$

of D_1, D_2, $\cdots$ have the limit $\sqrt{2}/3$, we sum these finite geometric series and find that

$$y_n = \frac{\sqrt{2}}{2}\left[\frac{1 - (-\frac{1}{2})^n}{1 + \frac{1}{2}}\right] = \frac{\sqrt{2}}{3}\left[1 - \left(-\frac{1}{2}\right)^n\right].$$

As $n \to \infty$, $(-\frac{1}{2})^n$ approaches zero so that the y_n have the limit $\sqrt{2}/3$.

From the fact that the abscissas have the limit $\sqrt{2}$ and the ordinates have the limit $\sqrt{2}/3$, we can prove that the sequence D_1, D_2, $\cdots$ has the limit Z: $(\sqrt{2}, \sqrt{2}/3)$ by the Pythagorean Theorem. We express the distance D_nZ by

$$(D_nZ)^2 = (\sqrt{2} - x_n)^2 + \left(\frac{\sqrt{2}}{3} - y_n\right)^2.$$

The terms on the right approach zero as $n \to \infty$, hence their squares approach zero, and so does the sum of their squares. Therefore the distances D_nZ approach zero and Z is indeed the limit of the sequence D_1, D_2, $\cdots$.

4.5 Denote the abscissas and ordinates of $D_1', D_2', \cdots$ by $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$ respectively. It is clear from the construction of Example 4′ and our knowledge of Example 4 that

$$x_1 = \frac{\sqrt{2}}{2}, \quad x_2 = \frac{\sqrt{2}}{2}\left(1 + \frac{1}{2}\right), \quad x_3 = \frac{\sqrt{2}}{2}\left(1 + \frac{1}{2} - \frac{1}{4}\right), \quad \cdots,$$

$$\begin{aligned} x_n &= \frac{\sqrt{2}}{2}\left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{8} + \cdots \pm \frac{1}{2^{n-1}}\right) \\ &= \frac{\sqrt{2}}{2}\left[\left(1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots \pm \frac{1}{2^{n-2}}\right)\right. \\ &\qquad \left. + \frac{1}{2}\left(1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots \pm \frac{1}{2^{n-2}}\right)\right] \\ &= \frac{\sqrt{2}}{2}\left[\frac{1 - (-\frac{1}{4})^{n/2}}{\frac{5}{4}} + \frac{1 - (-\frac{1}{4})^{n/2}}{2\cdot(\frac{5}{4})}\right] \\ &= \frac{3\sqrt{2}}{5}\left[1 - \left(-\frac{1}{4}\right)^{n/2}\right], \qquad \text{for } n \geq 2,\ n \text{ even,} \end{aligned}$$

and

$$x_{n+1} = x_n \pm \frac{\sqrt{2}}{2}\left(\frac{1}{4}\right)^{n/2}, \qquad \text{so that} \qquad \lim_{n\to\infty} x_n = \frac{3\sqrt{2}}{5},$$

and

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} \frac{\sqrt{2}}{2}\left(\frac{1}{4}\right)^{n/2} = \frac{3\sqrt{2}}{5}.$$

For the ordinates, we have

$$y_1 = \frac{\sqrt{2}}{2}, \quad y_2 = \frac{\sqrt{2}}{2}\left(1 - \frac{1}{2}\right), \quad y_3 = \frac{\sqrt{2}}{2}\left(1 - \frac{1}{2} - \frac{1}{4}\right), \cdots,$$

$$\begin{aligned} y_n &= \frac{\sqrt{2}}{2}\left(1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \cdots \pm \frac{1}{2^{n-1}}\right) \\ &= \frac{\sqrt{2}}{2}\left[\left(1 - \frac{1}{4} + \frac{1}{16} - \cdots \pm \frac{1}{2^{n-2}}\right)\right. \\ &\qquad \left. - \frac{1}{2}\left(1 - \frac{1}{4} + \frac{1}{16} - \cdots \pm \frac{1}{2^{n-2}}\right)\right] \\ &= \frac{\sqrt{2}}{2}\left[\frac{1 - (-\frac{1}{4})^{n/2}}{\frac{5}{4}} - \frac{1 - (-\frac{1}{4})^{n/2}}{2\cdot(\frac{5}{4})}\right] \\ &= \frac{\sqrt{2}}{5}\left[1 - \left(-\frac{1}{4}\right)^{n/2}\right], \qquad \text{for } n \geq 2,\ n \text{ even,} \end{aligned}$$

and

$$y_{n+1} = y_n \pm \frac{\sqrt{2}}{2}\left(\frac{1}{4}\right)^{n/2},$$

so that

$$\lim_{n\to\infty} y_n = \frac{\sqrt{2}}{5} = \lim_{n\to\infty} y_{n+1}.$$

As we have seen in the solution to Problem 4.4, this implies that the point $(3\sqrt{2}/5, \ \sqrt{2}/5)$ is the limit point of the sequence D_1', D_2', $\cdots$.

4.6 Project each point E_1, E_2, E_3, and so on, perpendicularly onto the y-axis (this is what we do, essentially, when we calculate the ordinate of a point). Call these points F_1, F_2, F_3, and so on. Now if we let f_n denote the ordinate of E_n, then f_n is also the length of OF_n. Notice that the sequence of numbers f_2, f_4, f_6, $\cdots$ is constantly increasing and is bounded above. By the Bolzano-Weierstrass principle this sequence has a limit f^*. We shall show in a moment, but the reader may prefer to prove it himself, that f^* is also the limit of the sequence of odd-numbered f's, f_1, f_3, f_5, $\cdots$, which approach it from above. Thus the entire sequence has f^* as a limit, but the approach to this limit is two-sided. To f^* there corresponds a point F^* (on the y-axis) which ought to be the projection of the limit of the points E_1, E_2, E_3, $\cdots$; but as we have seen, the points E_1, E_2, $\cdots$ have no limit.

A proof that the sequence of odd-numbered f's has f^* as a limit follows. For every n,

$$f_{2n-1} - f^* = (f_{2n-1} - f_{2n}) + (f_{2n} - f^*),$$

the second term in parentheses being negative (see Figure 4.10). The first term on the right is precisely $\sqrt{2}/(2n)$. We have no formula for the second term; *but* since the sequence f_{2n} $(n = 1, 2, 3, \cdots)$ converges to f^*, we know that all such terms are small when n is large enough, by the very definition of limit. Thus we can be sure that when n is large enough, the right hand side is the difference of two small numbers and is small. This shows that f_{2n-1} is near to f^* (for large n) and concludes the proof.

4.7 Example 5 showed (see solution to previous problem) that a sequence of points in the plane may have no limit point although the sequence of their projections has a limit point. Assertion (a) would be correct if one added "if the given sequence of points has a limit"; we shall demonstrate this in a moment.

Assertion (b) is true. Let Q_n be the points in the given sequence, Q its limit point, P_n the projections of Q_n, and P the projection of Q. Then $P_nP = Q_nQ \cos\alpha_n$ where α_n is the angle between the segment Q_nQ and the line which carries the projections. Since $|\cos\alpha_n| \leq 1$ it follows that $|P_nP| \leq |Q_nQ|$ and since $\lim_{n\to\infty} Q_nQ = 0$, it follows that $\lim_{n\to\infty} |P_nP| = 0$ so that P, the projection of Q, is indeed the limit point of the sequence P_n.

This also proves that the limit of the projections is the projection of the limit of a sequence of points, provided they have a limit. If P^* is the limit of the projections, P the projection of the limit Q, then $\lim_{n\to\infty} P_nP^* = 0$ and $\lim_{n\to\infty} P_nP = 0$ imply that P and P^* coincide.

4.8 Let

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

Since for $k > 1$ we have $k > k - 1$, it follows that

$$\frac{1}{k} < \frac{1}{k-1} \quad \text{and} \quad \frac{1}{k^2} < \frac{1}{k(k-1)} \qquad \text{for all } k > 1.$$

Therefore

$$(1) \qquad S_n < 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n}.$$

Now we observe that

$$\frac{1}{(k-1)k} = \frac{1}{k-1} - \frac{1}{k} \qquad \text{for } k = 2, 3, \cdots$$

and re-write the right member of (1) in the form

$$S_n < 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= 2 - \frac{1}{n}.$$

This proves that $S_n < 2$, for $n = 1, 2, \cdots$.

4.9 By Pythagoras' theorem, we may express any segment OP_k in terms of the previous one:

$$(1) \qquad OP_k^2 = OP_{k-1}^2 + \frac{1}{(k-1)^2}.$$

Next, we express OP_{k-1} in terms of the previous one and substitute in (1):

$$OP_k^2 = OP_{k-2}^2 + \frac{1}{(k-2)^2} + \frac{1}{(k-1)^2}.$$

Continuing in this manner we find that

$$OP_k^2 = OP_1^2 + 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(k-1)^2},$$

and, since $OP_1 = 1$, we have (in the notation used in the solution to the previous problem)

$$OP_k^2 = 1 + S_{k-1}.$$

We have seen that $S_n < 2$ for all n. Hence $OP_n^2 < 1 + 2 = 3$ and

$$(2) \qquad OP_n < \sqrt{3} \qquad \text{for all } n.$$

The length of the zig-zag $P_1P_2\cdots P_n$ is

$$L_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

and we have already seen (page 65) that the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ has no limit.

To prove that the projections Q_k (on a circle of radius 3 and center O) of our points P_k wind around indefinitely often, it suffices to show that the sum of the angles α_n between OP_n and OP_{n+1} becomes arbitrarily large. To see this, consider

$$\sin \alpha_n = \frac{\frac{1}{n}}{OP_{n+1}} = \frac{1}{n\, OP_{n+1}}.$$

By our result (2), we see that

$$\frac{1}{n\, OP_{n+1}} > \frac{1}{n\sqrt{3}}, \qquad \text{for } n = 1, 2, \cdots;$$

Moreover, for any acute angle α, we have $\sin\alpha < \alpha$ (see Figure 3.15). Thus

$$\alpha_1 + \alpha_2 + \cdots > \sin \alpha_1 + \sin \alpha_2 + \cdots > \frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{3}} + \cdots$$

$$= \frac{1}{\sqrt{3}}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right),$$

so that the sum of the angles exceeds the harmonic series (multiplied by the constant factor $1/\sqrt{3}$) and hence is infinite.

The sequence $P_1, P_2, \cdots$ clearly cannot have a limit point for, if it did, all points after a certain point (say P_N) on would have to be in some small sector of the circle, and this is clearly not the case.

4.10 (a) $$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots = \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots\right].$$

The quantity in brackets is the harmonic series treated earlier. It was found to be infinite. Hence, a constant times the harmonic series is infinite, and the series (a) diverges.

(b) $$\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \cdots = \frac{1}{4\cdot 1 - 1} + \frac{1}{4\cdot 2 - 1} + \frac{1}{4\cdot 3 - 1} + \cdots + \frac{1}{4n - 1} + \cdots$$

since

$$\frac{1}{4n - 1} \geq \frac{1}{4n} \qquad \text{for } n = 1, 2, \cdots,$$

each term of the series (b) is greater than the corresponding term in the diverging series

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \cdots \frac{1}{4n} + \cdots = \frac{1}{4}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots\right],$$

and hence the series (b) diverges.

(c) Since for $n > 1$, $n > \sqrt{n}$, we have

$$\frac{1}{\sqrt{n}} > \frac{1}{n}$$

and

$$\frac{1}{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$$
$$> \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

Hence the sequence (c) diverges.

(d) The terms of this sequence are even larger than the corresponding terms of (c) and therefore (d) certainly diverges.

4.11 (a) If for every line in the plane the projection of P is the limit of the projections of P_n, then this is true, in particular, for the two perpendicular axes of a coordinate system. Denote the projections of P_n on the x-axis and on the y-axis by x_n, y_n respectively, and those of P by x and y. Then, see Figure 4.15(a),

$$(P_nP)^2 = (x_n - x)^2 + (y_n - y)^2,$$

and since the x_n approach x and the y_n approach y,

$$\lim_{n\to\infty} (P_nP)^2 = 0$$

and the P_n approach P.

(b) Clearly, this result cannot be deduced from the fact that the given data are true for just one line, as Example 5 (page 68) shows.

(c) If the given data are true for any two non-parallel lines, say l_1 and l_2, take one (say l_1) to be the x-axis. It can be shown (by methods of analytic geometry or linear algebra) that any line in the plane, for example the y-axis, can be expressed as a linear combination of two given non-parallel lines. Moreover, the projections y_n of P_n on the y-axis can be expressed in terms of the x_n and the projections z_n of P_n on the line l_2 [see Figure 4.15(b)], and the y_n have a limit y if the x_n and the z_n have limits. Thus the problem that P is the limit of the P_n can be reduced to the problem solved in (a).

CHAPTER FIVE

5.1 Assume that $\sqrt{2}$ is rational, i.e. that the diagonal of a unit square has length p_1/q_1 where p_1 and q_1 are integers. Then a square whose sides are q_1 units long has a diagonal of length p_1.

Now construct the following sequence of right isosceles triangles: The first has legs of length q_1 and a hypotenuse of length p_1, see Figure 5.4(b). Erect a perpendicular to the hypotenuse at a point which divides it into segments of lengths q_1 and $p_1 - q_1$. This perpendicular cuts off a corner of the first triangle, and this corner is our second triangle, clearly similar to the first, with leg of length q_2 and hypotenuse of length p_2. We observe [see Figure 5.4(b)] that

$$q_2 = p_1 - q_1 \quad \text{and} \quad p_2 = q_1 - q_2 = q_1 - (p_1 - q_1) = 2q_1 - p_1 .$$

Now we repeat the construction and cut off the next corner triangle Its legs have length

$$q_3 = p_2 - q_2 = 2q_1 - p_1 - (p_1 - q_1) = 3q_1 - 2p_1$$

and its hypotenuse has length

$$p_3 = q_2 - q_3 = p_1 - q_1 - (3q_1 - 2p_1) = 3p_1 - 4q_1 .$$

We continue cutting off corners, always obtaining an isosceles right triangle similar to all the previous ones. The leg of the nth triangle has length q_n, its hypotenuse has length p_n, and these lengths satify the relations

$$q_n = p_{n-1} - q_{n-1} , \qquad p_n = q_{n-1} - q_n .$$

Since $p_{n-1} = q_{n-2} - q_{n-1}$ we may express the length q_n of the nth leg by

$$q_n = q_{n-2} - 2q_{n-1} , \qquad n > 2,$$

that is, in terms of the lengths of the legs of the previous two triangles.

Now consider the sequence $q_1, q_2, q_3, \cdots$. Since p_1 and q_1 are integers, $q_2 = p_1 - q_1$ is an integer, $q_3 = q_1 - 2q_2$ is an integer and, in general, $q_n = q_{n-2} - 2q_{n-1}$ is an integer for all $n > 2$. It is clear from our construction that the legs of subsequent triangles decrease in length, i.e. that

$$q_1 > q_2 > q_3 > \cdots .$$

Thus the assumption that $\sqrt{2} = p_1/q_1$ is rational has led to an infinite decreasing sequence of positive integers, and no such sequence exists. We conclude that $\sqrt{2}$ is irrational.

In order to apply this method to $\sqrt{5}$, assume that $\sqrt{5} = r_1/s_1$ where r_1 and s_1 are integers. Blow up the rectangle of Figure 5.5 so that its sides are s_1, $2s_1$; then its diagonal is r_1. Our construction will lead to similar right triangles with legs s_n, $2s_n$ and hypotenuse r_n. The recursion relations will be

$$s_n = r_{n-1} - 2s_{n-1} , \qquad r_n = s_{n-1} - 2s_n ,$$

so that

$$s_n = s_{n-2} - 2s_{n-1} - 2s_{n-1} = s_{n-2} - 4s_{n-1} ,$$

and the sequence $s_1, s_2, s_3, \cdots$ of lengths of shorter legs of the similar triangles is again a decreasing infinite sequence of integers.

These examples show how this method can be used to prove the irrationality of $\sqrt{k}$ for any integer k which can be written as the sum of the squares of two integers: $k = a^2 + b^2$. We have used it for $k = 1^2 + 1^2$, and for $k = 2^2 + 1^2$. The details of this generalization are left to the reader.

5.3 The kth fraction, F_k, is formed from the previous fraction, F_{k-1}, as follows:

$$F_k = \frac{1}{1 + F_{k-1}}.$$

If $\quad F_{k-1} = \frac{p}{q}, \quad$ then $\quad F_k = \frac{1}{1 + \frac{p}{q}} = \frac{q}{p + q}.$

5.4 (a) A sequence of finite parts of the expression $\sqrt{1 - \sqrt{1 - \sqrt{1 - \cdots}}}$ is formed in the following manner:

$$\sqrt{1}, \quad \sqrt{1 - \sqrt{1}}, \quad \sqrt{1 - \sqrt{1 - \sqrt{1}}}, \quad \cdots.$$

When we compute these numbers we see that this is the sequence 1, 0, 1, 0, $\cdots$, which has no limit.

(b) Since m satisfies $m^2 + m = 1$, its reciprocal satisfies

$$\frac{1}{\tau^2} + \frac{1}{\tau} = 1 \qquad \text{or} \qquad 1 + \tau = \tau^2.$$

The terms a_i of the sequence of finite parts

$$\sqrt{1}, \quad \sqrt{1 + \sqrt{1}}, \quad \sqrt{1 + \sqrt{1 + \sqrt{1}}}, \quad \cdots$$

obey the recursion formula

$$(1) \qquad a_1 = \sqrt{1}, \qquad a_n = \sqrt{1 + a_{n-1}}, \quad \text{for } n = 2, 3, \cdots.$$

We shall show that the increasing sequence a_1, a_2, $\cdots$ has a limit by applying the Bolzano-Weierstrass Theorem, see Section 3.8. In order to do this, we must find a bound B such that $a_1 < a_2 < \cdots < B$.

The fact that the a_i increase implies that $a_{n+1} - a_n > 0$ for $n = 1, 2, \cdots$. From (1) we have

$$\begin{aligned} a_{n+1} &- a_n \\ &= [\sqrt{1 + a_n} - \sqrt{1 + a_{n-1}}]\,\frac{\sqrt{1 + a_n} + \sqrt{1 + a_{n-1}}}{\sqrt{1 + a_n} + \sqrt{1 + a_{n-1}}} \\ &= \frac{(1 + a_n) - (1 + a_{n-1})}{\sqrt{1 + a_n} + \sqrt{1 + a_{n-1}}} = \frac{a_n - a_{n-1}}{\sqrt{1 + a_n} + \sqrt{1 + a_{n-1}}}. \end{aligned}$$

Since $a_i > 0$ for all i, the denominator in the last expression is greater than 2. Therefore

$$a_{n+1} - a_n < \frac{1}{2}\,(a_n - a_{n-1}) \qquad \text{for all } n,$$

and

$$(2) \qquad a_{n+1} - a_n < \frac{1}{2}(a_n - a_{n-1}) < \frac{1}{2}\left[\frac{1}{2}(a_{n-1} - a_{n-2})\right] < \cdots < \frac{1}{2^{n-1}}(a_2 - a_1).$$

Next we write a_{n+1} in the form

$$a_{n+1} = (a_{n+1} - a_n) + (a_n - a_{n-1}) + \cdots + (a_2 - a_1) + a_1$$

and apply inequality (2) to each expression in parentheses:

$$a_{n+1} < \frac{1}{2^{n-1}}(a_2 - a_1) + \frac{1}{2^{n-2}}(a_2 - a_1) + \cdots + (a_2 - a_1) + a_1$$

$$= (a_2 - a_1)\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right] + a_1 .$$

The expression in brackets never exceeds 2, so

$$a_{n+1} < 2(a_2 - a_1) + a_1 = 2(\sqrt{2} - 1) + 1 = 2\sqrt{2} - 1,$$

and this number bounds all terms of our sequence.

Observe that we did not make use of the fact that the limit of this sequence is $\tau = 1/m = 1 + m = 1.618 \cdots$. The bound

$$B = 2\sqrt{2} - 1 = 1.828 \cdots$$

which we constructed is somewhat larger than this limit.

5.5 The values of these ratios, calculated to six decimal places, are

$$\frac{5}{8} \approx .625000; \qquad \frac{8}{13} \approx .615385; \qquad \frac{13}{21} \approx .619048;$$

$$\frac{21}{34} \approx .617647; \qquad \frac{34}{55} \approx .618182; \qquad \frac{55}{89} \approx .617978.$$

The fractions

$$\frac{377}{610} \approx .618033 \qquad \text{and} \qquad \frac{610}{987} \approx .618034$$

are the 17th and 18th terms of the sequence.

5.6 Each successive fraction in the sequence is numerically closer to m. The approximate differences between m and the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{8}$, $\frac{8}{13}$ are, respectively, .118034, .048633, .018034, .006966, and .002649. For a general proof of the fact that each convergent to an infinite continued fraction is closer to it than the previous convergent, see for example Chapter 3 (particularly Theorem 3.7) of the book by C. D. Olds, *Continued Fractions*, to appear in this series.

5.7 $m^5 = 2m - 3m^2 = 5m - 3$; $m^6 = 5m^2 - 3m = 5 - 8m$;

$m^7 = 5m - 8m^2 = 13m - 8$; $m^8 = 13m^2 - 8m = 13 - 21m; \cdots$.

If f_n is the nth term of the Fibonacci Sequence, the formula for the nth power of m is

$$m^n = (-1)^n(f_{n-1} - f_n m).$$

The corresponding situation for τ is

$$\tau^4 = 3\tau + 2; \quad \tau^5 = 5\tau + 3; \quad \tau^6 = 8\tau + 5; \cdots; \quad \tau^n = f_n\tau + f_{n-1}.$$

5.8 A complete solution to Problem 5.8 is given in Chapter 6, pp. 114–117.

5.9 The way the vertices are ordered in successive rectangles reflects the fact that the shorter side of each rectangle (i.e. the line joining the 2 vertices named last in the ordering) is the longer side of the next one (i.e. the line between the vertices listed in the middle position). In each case the vertex named first is the one from which the 45° line is drawn to the point that is the first named vertex of the next rectangle.

5.10 The length of each successive segment of this zig-zag is the length of the preceding segment reduced by the factor $m < 1$. Hence, the length of the zig-zag is the sum of the infinite geometric series

$$\sqrt{2} + \sqrt{2} \cdot m + \sqrt{2} \cdot m^2 + \sqrt{2} \cdot m^3 + \cdots$$

$$= \frac{\sqrt{2}}{1 - m} = \frac{\sqrt{2}}{m^2}.$$

The solution to Problem 3.13 (p. 52) proves that the formula for the sum of *any* infinite geometric series with first term a and ratio $r < 1$ is $a/(1 - r)$.

5.11

Number of quarter-turns (degrees) about T		Distance from T to point on the spiral
1/6	(15°)	$AT \cdot m^{1/6}$
1/3	(30°)	$AT \cdot m^{1/3}$
1/2	(45°)	$AT \cdot m^{1/2}$
5/6	(75°)	$AT \cdot m^{5/6}$
4/3	(120°)	$AT \cdot m^{4/3}$
3/2	(135°)	$AT \cdot m^{3/2}$
5/3	(150°)	$AT \cdot m^{5/3}$
5/2	(225°)	$AT \cdot m^{5/2}$
...	...	...
$\frac{2n+1}{2}$	$\left(\frac{2n+1}{2} \cdot 90°\right)$	$AT \cdot m^{(2n+1)/2}$

5.12 If t takes on negative values, we get a continuation of the spiral in a counter-clockwise direction from AT. As the values of t become smaller and smaller (i.e. as t approaches $-\infty$), R increases without limit.

5.13 To multiply a number R_1 by a number R_2 by means of the spiral in Figure 5.11 (where the distance AT is now taken to be the unit of measurement) we use the ruler to locate those points P_1 and P_2 on the spiral which have distances R_1 and R_2 from T:

$$P_1T = R_1, \qquad P_2T = R_2.$$

Now we follow the spiral from the point A to the point P_1 and denote by α_1 the angle through which the radius vector to the spiral must rotate to get from TA to TP_1. (Observe that if $R_1 < 1$, then we reach P_1 by going in the clockwise direction and α_1 will be taken to be positive; if $R_1 > 1$, then we reach P_1 by going counter-clockwise and α_1 will be taken negative.) Next, we follow the spiral from A to P_2 and measure the angle α_2 by which the radius vector must be rotated to get from TA to TP_2. Now we add the angles α_1 and α_2, and rotate the line AT through the angle $\alpha_1 + \alpha_2$ always following the spiral from the point A on. This will lead to a point P_3 on the spiral whose distance from T is

$$P_3T = R_1 \cdot R_2 \cdot$$

This method is just a geometric interpretation of the law of exponents: Given

$$R_1 = m^{\alpha_1}, \qquad R_2 = m^{\alpha_2},$$

we have found

$$R_1 \cdot R_2 = m^{\alpha_1 + \alpha_2}.$$

5.14 If the radius vectors $TP_1, TP_2, \cdots, TP_n$ have the same direction but different magnitudes $R_1, R_2, \cdots, R_n$, then the angles of rotation $\alpha_1, \alpha_2, \cdots, \alpha_n$, measured from the line through T and A as this line passes through each point of the curve from A to P_1, to $P_2, \cdots$, to P_n, differ only by multiples of 2π radians (one full turn about T, i.e. 4 quarter-turns). This property corresponds to the fact that the logarithms of the numbers represented by the lengths $R_1, R_2, \cdots, R_n$ would differ only in their characteristics, i.e. in the integer part of the logarithm. (If α is measured in quarter-turns, these logarithms would differ by multiples of 4.) If α is between $4k$ and $4(k+1)$ quarter-turns, $\alpha - 4k$ would correspond to the mantissa and would determine the direction of the line TP, while the characteristic $4k$ would determine on which "ring" of the spiral the point P lies.

CHAPTER SIX

6.1 Assume to the contrary that there exist integers a and b such that

$$\frac{a}{b} \cdot (1 + \sqrt{2}) = 1.$$

Then

$$1 + \sqrt{2} = \frac{b}{a},$$

or

$$\sqrt{2} = \frac{b}{a} - 1 = \frac{b - a}{a}.$$

But if b and a are integers, $b - a$ is also an integer, and the last equality states that $\sqrt{2}$ is rational, which is false. Therefore the reciprocal of $1 + \sqrt{2}$ is not rational.

6.2 Let d be the highest common factor of a and b, and let x and y be integers. Then the integer

$$ax + by = c = a'dx + b'dy = d(a'x + b'y)$$

is clearly divisible by d.

Conversely, if c is divisible by d, the highest common factor of a and b, then we can find integers x and y such that

$$ax + by = c$$

in the following way. We divide the equation by d obtaining

$$a'x + b'y = c',$$

where a' and b' are relatively prime. In this case it is known (see e.g. the discussion of Euclid's algorithm in *Continued Fractions* by C. D. Olds, to appear in this series) that there exist integers x_1 and y_1 such that

$$a'x_1 + b'y_1 = 1;$$

then the integers $x = c'x_1$, $y = c'y_1$ will satisfy

$$a'x + b'y = c',$$

and hence also $ax + by = c$.

6.3 2^{n-1}, $n = 1, 2, 3, \cdots$.

6.4 When $N = 1$, N and the sum of its digits clearly have the same residue modulo 3. This proves the first step in the induction.

Suppose that k is an integer such that k and the sum of its digits have the same residue modulo 3, i.e. such that

$$k = 3q + r, \qquad 0 \leq r < 3,$$

and the sum of the digits of k is given by

$$3s + r, \qquad 0 \le r < 3.$$

To prove the inductive step, we must show that $k + 1$ and the sum of its digits have the same remainder when divided by 3. When $0 \le r + 1 < 3$, we have

$$k + 1 = 3q + (r + 1);$$

otherwise

$$k + 1 = 3(q + 1).$$

If the last m ($m \ge 0$) digits of a number k are all 9's, these 9's will become 0's when 1 is added to k, but the first digit which is not a 9 will be increased by 1. Since the sum of the digits of k is $3s + r$, we may write the sum of the digits of $k + 1$ in the form

$$(3s + r) + 1 - 9m,$$

which is equivalent to

$$3(s - 3m) + (r + 1).$$

Thus $k + 1$ and the sum of the digits of $k + 1$ have the same residue modulo 3.

6.5 The assertion is true for $n = 1$. Assume that for $n = k$,

$$1 + 2 + \cdots + k = \tfrac{1}{2}k(k + 1),$$

and consider the case for $n = k + 1$. By applying the inductive hypothesis we get

$$1 + 2 + \cdots + k + k + 1 = \tfrac{1}{2}k(k + 1) + k + 1,$$

which can be written

$$\tfrac{1}{2}k^2 + \tfrac{3}{2}k + 1 = \tfrac{1}{2}(k^2 + 3k + 2) = \tfrac{1}{2}(k + 1)(k + 2).$$

Since this is of the form $\frac{1}{2}n(n + 1)$, the proof is complete.

6.6 For all integers n, 2^{n+9} exceeds $(n + 9)^3$.

6.7 It is true that, when k is an integer greater than 2, then

$$2k^2 > k^2 + 2k + 2 > (k + 1)^2;$$

to show this, note that when $k > 2$, then $k - 2 > 0$, and since k is an integer, $k - 2 \ge 1$, $k \ge 3$ so that $k(k - 2) > 1$ or $k^2 > 2k + 1$. Hence

$$2k^2 = k^2 + k^2 > k^2 + 2k + 1,$$

that is, $2k^2 > (k + 1)^2$.

This fact does not enable us to prove that 2^n exceeds n^2 for all $n > 2$ because, in order to use that $2^{k+1} > 2k^2$, we had to assume that $2^k > k^2$, and it is *not* true that 2^3 exceeds 3^2.

6.8 (a) For $N = 1$ we have $(2^1)^1 = 2 = 2^{(1^2)}$. If, when $N = k$,

$$(2^k)^k = 2^{(k^2)},$$

then

$$\begin{aligned}(2^{k+1})^{k+1} &= (2^k \cdot 2)^{k+1} = (2^k \cdot 2)^k (2^k \cdot 2)\\ &= (2^k)^k \cdot 2^{2k} \cdot 2\\ &= 2^{(k^2+2k+1)}\\ &= 2^{[(k+1)^2]}\end{aligned}$$

(b) From $2^N > N^2$ if $N > 4$ we get

$$(2^N)^N > (N^2)^N \qquad \text{or} \qquad 2^{(N^2)} > (N^2)^N.$$

Thus, if we take $n = N^2$, we have that for $N > 4$, $2^n > n^N$. But from the proofs of Theorems 2 and 3 we know that for $N = 2$ and $N = 3$, $2^n > n^N$ only if n exceeds N^2. This suggests that we can prove the inductive step of Therorem N by showing that for all $n > N^2$, $2^n > n^N$.

6.9 If $2^k > k^N$, and $k > N^2$, then

$$2^{k+1} = 2 \cdot 2^k > 2k^N = k^N + k^N > k^N + N^2 k^{N-1}$$

so that

$$\begin{aligned}2^{k+1} &> k^N + Nk^{N-1} + N(N-1)k^{N-1}\\ &\geq k^N + Nk^{N-1} + N(N-1)k^{N-2}\\ &= k^N + Nk^{N-1} + \frac{N(N-1)}{2}k^{N-2} + \frac{N(N-1)}{2}k^{N-2}\\ &\geq k^N + Nk^{N-1} + \frac{N(N-1)}{2}k^{N-2} + \frac{N(N-1)(N-2)}{3}k^{N-3}\\ &= k^N + Nk^{N-1} + \frac{N(N-1)}{2}k^{N-2} + \frac{N(N-1)(N-2)}{2 \cdot 3}k^{N-3}\\ &\qquad + \frac{N(N-1)(N-2)}{2 \cdot 3}k^{N-3}\\ &\geq \cdots\\ &\geq k^N + Nk^{N-1} + \frac{N(N-1)}{2!}k^{N-2} + \frac{N(N-1)(N-2)}{3!}k^{N-3}\\ &\qquad + \cdots + \frac{N(N-1)\cdots[N-(N-2)]}{(N-1)!}k^{N-(N-1)} + 1\\ &= (k+1)^N.\end{aligned}$$

6.10 We have proved (Theorem 1) that when $N = 1$,

$$2^n > n^N = n \qquad \text{for all integers } n.$$

Assume that when $N = k$ and n exceeds k^2, it is true that $2^n > n^k$. It follows from Problem 6.9 that $2^n > n^{k+1}$ provided that $n > (k+1)^2$, which is all we need to complete the proof that $2^n > n^N$ for all integers N and n such that $n > N^2$.

6.11 Let us try to imitate the proof of Lagrange's Theorem (pages 115–117) in the present case and let us observe what modifications will be necessary.

The box principle tells us that any sequence of residues (mod N) has a repeating consecutive pair within $N^2 + 2$ terms. If the pairs a_i, a_{i+1} and a_k, a_{k+1} have the same residues, then from

$$a_i \equiv a_k (\text{mod } N) \qquad \text{and} \qquad a_{i+1} \equiv a_{k+1} (\text{mod } N)$$

we get

$$3a_i \equiv 3a_k (\text{mod } N) \qquad \text{and} \qquad 2a_{i+1} \equiv 2a_{k+1} (\text{mod } N).$$

It follows that

$$2a_{i+1} + 3a_i \equiv 2a_{k+1} + 3a_k (\text{mod } N),$$

or

$$a_{i+2} \equiv a_{k+2} (\text{mod } N).$$

By the same argument

$$a_{i+3} \equiv a_{k+3} (\text{mod } N),$$

$$a_{i+4} \equiv a_{k+4} (\text{mod } N),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

which shows that the sequence of residues (mod N) of the sequence given by $a_{n+1} = 2a_n + 3a_{n-1}$ is periodic.

Let the period of the sequence be p. Then

$$a_j \equiv a_{j+p} \ (\text{mod } N) \qquad (j \geq T)$$

from some j on, say $j = T$. Suppose a_T is not the first term of the sequence. From the recursion formula we have

$$\begin{aligned} 3a_{T-1} &= a_{T+1} - 2a_T \\ &\equiv a_{T+1+p} - 2a_{T+p} \ (\text{mod } N) \\ &= 3a_{T+p-1}\,. \end{aligned}$$

Hence,

$$3a_{T-1} \equiv 3a_{T-1+p} \ (\text{mod } N),$$

or

$$3(a_{T-1} - a_{T-1+p}) \equiv 0 \ (\text{mod } N).$$

Clearly, we can conclude that

$$a_{T-1} \equiv a_{T-1+p} \pmod{N}$$

only if we assume that 3 and N are relatively prime; otherwise, it does not necessarily follow that N is a factor of $a_{T-1} - a_{T-1+p}$. Since T is a a finite integer, this process applied successively to $T - 1$, $T - 2$, $T - 3$, $\cdots$, eventually must lead to

$$a_1 \equiv a_{1+p} \pmod{N}.$$

Thus, the sequence of residues (mod N), $N \geq 2$, of the sequence defined by $a_{n+1} = 2a_n + 3a_{n-1}$, $n \geq 2$ (where the initial values a_1 and a_2 may be any given integers) is periodic. If 3 and N are relatively prime, then the periodic part begins with the residue of a_1.

6.12 The residues (mod$N$), $N \geq 2$, of any sequence defined by

$$a_{n+1} = \alpha a_n + \beta a_{n-1}$$

with arbitrary initial integers a_1 and a_2 are periodic; the repetition of a pair occurs within at most $N^2 + 2$ terms. If N and β are relatively prime, then the periodic part begins with the residue of a_1.

6.13 The sequence of residues (mod$N$) of any sequence defined by

$$a_{n+1} = \alpha a_n + \beta a_{n-1} + \gamma a_{n-2}, \qquad n \geq 3,$$

has a repeated consecutive triplet within $N^3 + 3$ terms. If N and γ are relatively prime, the sequence of residues (mod$N$) is periodic from the beginning on.

6.14 In general, the sequence of residues (mod$N$) of a sequence a_1, a_2, $\cdots$, a_n, $\cdots$ built (after the nth term) on a rule expressing the $(n + 1)$th term as a linear combination of the preceding n terms is periodic from the beginning on and will repeat within $N^n + n$ terms whenever N has no factors greater than 1 in common with the coefficient of the earliest term in the recursion formula.

6.15 The lines $x = a$ for all rational a constitute a countable infinity of lines since the set of all rational numbers is countable. The same is true for the sets $y = b$, $x = c$, $y = d$ for rational b, c, d. Since all special rectangles are formed by combining 4 sides, each from one of these sets, we obtain $\aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdot \aleph_0 = \aleph_0$ possible special rectangles. (This even includes the degenerate rectangles in which a pair of opposite sides coincides. Therefore, the non-degenerate special rectangles certainly constitute a countable set.)

6.16 Let P be the point (x_0, y_0), and let d be the minimum distance from P to any point on the given rectangle. Then there exist rational numbers $\delta_1 < \frac{1}{2}d$ and $\delta_2 < \frac{1}{2}d$ such that

$$x = x_0 + \delta_1 = a \qquad \text{and} \qquad x = x_0 - \delta_2 = a'$$

are rational, and numbers $\epsilon_1 < \frac{1}{2}d$ and $\epsilon_2 < \frac{1}{2}d$ such that

$$y = y_0 + \epsilon_1 = b \qquad \text{and} \qquad y = y_0 - \epsilon_2 = b'$$

are rational. It follows that the sides of a special rectangle R lie on the lines $x = a$, $x = a'$, $y = b$, $y = b'$, and the point P is inside of this rectangle. Furthermore, since the length of the diagonal of R is

$$\sqrt{(\delta_1 + \delta_2)^2 + (\epsilon_1 + \epsilon_2)^2} < d,$$

the distance from P to the farthest point on R is less than the minimum distance from P to the given rectangle. Hence, the special rectangle lies entirely within the given one.

6.17 If we assume that there is no point P in the set X such that every rectangle containing P contains uncountably many points of X, then every point in the set X must be inside at least one rectangle containing a countable set of points of X. In this case, the solution to Problem 6.16 shows that every point of X is inside of a special rectangle which is entirely within the rectangle containing a countable set of points belonging to X, and so also contains at most a countable infinity of points of X. Now, we have proved (Problem 6.15) that the set of all special rectangles is of power $\aleph_0$; therefore, the set of special rectangles with which we are concerned is certainly countable. Moreover, since each of these special rectangles contains a countable set of points of X, it follows from $\aleph_0 \cdot \aleph_0 = \aleph_0$ that the set X is countable. But this contradicts the hypothesis that the given set is uncountable; hence, there must exist some point P in X such that every rectangle containing P contains uncountably many points belonging to the set X.

6.18 Take the point P obtained in the solution to Problem 6.17, and a sequence of decreasing intervals (rectangles in the case of the plane) closing down on P. In each of these intervals pick one point of X from among the uncountably many which are available. This gives a sequence P_1, P_2, P_3, $\cdots$ of points of X which form the desired convergent sequence. A proof such as this is called "non-constructive" because no mechanism is provided for actually defining each point P_n. Since we know nothing about X except that it is uncountable, no method of selection is available to us.

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Uses of Infinity

The word "infinity" usually elicits feelings of awe, wonder, and admiration; the concept infinity has fascinated philosophers and theologians. The author shows how professional mathematicians tame this unwieldy concept, come to terms with it, and use its various aspects as their most powerful tools of the trade.

The early chapters are descriptive and intuitive, full of examples that not only illustrate some infinite processes, but that are worth studying for their own sake. Many questions are raised in the beginning, partially answered in various contexts throughout the book, and finally treated with the precision necessary to give the reader an excellent grasp of the fundamental notions used in the calculus as well as in virtually all other mathematical disciplines. The text is peppered with challenging problems whose solutions appear at the end of the book.

Leo Zippin was born in New York City in 1905. He received his PhD from the University of Pennsylvania in 1929. Zippin joined the Queens faculty of CUNY in 1938. He helped to create the doctoral program in mathematics at CUNY and served as its first executive officer, from 1964 to 1968. He retired in 1971. He became known internationally in the 1950's for having helped solve "the fifth problem of Hilbert," a poser regarding locally Euclidean topological groups. He passed away in 1995.

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