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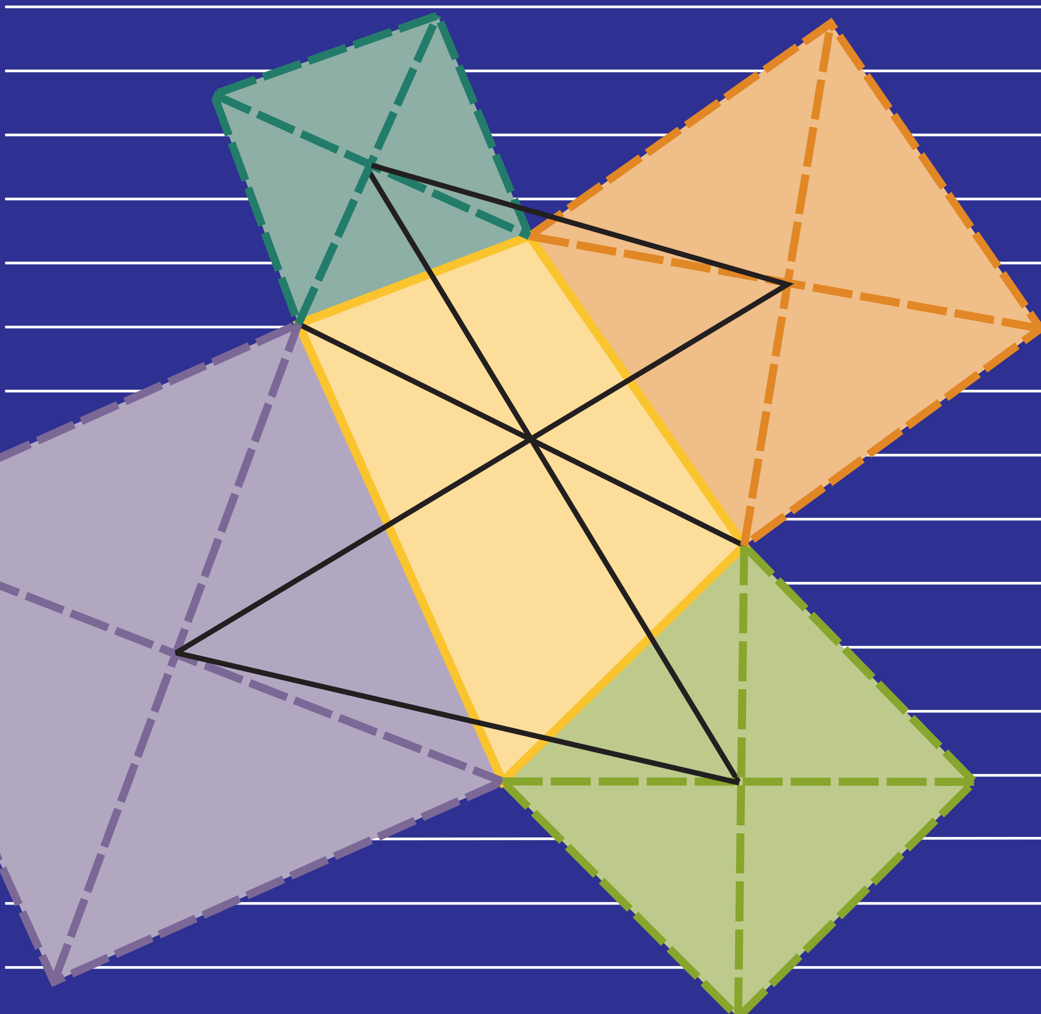
Geometric Transformations I

I. M. Yaglom

Translated from the Russian by
Allen Shields



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Geometric Transformations I

Anneli Lax New Mathematical Library
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Geometric Transformations I

by

I. M. Yaglom

translated from the Russian by

Allen Shields
University of Michigan

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Note to the Reader

This book is one of a series written by professional mathematicians in order to make some important mathematical ideas interesting and understandable to a large audience of high school students and laymen. Most of the volumes in the *New Mathematical Library* cover topics not usually included in the high school curriculum; they vary in difficulty, and, even within a single book, some parts require a greater degree of concentration than others. Thus, while the reader needs little technical knowledge to understand most of these books, he will have to make an intellectual effort.

If the reader has so far encountered mathematics only in classroom work, he should keep in mind that a book on mathematics cannot be read quickly. Nor must he expect to understand all parts of the book on first reading. He should feel free to skip complicated parts and return to them later; often an argument will be clarified by a subsequent remark. On the other hand, sections containing thoroughly familiar material may be read very quickly.

The best way to learn mathematics is to *do* mathematics, and each book includes problems, some of which may require considerable thought. The reader is urged to acquire the habit of reading with paper and pencil in hand; in this way mathematics will become increasingly meaningful to him.

For the authors and editors this is a new venture. They wish to acknowledge the generous help given them by the many high school teachers and students who assisted in the preparation of these monographs. The editors are interested in reactions to the books in this series and hope that readers will write to: Editorial Committee of the NML series, NEW YORK UNIVERSITY, THE COURANT INSTITUTE OF MATHEMATICAL SCIENCES, 251 Mercer Street, New York, N. Y. 10012.

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GEOMETRIC TRANSFORMATIONS I

Translator's Preface

The present volume is Part I of *Geometric Transformations* by I. M. Yaglom. The Russian original appeared in three parts; Parts I and II were published in 1955 in one volume of 280 pages. Part III was published in 1956 as a separate volume of 611 pages. In the English translation Parts I and II are published as two separate volumes: NML 8 and NML21. The first chapter of Part III, on projective and some non-Euclidean geometry, was translated into English and published in 1973 as NML vol. 24; the balance of Part III, on inversions, has not so far been published in English.

In this translation most references to Part III were eliminated, and Yaglom's "Foreword" and "On the Use of This Book" appear, in greatly abbreviated form, under the heading "From the Author's Preface".

This book is not a text in plane geometry. On the contrary, the author assumes that the reader is already familiar with the subject. Most of the material could be read by a bright high school student who has had a term of plane geometry. However, he would have to work; this book, like all good mathematics books, makes considerable demands on the reader.

The book deals with the fundamental transformations of plane geometry, that is, with distance-preserving transformations (translations, rotations, reflections) and thus introduces the reader simply and directly to some important group theoretic concepts.

The relatively short basic text is supplemented by 47 rather difficult problems. The author's concise way of stating these should not discourage the reader; for example, he may find, when he makes a diagram of the given data, that the number of solutions of a given problem depends on the relative lengths of certain distances or on the relative positions of certain given figures. He will be forced to discover for himself the conditions under which a given problem has a unique solution. In the second half of this book, the problems are solved in detail and a discussion of the conditions under which there is no solution, or one solution, or several solutions is included.

The reader should also be aware that the notation used in this book may be somewhat different from the one he is used to. For example, if two lines l and m intersect in a point O, the angle between them is often referred to as $\sphericalangle lOm$; or if A and B are two points, then "the line AB" denotes the line through A and B, while "the line segment AB" denotes the finite segment from A to B.

The footnotes preceded by the usual symbol † were taken over from the Russian version of this book while those preceded by the symbol T have been added in this translation.

I wish to thank Professor Yaglom for his valuable assistance in preparing the American edition of his book. He read the manuscript of the translation and made a number of suggestions. He has expanded and clarified certain passages in the original, and has added several problems. In particular, Problems 4, 14, 24, 42, 43, and 44 in this volume were not present in the original version while Problems 22 and 23 of the Russian original do not appear in the American edition. In the translation of the next part of Yaglom's book, the problem numbers of the American edition do not correspond to those of the Russian edition. I therefore call to the reader's attention that all references in this volume to problems in the sequel carry the problem numbers of the Russian version. However, NML 21 includes a table relating the problem numbers of the Russian version to those in the translation (see p. viii of NML 21).

The translator calls the reader's attention to footnote † on p. 20, which explains an unorthodox use of terminology in this book.

Project for their advice and assistance. Professor H. S. M. Coxeter was particularly helpful with the terminology. Especial thanks are due to Dr. Anneli Lax, the technical editor of the project, for her invaluable assistance, her patience and her tact, and to her assistants Carolyn Stone and Arlys Stritzel.

Allen Shields

From the Author's Preface

This work, consisting of three parts, is devoted to elementary geometry. A vast amount of material has been accumulated in elementary geometry, especially in the nineteenth century. Many beautiful and unexpected theorems were proved about circles, triangles, polygons, etc. Within elementary geometry whole separate "sciences" arose, such as the geometry of the triangle or the geometry of the tetrahedron, having their own, extensive, subject matter, their own problems, and their own methods of solving these problems.

The task of the present work is not to acquaint the reader with a series of theorems that are new to him. It seems to us that what has been said above does not, by itself, justify the appearance of a special monograph devoted to elementary geometry, because most of the theorems of elementary geometry that go beyond the limits of a high school course are merely curiosities that have no special use and lie outside the mainstream of mathematical development. However, in addition to concrete theorems, elementary geometry contains two important general ideas that form the basis of all further development in geometry, and whose importance extends far beyond these broad limits. We have in mind the deductive method and the axiomatic foundation of geometry on the one hand, and geometric transformations and the group-theoretic foundation of geometry on the other. These ideas have been very fruitful; the development of each leads to non-Euclidean geometry. The description of one of these ideas, the idea of the group-theoretic foundation of geometry, is the basic task of this work....

Let us say a few more words about the character of the book. It is intended for a fairly wide class of readers; in such cases it is always necessary to sacrifice the interests of some readers for those of others. The author has sacrificed the interests of the well prepared reader, and has striven for simplicity and clearness rather than for rigor and for logical exactness. Thus, for example, in this book we do not define the general concept of a geometric transformation, since defining terms that are intuitively clear always causes difficulties for inexperienced readers. For the same reason it was necessary to refrain from using directed angles and to postpone to the second chapter the introduction of directed segments, in spite of the disadvantage that certain arguments in the basic text and in the solutions of the problems must, strictly speaking, be considered incomplete (see, for example, the proof on page 50). It seemed to us that in all these cases the well prepared reader could complete the reasoning for himself, and that the lack of rigor would not disturb the less well prepared reader. . . .

The same considerations played a considerable role in the choice of terminology. The author became convinced from his own experience as a student that the presence of a large number of unfamiliar terms greatly increases the difficulty of a book, and therefore he has attempted to practice the greatest economy in this respect. In certain cases this has led him to avoid certain terms that would have been convenient, thus sacrificing the interests of the well prepared reader. . . .

The problems provide an opportunity for the reader to see how well he has mastered the theoretical material. He need not solve all the problems in order, but is urged to solve at least one (preferably several) from each group of problems; the book is constructed so that, by proceeding in this manner, the reader will not lose any essential part of the content. After solving (or trying to solve) a problem, he should study the solution given in the back of the book.

The formulation of the problems is not, as a rule, connected with the text of the book; the solutions, on the other hand, use the basic material and apply the transformations to elementary geometry. Special attention is paid to methods rather than to results; thus a particular exercise may appear in several places because the comparison of different methods of solving a problem is always instructive.

There are many problems in construction. In solving these we are not interested in the "simplest" (in some sense) construction—instead the author takes the point of view that these problems present mainly a logical interest and does not concern himself with actually carrying out the construction.

No mention is made of three-dimensional propositions; this restriction does not seriously affect the main ideas of the book. While a section of problems in solid geometry might have added interest, the problems in this book are illustrative and not at all an end in themselves.

The manuscript of the book was prepared by the author at the Orekhovo-Zuevo Pedagogical Institute . . . in connection with the author's work in the geometry section of the seminar in secondary school mathematics at Moscow State University.

I. M. Yaglom

Solutions

Chapter One. Displacements

1. Translate the circle S_1 a distance a in the direction l, and let S_1' be its new position; let A' and B' be the points of intersection of S_1' with the circle S_2 (see Figure 60). The two lines parallel to l, one through the point A' and the other through the point B' will each solve the problem (the segments AA' and BB' in Figure 60 are each equal to the distance a of the translation). One can find two additional solutions by translating S_1 in the opposite direction a distance a parallel to l into the new position S_1''.

Depending on the number of points of intersection of the circles S_1' and S_1'' with S_2, the problem may have infinitely many solutions, four solutions, three solutions, two solutions, one solution, or no solution at all. In the case shown in Figure 60 the problem has three solutions.

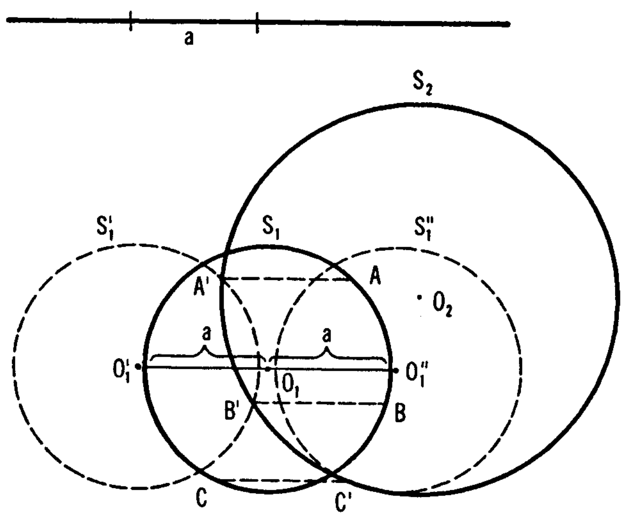


Figure 60

2. (a) Assume that the problem has been solved, and translate the segment MN into a new position AN' in such a manner that the point M is carried into the point A (Figure 61a). Then $AM = N'N$, and therefore

$$AM + NB = N'N + NB.$$

Thus the path $AMNB$ will be the shortest path if and only if the points N', N, and B lie on one line.

Thus we have the following construction: From the point A lay off a segment AN' equal in length to the width of the river, perpendicular to the river, and directed toward it; pass a line through the points N' and B; let N be the point of intersection of this line with the river bank nearest to B; build the bridge across the river at the point N.

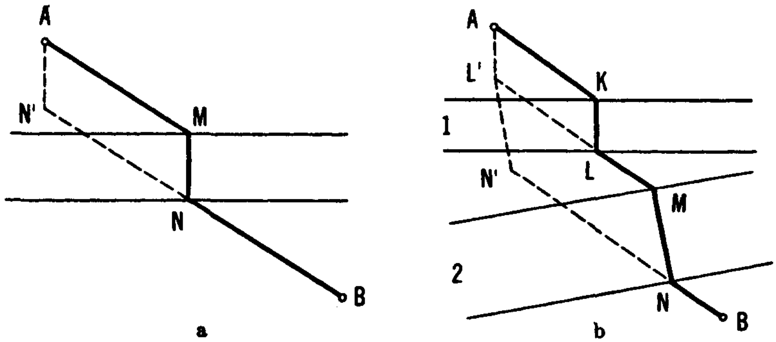


Figure 61

(b) For simplicity we consider the case of two rivers. Assume that the problem has been solved, and let KL and MN be the two bridges across the rivers. Translate the segment KL to a new position AL' in such a manner that the endpoint K is taken into the point A (Figure 61b). Then $AK = L'L$ and

$$AK + LM + NB = L'L + LM + NB.$$

If $AKLMNB$ is the shortest path from A to B, then $L'LMNB$ will be the shortest path from L' to B and $LMNB$ the shortest path from L to B. But L and B are only separated by the second river, and so from part (a) we know how to construct the shortest path between them.

Thus we have the following construction: From the point A lay off a segment AL' equal in length to the width of the first river, perpendicular to it, and directed toward it; from the point L lay off a segment $L'N'$ equal in length to the width of the second river, perpendicular to it, and directed toward it. Pass a line through the points N' and B; let N be the point of intersection of this line with the bank of the second river nearest to B. The bridge across the second river should be built at N. Let M be the other endpoint of this bridge. Pass a line through the point M parallel to the line $N'B$, and let L be the point of intersection of this line with the bank of the first river nearest to M. The first bridge should be built at L.

3. (a) Let M be a point in the plane for which $MP + MQ = a$, where P and Q are the feet of the perpendiculars from M to the lines l_1 and l_2, respectively (Figure 62a). Translate the line l_2 a distance a in the direction QM. If l_2' is the new line obtained by this translation, then it is clear that the distance MQ' of the point M from the line l_2' is equal to $a - MQ = MP$. Consequently M is on the bisector of one of the angles between the lines l_1 and l_2'.

From this it is clear that all points of the desired locus lie on the bisectors of the angles formed by the line l_1 with the lines l_2' and l_2'', obtained from l_2 by translation through a distance a in the direction perpendicular to l_2. However, not all the points on these four bisectors are points of our locus. From Figure 62a it is not difficult to see that only the points on the rectangle $ABCD$ formed by the intersections of the four bisectors will be points of the locus.

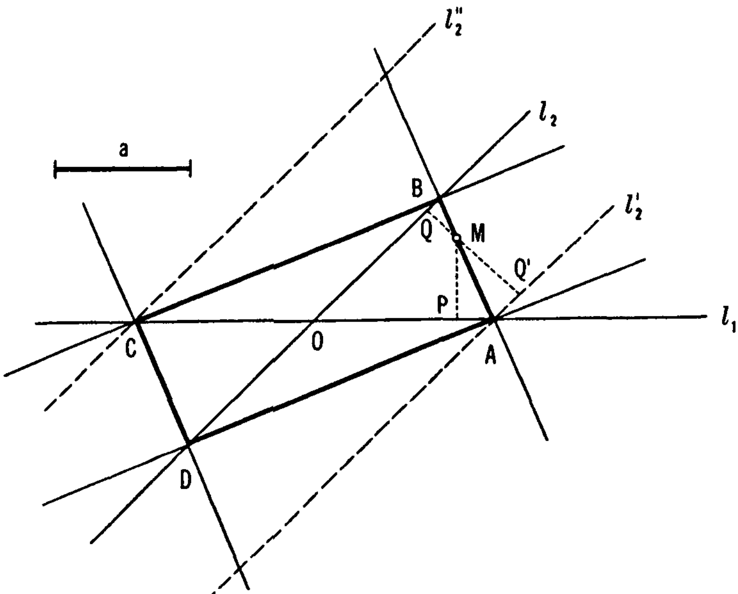


Figure 62a

(b) Let M be a point of the plane satisfying one of the following two equations:

$$MP - MQ = a \quad \text{or} \quad MQ - MP = a,$$

where P and Q are the feet of the perpendiculars from M to the lines l_1 and l_2 (in Figure 62b, the point M satisfies the second equation). Translate the line l_2 a distance a in the direction QM, and let l_2' be the new line. Just as in part (a) one can show that M is equidistant from l_1 and l_2' (see Figure 62b, where $MQ - MP = a$, $M_1P_1 - M_1Q_1 = a$). It follows that all points of the desired locus lie on the bisectors of the four angles formed by the line l_1 with the lines l_2' and l_2''; however in the present case only points lying on the *extensions* of the sides of the rectangle $ABCD$ will be points of the locus (the equation $MP - MQ = a$ is satisfied by the points on HBG and LDN, while the equation $MQ - MP = a$ is satisfied by the points on EAF and ICK).

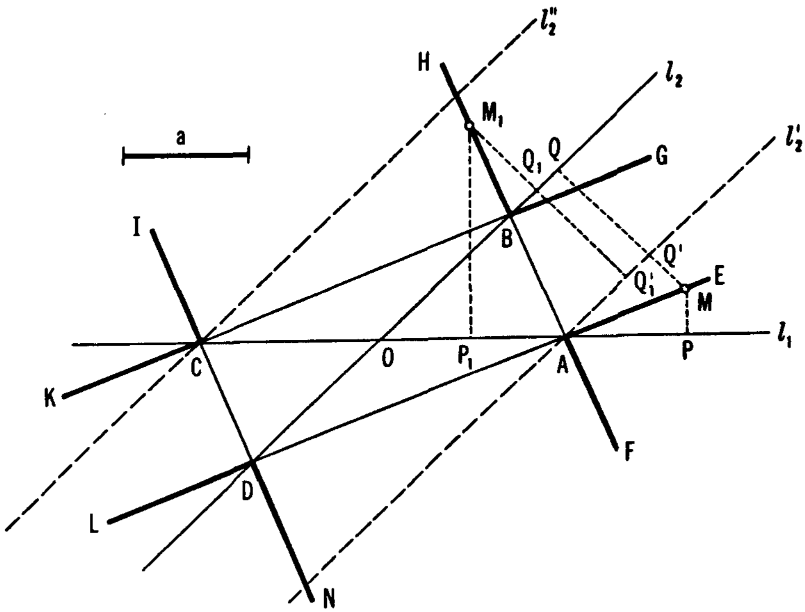


Figure 62b

4. Observe that triangle BDE is obtained from triangle DAF by a translation (in the direction AB through a distance AD); thus the line segments joining pairs of corresponding points in these two figures are equal and parallel to one another. Therefore

$$O_1O_2 = Q_1Q_2, \qquad O_1O_2 \parallel Q_1Q_2.$$

Similarly one has

$$O_2O_3 = Q_2Q_3, \qquad O_2O_3 \parallel Q_2Q_3,$$

and

$$O_3O_1 = Q_3Q_1, \qquad O_3O_1 \parallel Q_3Q_1.$$

Therefore triangles $O_1O_2O_3$ and $Q_1Q_2Q_3$ are congruent (in fact, their corresponding sides are parallel, that is, the triangles are obtained from one another by a translation—see pages 18–19).

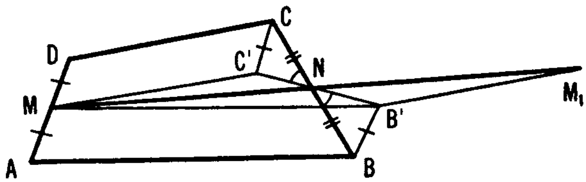


Figure 63

5. Translate the sides AB and DC of the quadrilateral $ABCD$ into the new positions MB' and MC' (Figure 63). The two quadrilaterals $AMB'B$ and $DMC'C$ thus formed will be parallelograms, and therefore

$$BB' \parallel AM \quad \text{and} \quad BB' = AM,$$
$$CC' \parallel DM \quad \text{and} \quad CC' = DM.$$

But $AM = MD$ (M is the midpoint of side AD); thus the segments BB' and CC' are equal and parallel. Since, in addition, $BN = NC$, it follows that

$$\triangle BNB' \cong \triangle CNC'.$$

Therefore $B'N = NC'$ and $\sphericalangle BNB' = \sphericalangle CNC'$, that is, the segments $B'N$ and NC' are extensions of each other.

Thus we have constructed a triangle $MB'C'$ in which, by the conditions of the problem, the median MN is equal to half the sum of the two adjacent sides MB' and MC' (since $MB' = AB$, $MC' = DC$). If we extend the median MN past the point N a distance $NM_1 = MN$ and join M_1 with B', we obtain a triangle MM_1B' in which the side $MM_1 = 2MN$ is equal to the sum of the sides MB' and $B'M_1 = MC'$, which is impossible. Consequently the point B must lie on the segment MM_1. But this means that

$$MB' \parallel MN \parallel MC';$$

therefore

$$AB \parallel MN \quad \text{and} \quad DC \parallel MN,$$

that is, the quadrilateral $ABCD$ is a trapezoid.

6. Assume that the problem has been solved. Translate the segment AX a distance $EF = a$ in the direction of the line CD, and let the new position be $A'X'$ (Figure 64).

Clearly $A'X'$ passes through the point F. Further

$$\measuredangle A'FB = \measuredangle AXB = \tfrac{1}{2}AmB;\dagger$$

therefore we may regard the angle $A'FB$ as known.

Thus we have the following construction: Translate the point A a distance a in the direction of the chord CD, and denote its new position by A'. Using the segment $A'B$ as a chord, construct a circular arc[T] that subtends an angle equal to $\measuredangle AXB$ (that is, if Y is any point on the circular arc, then $\measuredangle A'YB = \measuredangle AXB = \tfrac{1}{2}AmB$).

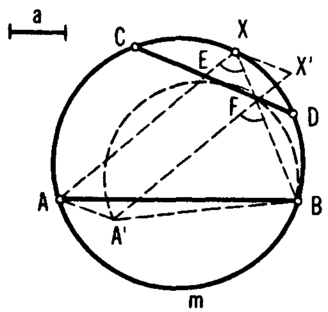


Figure 64

If this circular arc intersects the chord CD in two points, either one of them may be taken as the point F, and the point X is obtained as the point of intersection of the original circle with the line BF. In this case the problem has two solutions.

If the circular arc is tangent to CD, the point of tangency must be taken as the point F, and the problem has just one solution.

If the arc does not intersect CD at all, the problem has no solution.

If one assumes that CD is intersected by the extensions of chords AX and BX (and that points E and F are outside the circle—on the extension of chord CD), then the problem can have up to four solutions. (This is due to the fact that A may be translated in either of two opposite directions.)[TT]

† AmB stands for arc AmB.

[T] For the details of this construction, see, for example, *Hungarian Problem Book 1* in this series, Problem 1895/2, Note.

[TT] The foregoing paragraphs concerning the number of solutions were added in translation.

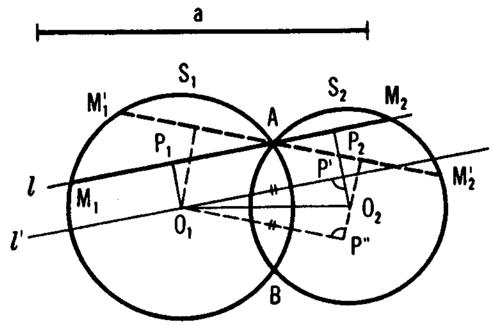


Figure 65

7. (a) Assume that the problem has been solved, i.e., that $M_1M_2 = a$ (Figure 65). From the centers O_1 and O_2 of the circles S_1 and S_2, drop perpendiculars O_1P_1 and O_2P_2 onto the line l; then

$$AP_1 = \tfrac{1}{2}AM_1, \qquad AP_2 = \tfrac{1}{2}AM_2,$$

and consequently,

$$P_1P_2 = \tfrac{1}{2}(AM_1 + AM_2) = \tfrac{1}{2}M_1M_2 = \tfrac{1}{2}a.$$

Translate the line l into a line l' passing through the point O_1; let P' be the point of intersection of l' with the line O_2P_2. Then

$$O_1P' = P_1P_2 = \tfrac{1}{2}a,$$

since the quadrilateral $P_1O_1P'P_2$ is a rectangle.

Thus we have the following construction: Construct a right triangle O_1O_2P' with O_1O_2 as hypotenuse and with side $O_1P' = \tfrac{1}{2}a$. The desired line l will be parallel to the line O_1P'.

If $O_1O_2 > \tfrac{1}{2}a$ the problem has two solutions (the construction of a second solution to the problem is indicated in dotted lines in Figure 65); if $O_1O_2 = \tfrac{1}{2}a$ there is one solution, and if $O_1O_2 < \tfrac{1}{2}a$ there are no solutions.

(b) Let M, N, P be the three given points and let ABC be the given triangle (Figure 66). On the segments MN and MP construct circular arcs subtending angles equal to $\measuredangle ACB$ and $\measuredangle ABC$, respectively. Thus we are led to the following problem: Pass a line B_1C_1 through the point M in such a way that the segment cut off by the two circular arcs has length BC, that is, we are led to Problem (a). The problem may have two solutions, or one solution, or no solutions at all (depending on which sides of the triangle are to pass through each of the three given points).

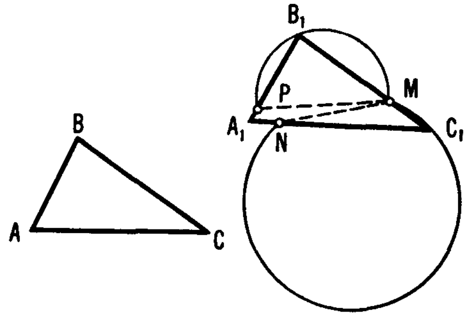


Figure 66

8. (a) Assume that the problem has been solved, and let the line l meet the circles S_1 and S_2 in points A, B and C, D (Figure 67a). Translate the circle S_1 a distance AC in the direction of the line l, and let S_1' be its new position. Since $AB = CD$, the segment AB will coincide with CD; therefore the centers O_2 and O_1' of the circles S_2 and S_1' will both lie on the perpendicular bisector of the segment CD.

Thus we have the following construction: Let m be the line perpendicular to l_1 and passing through the center O_2 of the circle S_2; let n be the line parallel to l_1 and passing through the center O_1 of the circle S_1; Let O_1' be the point of intersection of these two lines. Translate S_1 into a new position S_1' with center at O_1'. The line through the points of intersection of S_2 and S_1' is the solution to the problem.

The problem can have one solution or no solution.

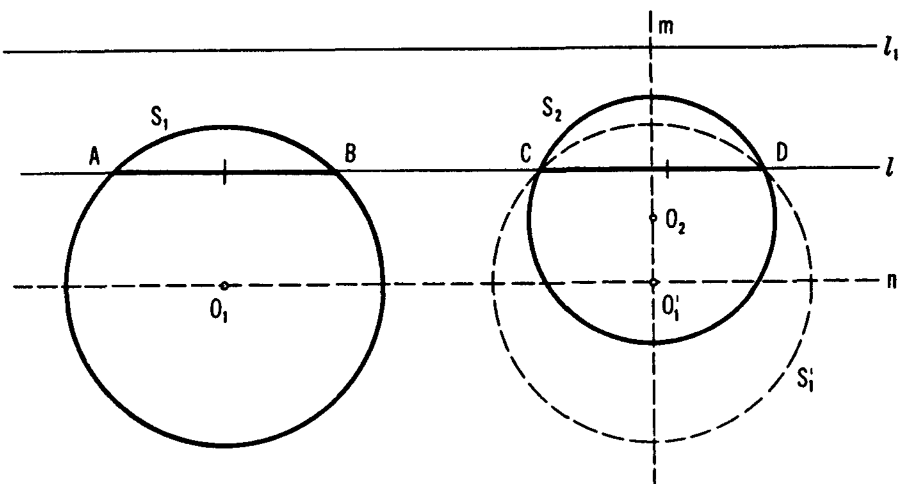


Figure 67a

(b) Assume that the problem has been solved and let the line l meet S_1 and S_2 in points A, B and C, D; then $AB + CD = a$ (Figure 67b). Translate the circle S_1 a distance a in the direction of l and denote its new position by S_1'; then

$$AA' = a = AB + CD,$$

that is, $BA' = CD$. Therefore, if we translate the circle S_2 in the direction of l into a new position S_2' whose center O_2' is on the perpendicular bisector m of the segment O_1O_1' (O_1 and O_1' are the centers of the circles S_1 and S_1'), then the chord CD of S_2 will be taken into BA'.

Thus we have the following construction: Translate the circle S_1 a distance a in the direction of the line l_1, and denote the new position by S_1'; then translate S_2 in the direction of l_1 into a new position S_2' whose center lies on the perpendicular bisector m of the segment O_1O_1'. The points of intersection of the circles S_1 and S_2' (in the diagram they are the points B and B_1) determine the desired lines. The problem has at most two solutions; the number of solutions depends upon the number of points of intersection of the circles S_1 and S_2' (a case when there are two solutions l and l' is shown in Figure 67b).

The other part of the problem, where the *difference* of the two chords cut off on the line l by the two circles is given, can be solved in a similar manner.

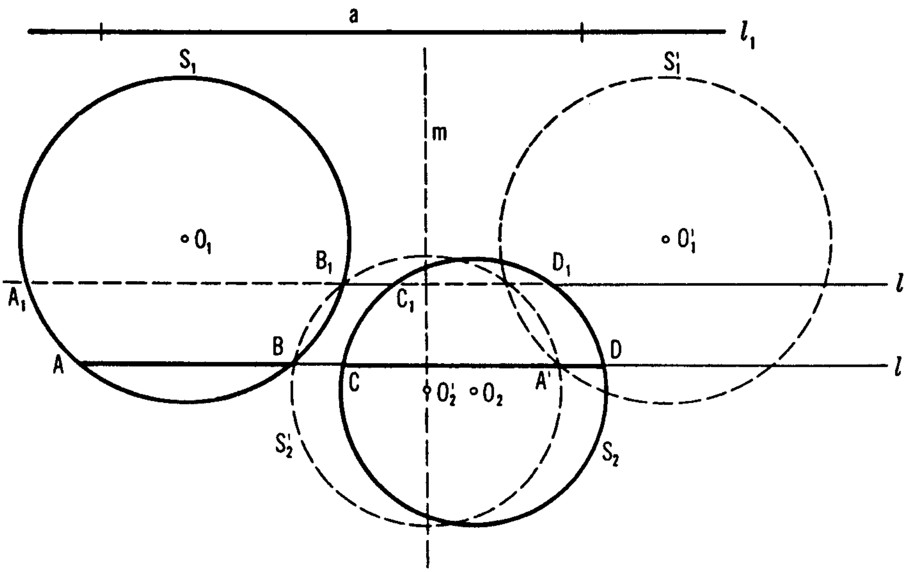


Figure 67b

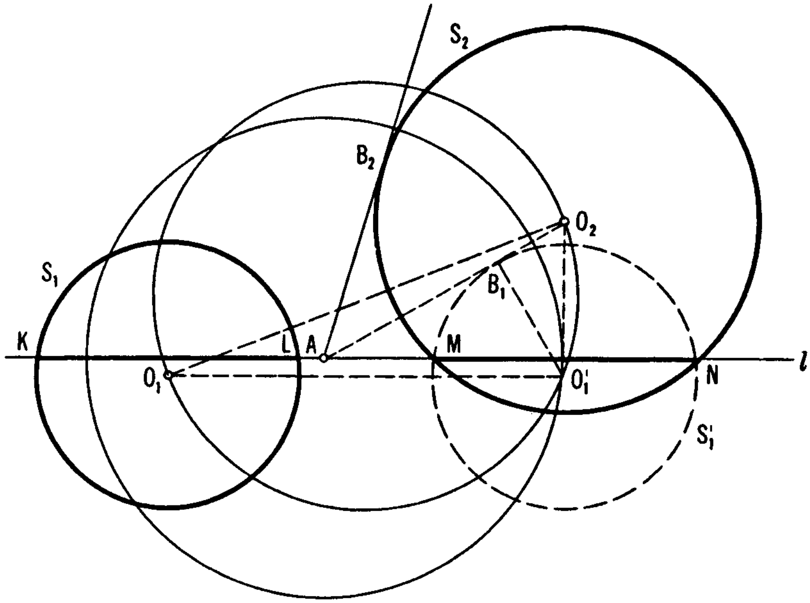


Figure 68

(c) Assume that the problem has been solved, and translate the circle S_1 in the direction of the line KN so that the segment KL coincides with MN; denote the new circle so obtained by S_1' (see Figure 68). Thus the circles S_2 and S_1' have the common chord MN.

Let AB_1 and AB_2 be tangents from the point A to the circles S_1' and S_2 respectively (the points of tangency are B_1 and B_2, respectively). Then

$$(AB_1)^2 = AM \cdot AN; \qquad (AB_2)^2 = AM \cdot AN$$

and therefore

$$(AB_1)^2 = (AB_2)^2.$$

We can now determine AO_1' (O_1' is the center of S_1'):

$$AO_1' = \sqrt{(O_1'B_1)^2 + (AB_1)^2} = \sqrt{r_1^2 + (AB_2)^2},$$

where r_1 is the radius of S_1; in addition, we know that $\measuredangle O_1O_1'O_2$ is a right angle, because $O_1'O_2$, through the centers of S_1' and S_2, is perpendicular to MN, their common chord, and therefore also to O_1O_1', which is parallel to l. This enables us to find the translation carrying S_1 into S_1'.

We use the following construction. With the point A as center, draw a circle of radius

$$\sqrt{r_1^2 + (AB_2)^2};$$

draw a second circle having the segment O_1O_2 as diameter. The intersection of these two circles determines the position of the center O_1' of the circle S_1' of radius r_1. Now find the points M and N of intersection of the circles S_2 and S_1' and draw the line MN, which will be the solution to the problem. Indeed, the point A lies on the line MN; for otherwise the equation $(AB_1)^2 = (AB_2)^2$ could not be satisfied [if the line AM were to intersect the circles S_2 and S_1' in distinct points N_2 and N_1, then we would have $(AB_2)^2 = AM \cdot AN_2$ and $(AB_1)^2 = AM \cdot AN_1$]. Also, O_2O_1' is perpendicular to MN, and O_1O_1' is perpendicular to O_2O_1'; therefore $O_1O_1' \parallel MN$, that is, the chords KL and MN of the circles S_1 and S_1' are at the same distance from the centers O_1 and O_1'. But this means that the chords KL and MN have the same length, which was to be proved.

The problem has at most two solutions.

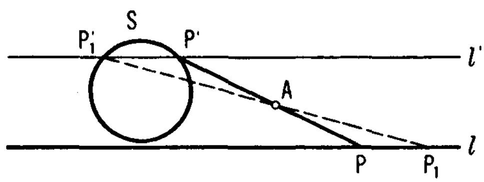


Figure 69

9. Draw the line l' obtained from l by a half turn about the point A (Figure 69); let P' be one of the points of intersection of this line with the circle S. Then the line $P'A$ is a solution to the problem, since the point P of intersection of this line with the line l is obtained from P' by a half turn about A, and therefore $P'A = AP$.

There are at most two solutions to this problem.

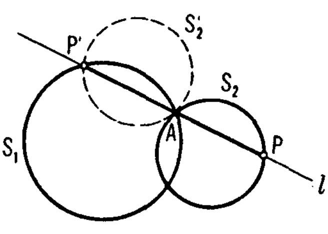


Figure 70a

10. (a) Draw the circle S_2' obtained from S_2 by a half turn about the point A (Figure 70a). The circles S_1 and S_2' intersect in the point A; let P' be their other point of intersection. Then the line $P'A$ will solve the problem, because the point P where this line meets the circle S_2 is obtained from P' by a half turn about A, and therefore $P'A = AP$.

If the circles S_1 and S_2 intersect in two points, then the problem has exactly one solution; if they are tangent, then there is no solution if the radii are different, and there are infinitely many solutions if the radii are equal.

Remark: This problem is a special case of Problem 8(c), and it has a much simpler solution.

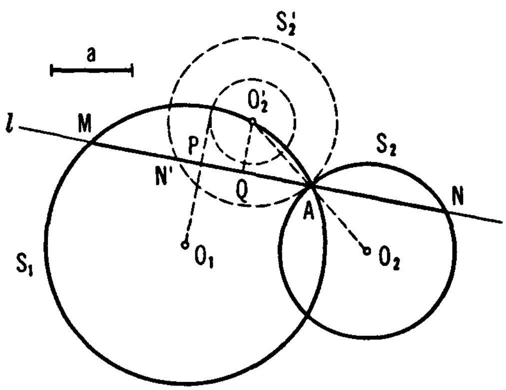


Figure 70b

(b) Draw the circle S_2' obtained from S_2 by a half turn about the point A. Assume that the problem has been solved and that the line MAN is the solution (Figure 70b). Let N' be the point where this line intersects the circle S_2'; then $MN' = a$. From the centers O_1 and O_2' of the circles S_1 and S_2', drop perpendiculars O_1P and $O_2'Q$ to the line MAN; then

$$PA = \tfrac{1}{2}MA, \qquad QA = \tfrac{1}{2}N'A$$

and

$$PQ = PA - QA = \tfrac{1}{2}(MA - N'A) = \tfrac{1}{2}a.$$

Thus the distance from the point O_2' to the line O_1P is equal to $\frac{1}{2}a$, that is, the line O_1P is tangent to the circle with center O_2' and radius $\frac{1}{2}a$. This enables us now to find the line O_1P without assuming that the solution to the whole problem is already known. Having found O_1P we can now easily construct $MAN \perp O_1P$.

There are at most two solutions to the problem.

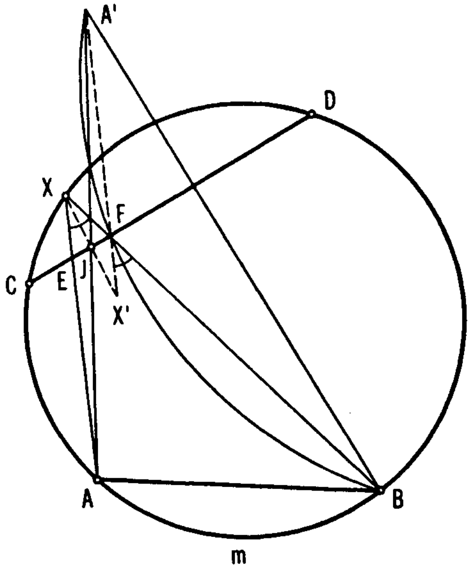


Figure 71

11. Assume that the problem has been solved (Figure 71), and let $A'X'$ be the segment obtained from AX by a half turn about the point J. Since AX passes through E, $A'X'$ will pass through F. Since $X'A' \parallel AX$, we see that

$$\sphericalangle X'FB = \sphericalangle AXB = \tfrac{1}{2}AmB;$$

therefore, $\sphericalangle A'FB = 180° - \sphericalangle X'FB$ and so we may regard

$$\sphericalangle A'FB = 180° - \tfrac{1}{2}AmB$$

as known.

Thus we have the following construction: Let A' be the point obtained from A by a half turn about J. On the segment $A'B$ construct the circle arc that subtends an angle of

$$180° - \tfrac{1}{2}AmB.$$

The point of intersection of this arc with the chord CD determines the point F, and the other intersection of the line BF with the circumference is the desired point X.

The problem has a unique solution; if one assumes that CD is intersected by the extensions of chords AX and BX, then there may be two solutions (cf. solution of Problem 6).

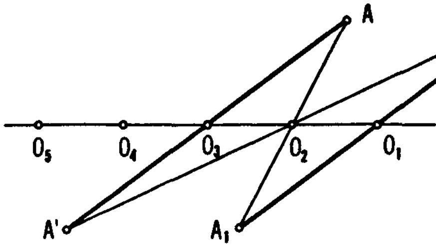


Figure 72

12. Assume that the figure F has two centers of symmetry, O_1 and O_2 (Figure 72). Then the point O_3, obtained from O_1 by a half turn about O_2 is also a center of symmetry of F. Indeed, if A is any point of F, then the points A_1, A_2, and A', where A_1 is obtained from A by a half turn about O_2, A_2 from A_1 by a half turn about O_1, and A' from A_2 by a half turn about O_2, will also be points of F (since O_1 and O_2 are centers of symmetry). But the point A' is also obtained from A by a half turn about O_3; indeed, the segments AO_3 and O_3A' are equal, parallel, and have opposite directions, since the pairs of segments AO_3 and A_1O_1, A_1O_1 and A_2O_1, A_2O_1 and $A'O_3$ are equal, parallel, and have opposite directions.

Thus if A is any point of F, then the symmetric point A' obtained from A by a half turn about O_3 is also a point of F, that is, O_3 is a center of symmetry of F.

Similarly one shows that the point O_4, obtained from O_2 by a half turn about O_3, and the point O_5, obtained from O_3 by a half turn about O_4, etc. are centers of symmetry. Thus we see that if the figure F has two distinct centers of symmetry then it has infinitely many.

13. (a) The segment A_nB_n is obtained from AB by n successive half turns about the points $O_1, O_2, \cdots, O_n$ (n even). But the sum of the half turns about O_1 and O_2 is a translation; the sum of the half turns about O_3 and O_4 is a translation; the sum of the half turns about O_5 and O_6 is a translation; $\cdots$; finally, the sum of the half turns about O_{n-1} and O_n is also a translation. Therefore A_nB_n is obtained from AB by $\frac{1}{2}n$ successive translations. Since any sum of translations is again a translation the segment A_nB_n is obtained from AB by a translation, and therefore $AA_n = BB_n$.

If n is odd the assertion of the problem is false, because the sum of an odd number of half turns is a translation plus a half turn, or, what is the same thing, is a half turn about some other point (see page 34); therefore, in general $AA_n \neq BB_n$ (although $AB_n = BA_n$).

(b) Since the sum of an odd number of half turns is a half turn [see the solution to Problem (a)], the point A_n obtained from A by the n successive half turns about the points $O_1, O_2, \cdots, O_n$ can also be obtained from A by a single half turn about some point O. The point A_{2n} is obtained from A_n by these same n half turns; therefore it can also be obtained from A_n by the single half turn about the point O. But this means that A_{2n} coincides with A.

If n is even then A_n is obtained from A by a translation, and A_{2n} is obtained from A_n by this same translation; therefore A_{2n} will not, in general, coincide with A. (It will coincide with A if this translation is the identity transformation, i.e., a translation through zero distance.[T])

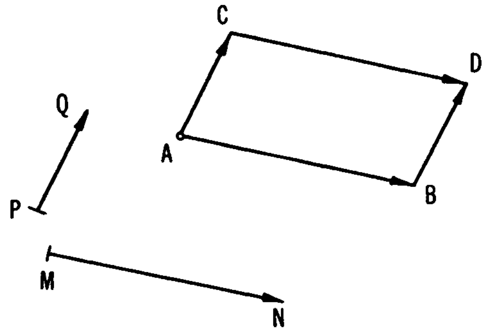


Figure 73

14. (a) The sum of the two half turns about the points O_1 and O_2 is a translation (see page 25) and the sum of the half turns about the points O_3 and O_4 is another translation (in general, different from the first). Thus the "first" point A_4 is obtained from A by performing two translations in succession; the "second" point (we denote it by A_4') is obtained from A by performing the same two translations *in the opposite order*. But *the sum of two translations is independent of the order in which they are performed*. (To prove this it is sufficient to consider Figure 73, where the points B and C are obtained from the point A by the translations indicated by the segments MN and PQ respectively. The point D is obtained from the point B by the translation PQ, and D is also obtained from C by the translation MN. From this the assertion of the theorem follows.)

(b) This problem is clearly the same as Problem 13(b) (for $n = 5$), since Problem 13(b) tells us that the point A_5, obtained from A by five successive half turns about the points O_1, O_2, O_3, O_4, O_5, is taken back into the point A by these same five half turns performed in the same order.

[T] This sentence was added in translation.

(c) Whenever n is odd, the final positions will be the same (see Problem 13).

[The two points obtained by the n half turns will also coincide in case $n = 2k$ is an even number and there exists a k-gon $M_1M_2\cdots M_k$, whose sides M_1M_2, M_2M_3, $\cdots$, M_kM_1 are equal to, parallel to, and have the same direction as the segments O_1O_2, O_3O_4, $\cdots$, $O_{n-1}O_n$ (in this case the sum of the n half turns about the points O_1, O_2, $\cdots$, O_n, carried out in either this order or in the reverse order, is a "translation through zero distance", that is, it is the identity transformation).]

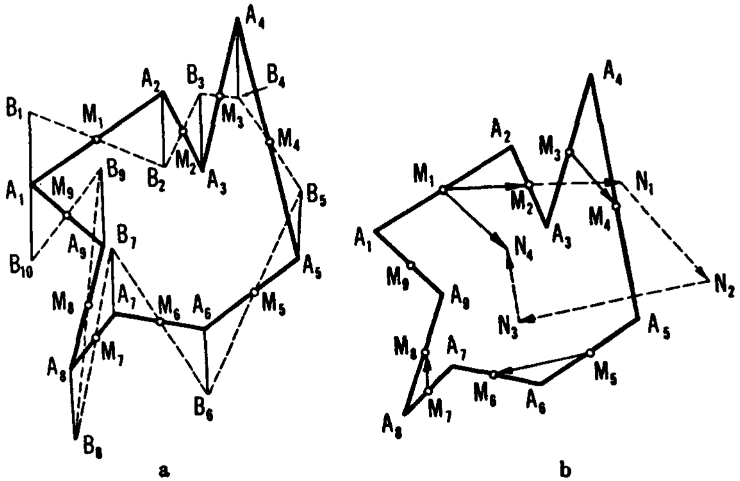


Figure 74

15. *First solution.* Assume that the problem has been solved and let

$$A_1A_2\cdots A_9$$

be the nine-gon, with M_1, M_2, $\cdots$, M_9 the centers of its sides (Figure 74a; here we are taking $n = 9$). Let B_1 be any point in the plane and let B_2 be obtained from it by a half turn about M_1. Let B_3 be obtained from B_2 by a half turn about M_2. Continue this until finally B_{10} is obtained from B_9 by a half turn about M_9. Since each of the segments A_2B_2, A_3B_3, $\cdots$, A_1B_{10} is obtained from the preceding one by a half turn, they are all parallel and have the same length, and each one has a direction opposite to the direction of the one before it. Therefore A_1B_1 and A_1B_{10} are equal and parallel and have opposite directions, which means that the point A_1 is the midpoint of the segment B_1B_{10}. This enables us to find A_1, since by starting with any point B_1 we can find B_{10}. The remaining vertices A_2, A_3, $\cdots$, A_9 are then found by successive half turns about M_1, M_2, $\cdots$, M_9.

The problem always has a unique solution; however, the nine-gon that is obtained need not be convex and may even intersect itself.

If n is even and if we repeat the same reasoning as before, i.e., if we assume that the problem has been solved, we see that A_1B_{n+1} and A_1B_1 are equal, parallel and have the same direction, that is, they coincide. Therefore if B_{n+1} does not coincide with B_1, then the problem has no solution. If B_{n+1} does coincide with B_1 then A_1B_1 will coincide with A_1B_{n+1} no matter where the point A_1 is chosen. In this case there are infinitely many solutions; any point in the plane can be taken for the vertex A_1.

Second solution. The vertex A_1 of the desired n-gon will be taken into itself by the sum of the half turns about the points $M_1, M_2, \cdots, M_n$, that is, A_1 is a fixed point of the sum of these n half turns (see Figure 74b) where the case $n = 9$ is shown). If n were even then the sum of n half turns would be a translation [see the solution to Problem 13(a)]. Since a translation has no fixed points, it follows that for n even the problem has, in general, no solution. The only exception occurs when the sum of the n half turns is the identity transformation (a translation through zero distance), which leaves all points in the plane fixed; in this case the problem has infinitely many solutions; any point in the plane can be taken for the vertex A_1.† If n is odd (for example, $n = 9$), then the sum of n half turns is a half turn. Since a half turn has exactly one fixed point, namely the center of symmetry, it follows that the vertex A_1 of the desired nine-gon must coincide with this center of symmetry; in this case the problem has a unique solution.

We now show how to construct the center of symmetry of the sum of the nine half turns about the points $M_1, M_2, \cdots, M_9$. The sum of the half turns about M_1 and M_2 is a translation in the direction M_1M_2 through a distance $2M_1M_2$; the sum of the half turns about M_3 and M_4 is a translation in the direction M_3M_4 through a distance $2M_3M_4$, etc. Thus the sum of the first eight half turns is the same as the sum of the four translations in the directions M_1M_2 (or M_1N_1), M_3M_4 ($\parallel N_1N_2$), M_5M_6 ($\parallel N_2N_3$) and M_7M_8 ($\parallel N_3N_4$) through distances $2M_1M_2$ ($= M_1N_1$), $2M_3M_4$ ($= N_1N_2$), $2M_5M_6$ ($= N_2N_3$), and $2M_7M_8$ ($= N_3N_4$) respectively (see Figure 74b), which is a single translation in the direction M_1N_4 through a distance M_1N_4. The point A_1 is the center of symmetry of the half turn that is the sum of a translation in the direction M_1N_4 through a distance M_1N_4 and a half turn about the point M_9. To find A_1 it is sufficient to lay off a segment M_9A_1 starting at M_9, parallel to N_4M_1 and of length $\frac{1}{2}M_1N_4$ (Figure 74b; compare this with Figure 18). Having found A_1, we have no difficulty in finding the remaining vertices of the nine-gon.

† See the note at the end of the solution of Problem 16(b) for a discussion of the conditions that the points $M_1, M_2, \cdots, M_n$ must satisfy in this case.

16. (a) If M, N, P, and Q are the midpoints of the sides of the quadrilateral $ABCD$ (see Figure 22a), then four half turns performed in succession about the points M, N, P, and Q will carry the point A into itself (compare with the solution to Problem 15). Now this is possible only in case the sum of the four half turns about the points M, N, P, and Q, which is equal to the sum of two translations in the directions MN and PQ through distances $2MN$ and $2PQ$ respectively, is the identity transformation. But this means that the segments MN and PQ are parallel, equal in length and oppositely directed, that is, the quadrilateral $MNPQ$ is a parallelogram.

(b) Just as in part (a), we conclude that the sum of the translations in the directions M_1M_2, M_3M_4, and M_5M_6 through distances $2M_1M_2$, $2M_3M_4$, and $2M_5M_6$ is the identity transformation. Therefore there is a triangle whose sides are parallel to M_1M_2, M_3M_4, and M_5M_6, and equal to $2M_1M_2$, $2M_3M_4$, and $2M_5M_6$; but this means that there is also a triangle whose sides are parallel to and have the same lengths as the segments M_1M_2, M_3M_4, M_5M_6.

In the same way one proves that there exists a triangle whose sides are parallel to, and have the same lengths as the segments M_2M_3, M_4M_5, M_6M_1.

Remark: Using the same method that was used in the solution of Problem 16(b) one can show that a set of $2n$ points M_1, M_2, $\cdots$, M_{2n} will be the midpoints of the sides of some $2n$-gon if and only if there exists an n-gon whose sides are parallel to and have the same lengths as the segments M_1M_2, M_3M_4, $\cdots$, $M_{2n-1}M_{2n}$; there will then also exist an n-gon whose sides are parallel to and have the same lengths as M_2M_3, M_4M_5, $\cdots$, $M_{2n}M_1$.

17. Rotate the line l_1 about the point A through an angle α, and let l_1' denote the new position of the line. Let M be the point of intersection of l_1' with the line l_2 (Figure 75). The circle having its center at A and passing through the point M will solve the problem, since the point of intersection M' of this circle with the line l_1 is taken into the point M by our rotation (that is, the central angle $MAM' = \alpha$).

The problem has two solutions (corresponding to rotations in the two directions), provided that neither of the angles between the lines l_1 and l_2 is equal to α; it has either exactly one solution or infinitely many solutions if one of the angles between the lines l_1 and l_2 is equal to α; it has either no solutions at all or infinitely many solutions if l_1 and l_2 are perpendicular and $\alpha = 90°$.

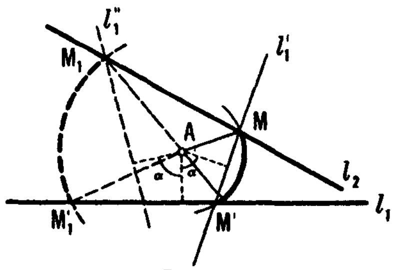


Figure 75

18. Assume that the problem has been solved and let ABC be the desired triangle whose vertices lie on the given lines l_1, l_2, and l_3 (Figure 76). Rotate the line l_2 about the point A through an angle of 60° in the direction from B to C; this will carry the point B into the point C.

Thus we have the following construction: Choose an arbitrary point A on the line l_1 and rotate l_2 about A through an angle of 60°. The point of intersection of the new line l_2' with l_3 is the vertex C of the desired triangle. The problem has two solutions since l_2 can be rotated through 60° in either of two directions; however, these two solutions are congruent.

The problem of constructing an equilateral triangle whose vertices lie on three given concentric circles is solved analogously.

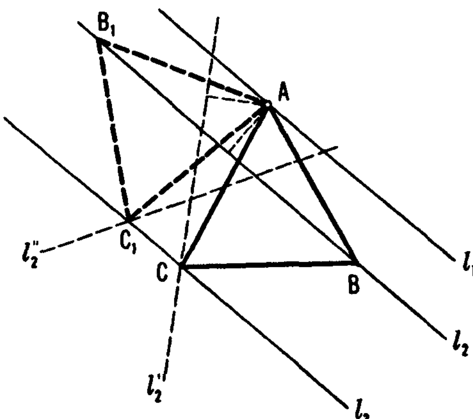


Figure 76

Remark: If we had chosen a different point A' instead of A on the line l_1, then the new figure would be obtained from Figure 76 by an isometry (more precisely, by a translation in the direction l_1 through a distance AA'). But in geometry we do not distinguish between such figures (see the introduction). For this reason we do not consider that the solution to the problem depends on the position of the point A on l_1. If the three lines l_1, l_2, and l_3 were not parallel, then the problem would be solved in exactly the same way; however now we would have to allow infinitely many different solutions corresponding to the different ways of choosing a point A on the line l_1 (since the triangles obtained would no longer be congruent).

In exactly the same way the problem of constructing an equilateral triangle ABC whose vertices lie on three concentric circles S_1, S_2, and S_3 can have at most four solutions (here the figures obtained by different choices of the point A on the circle S_1 will also be the same—they are all obtained from one another by a rotation about the common center of the three circles S_1, S_2, and S_3). On the other hand, if the circles S_1, S_2, and S_3 are not concentric, then the problem will have infinitely many solutions (different choices of the point A on the circle S_1 will correspond to essentially different solutions).

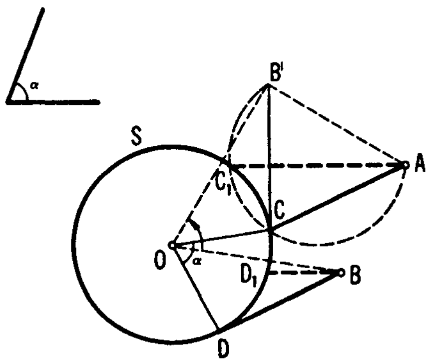


Figure 77

19. Let us assume that the arc CD has been found (Figure 77). Rotate the segment BD about the center O of the circle S through an angle α; it will be taken into a new segment $B'C$ that makes an angle $ACB' = \alpha$ with the segment AC.

Thus we have the following construction: Rotate the point B about O through an angle α into a new position B'. Through the points A and B' pass a circular arc subtending an angle α (that is, if C is any point on the circular arc, then $\measuredangle ACB' = \alpha$). The intersection of this circular arc with the circle S determines the point C.

The problem can have up to four solutions (the arc can meet the circle in two points, and the point B can be rotated about the point O in two directions).

20. Assume that the problem has been solved. Rotate the circle S_1 about A through an angle α into the position S_1' (Figure 78). The circles S_2 and S_1' will cut off equal chords on the line l_2. Thus the problem has

been reduced to Problem 8(c). In other words, a line l_2 must be passed through A so that it cuts off equal chords on S_1' and S_2. Then l_1 can be obtained by a rotation of l_2 about A through an angle α, and S_1 will cut the desired segment from l_1.

The problem can have up to four solutions. [Since S_1 can be rotated about A in either of two directions, there are two ways of reducing the problem to Problem 8(c) which, in turn may have two solutions.]

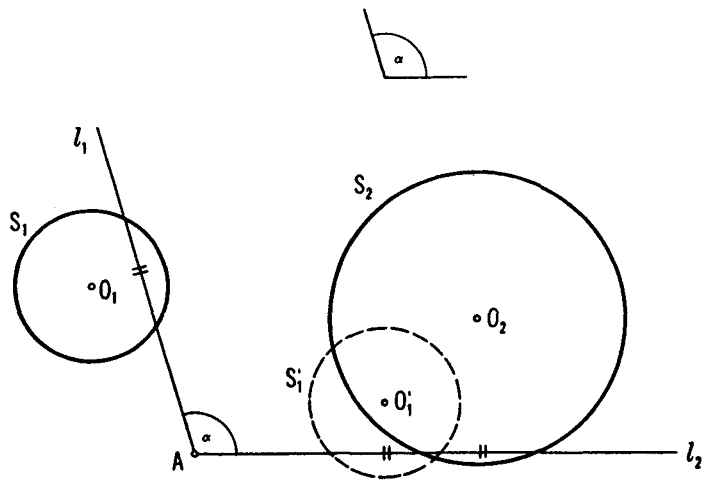


Figure 78

21. *First solution* (compare with the first solution of Problem 15). Assume that the problem has been solved and that $A_1A_2\cdots A_n$ is the desired n-gon (see Figure 79, where $n = 6$). Choose an arbitrary point B_1 in the plane. The sequence of rotations, first about M_1 through an angle α_1, then about M_2 through an angle α_2, etc., and finally about M_n through an angle α_n carries the segment A_1B_1 first into a segment A_2B_2, then carries A_2B_2 into a segment A_3B_3, $\cdots$ and finally carries A_nB_n into A_1B_{n+1}. All these segments are equal and therefore the vertex A_1 of the n-gon is equidistant from the points B_1 and B_{n+1} (where B_{n+1} is obtained from B_1 by these n rotations). Now choose a second point C_1 in the plane, and rotate it successively about the points $M_1, M_2, \cdots, M_n$ through angles $\alpha_1, \alpha_2, \cdots, \alpha_n$. Thus we obtain a second pair of points C_1 and C_{n+1} equidistant from A_1. Thus the vertex A_1 of the n-gon can be found as the intersection of the perpendicular bisectors of the segments B_1B_{n+1} and C_1C_{n+1}. Having found A_1 we obtain A_2 by rotating A_1 about M_1 through an angle α_1; A_3 is obtained by rotating A_2 about M_2 through an

angle α_2, etc. The problem has a unique solution provided that the perpendicular bisectors to B_1B_{n+1} and to C_1C_{n+1} do intersect (that is, the segments B_1B_{n+1} and C_1C_{n+1} are not parallel). If the perpendicular bisectors are parallel then the problem has no solution, and if they coincide then the problem has infinitely many solutions.

The polygon obtained as the solution to the problem need not be convex and may even intersect itself.

Second solution (compare with the second solution of Problem 15). The vertex A_1 is a fixed point of the sum of the n rotations with centers $M_1, M_2, \cdots, M_n$ and angles $\alpha_1, \alpha_2, \cdots, \alpha_n$ (these rotations take A_1 into A_2, A_2 into A_3, A_3 into A_4, etc. and, finally, A_n into A_1). But the sum of n rotations through the angles $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a rotation through the angle

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

provided that $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ is not a multiple of 360°; it is a translation otherwise (this follows from the theorem on the sum of two rotations). The only fixed point of a rotation is the center of rotation. Therefore if

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n$$

is not a multiple of 360°, then A_1 is found as the center of the rotation, that is, the sum of the rotations about the points $M_1, M_2, \cdots, M_n$ through angles $\alpha_1, \alpha_2, \cdots, \alpha_n$. Actually to find A_1 we may apply repeatedly the method given in the text to find the center of the sum of two rotations.†

A translation has no fixed points whatever. Therefore if

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n$$

is a multiple of 360° then the problem has no solution in general. However, in the special case when the sum of the rotations about the points $M_1, M_2, \cdots, M_n$ through the angles $\alpha_1, \alpha_2, \cdots, \alpha_n$ (where the sum $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ is a multiple of 360°) is the identity transformation, the problem has infinitely many solutions (any point in the plane may be chosen for the vertex A_1).

Thus, if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 180°$ (this is the case considered in Problem 15), the problem has a unique solution when n is odd and has no solution or has infinitely many solutions when n is even.

† It may happen that in the construction we shall have to find the center of a rotation that is the sum of a translation and a rotation. In this connection one should consult the text in fine print on page 36 or on page 51.

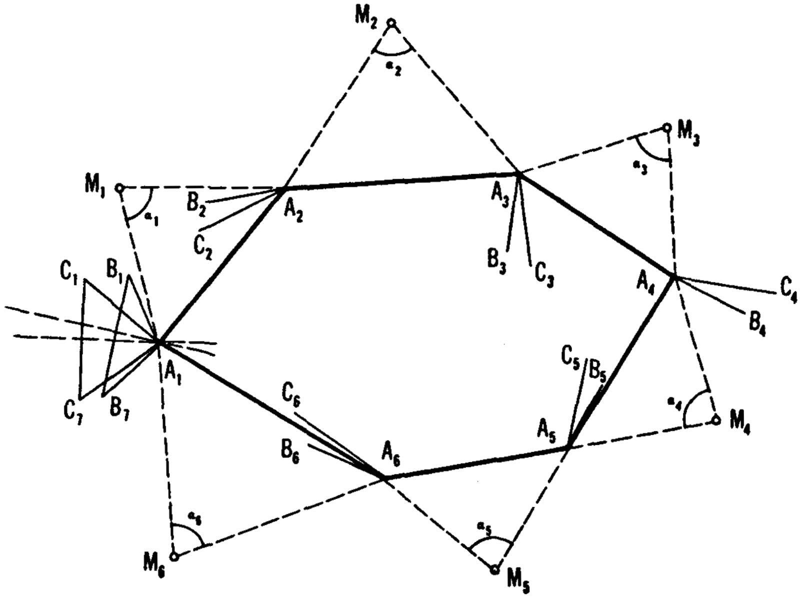


Figure 79

22. (a) Consider the sequence of three rotations, each through 120°, about the points O_1, O_2, O_3 (see Figure 31 in the text). The first of these rotations carries A into B, the second carries B into C, and the third carries C into A.

Thus the point A is a fixed point of the sum of these three rotations. But the sum of three rotations through 120° is, in general, a translation, and therefore has no fixed points. From the fact that A is a fixed point we see that the sum of these three rotations must be the identity transformation (translation through zero distance). The sum of the first two rotations is a rotation through 240° about the point O of intersection of two lines, one through O_1 and the other through O_2, each making an angle of 60° with O_1O_2. Therefore the triangle O_1O_2O is equilateral. Since the sum of this rotation and the rotation about O_3 through 120° is the identity transformation, the point O must coincide with O_3. Thus the triangle $O_1O_2O_3$ is equilateral, which was to be proved.

In the same way one can show that the centers O_1', O_2', O_3' of the equilateral triangles constructed on the sides of the given triangle ABC, but lying towards the interior of ABC, also form an equilateral triangle (Figure 80).

(b) The solution to this problem is similar to that of (a). Since the point A is taken into itself by the sum of the three rotations through angles β, α, and γ $(\alpha + \beta + \gamma = 360°)$ about the centers B_1, A_1, and C_1, we see that the sum of these rotations is the identity transformation. But this is possible only if C_1 coincides with the center of the rotation which is the sum of the two rotations through angles β and α about the centers B_1 and A_1, that is, if C_1 is the point of intersection of the two lines through B_1 and A_1 that make angles $\frac{1}{2}\beta$ and $\frac{1}{2}\alpha$ with the line B_1A_1. From this the assertion of the problem follows.

In the same way it can be shown that the vertices A_1', B_1', C_1' of the isosceles triangles ABC_1', BCA_1', and ACB_1' with vertex angles α, β, and γ, respectively, $(\alpha + \beta + \gamma = 360°)$ constructed on the sides of the given triangle ABC but lying towards the interior of ABC, also form a triangle with angles $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, $\frac{1}{2}\gamma$.

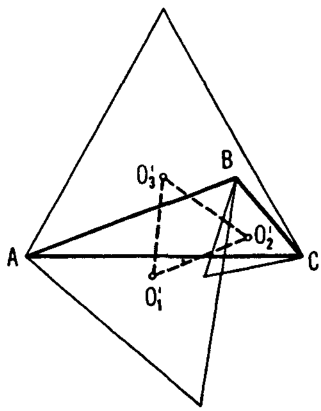


Figure 80

23. The sequence of three rotations in the same direction through angles of 60°, 60°, and 240° about the points A_1, B_1, and M takes the point B into itself (see Figure 32 in the text). Therefore the sum of these three rotations is the identity transformation, and thus the sum of the first two rotations is a rotation with center M. From this the assertion of the problem follows. (Compare with the solution to Problem 22.)

24. (a) The sum of the four rotations with centers M_1, M_2, M_3, and M_4, each through an angle of 60°, where the direction of the first and third rotations is opposite to that of the second and fourth, carries the

vertex A of the quadrilateral into itself (see Figure 33a, in the text). But the sum of the two rotations about M_1 and M_2 is a translation given by the segment M_1M_1', where M_1' is a vertex of the equilateral triangle $M_1M_2M_1'$ ($M_2M_1 = M_2M_1'$, $\measuredangle M_1M_2M_1' = 60°$, and the direction of rotation from M_2M_1 to M_2M_1' coincides with the direction of rotation from M_2B to M_2C; see Figure 81a, and Figure 28b in the text). Similarly the sum of the rotations about M_3 and M_4 is a translation given by the segment M_3M_3', where triangle $M_3M_4M_3'$ is equilateral (and the direction of rotation from M_4M_3 to M_4M_3' is the same as the direction of rotation from M_4D to M_4A). Thus the sum of two translations—given by the segments M_1M_1' and M_3M_3'—carries the point A into itself. But if the sum of two translations leaves even one point fixed, then this sum must be the identity transformation, that is, the two segments that determine the two translations must be equal, parallel, and oppositely directed. But if the equilateral triangles $M_1M_2M_1'$ and $M_3M_4M_3'$ are so situated that

$$M_1M_1' = M_3M_3', \qquad M_1M_1' \parallel M_3M_3'$$

and if M_1M_1' and M_3M_3' are oppositely directed, then the sides M_1M_2 and M_3M_4 are also equal, parallel, and oppositely directed, from which it follows that the quadrilateral $M_1M_2M_3M_4$ is a parallelogram (see Figure 81a).

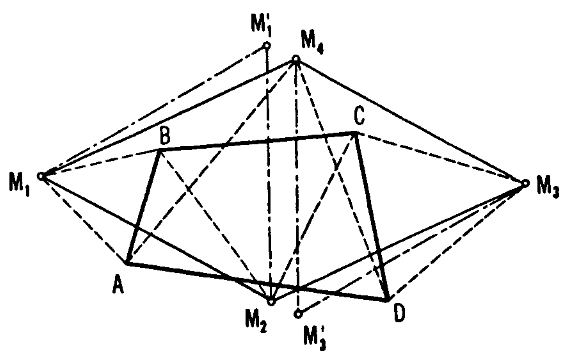


Figure 81a

(b) The sum of the four rotations about the points M_1, M_2, M_3, and M_4, each through an angle of 90°, clearly carries the vertex A of the quadrilateral into itself. It follows that this sum of four rotations is the identity transformation [compare the solution of Problem (a)]. But the sum of the rotations about M_1 and M_2 is a half turn about a point O_1—the vertex of an isosceles right triangle $O_1M_1M_2$ (since

$$\sphericalangle O_1M_1M_2 = \sphericalangle O_1M_2M_1 = 45°;$$

compare Figure 81b with Figure 28a in the text). Similarly the sum of rotations about M_3 and M_4 is a half turn about the vertex O_2 of an isosceles right triangle $O_2M_3M_4$. From the fact that the sum of the half turns about O_1 and O_2 is the identity transformation it clearly follows that these two points coincide. But this means that triangle $O_1M_1M_3$ is obtained from triangle $O_1M_2M_4$ by a rotation through 90° about the point $O_1 = O_2$, and therefore the segments M_1M_3 and M_2M_4 are equal and perpendicular.

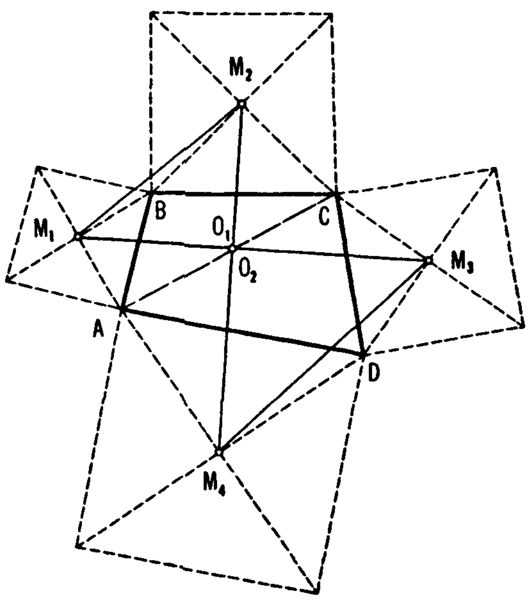


Figure 81b

(c) By what has already been proved [see the solution to Problem (b)], the diagonals M_1M_3 and M_2M_4 of the quadrilateral $M_1M_2M_3M_4$ are equal and mutually perpendicular. Further, since the point O of intersection of the diagonals of the parallelogram $ABCD$ is its center of symmetry, it is also the center of symmetry for all of Figure 81c, and in particular it is the center of symmetry for the quadrilateral $M_1M_2M_3M_4$ (which must, therefore, be a parallelogram—since the parallelogram is the only quadrilateral that has a center of symmetry). But a parallelogram whose diagonals are equal and perpendicular must be a square.

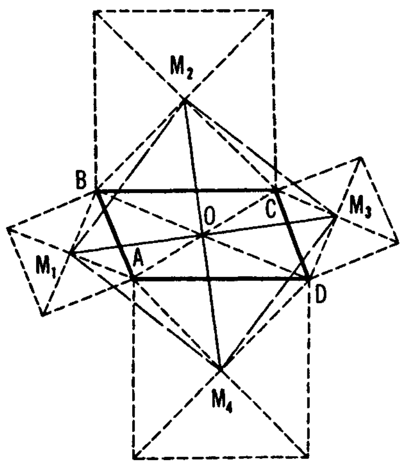


Figure 81c

In the same way it can be shown that if the four squares are constructed in the interior of the parallelogram, then their centers again form a square (Figure 81d).

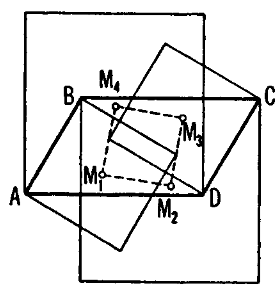


Figure 81d

Chapter Two. Symmetry

25. (a) Let us assume that the point X has been found, that is, that

$$\sphericalangle AXM = \sphericalangle BXN$$

(Figure 82a). Let B' be the image of B in the line MN; then

$$\sphericalangle B'XN = \sphericalangle BXN = \sphericalangle AXM,$$

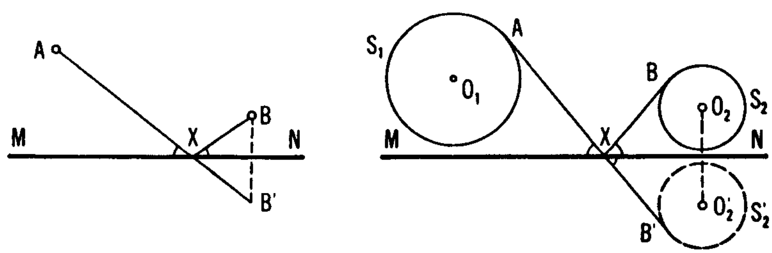
that is, the points A, X, B' lie on a line. From this it follows that X is the point of intersection of the lines MN and AB'.

(b) Let us assume that the point X has been found and let S_2' be the image of the circle S_2 in the line MN (Figure 82b).

If XA, XB, and XB' are tangents from the point X to the circles S_1, S_2, and S_2' then

$$\sphericalangle B'XN = \sphericalangle BXN = \sphericalangle AXM,$$

that is, the points A, X, and B' lie on a line. Therefore X is the point of intersection of the line MN with the common tangent line AB' to the circles S_1 and S_2'. The problem can have at most four solutions (there are at most four common tangents to two circles).



a b

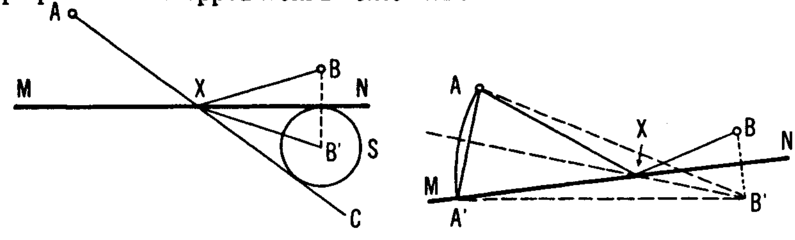
Figure 82

(c) *First solution.* Assume X has been found. Let B' be the image of B in MN and let XC be the continuation of the segment AX past the point X (Figure 83a). Then

$$\sphericalangle CXN = 2\sphericalangle BXN = 2\sphericalangle B'XN,$$

and therefore the ray XB' bisects the angle NXC. Thus the line AXC is tangent to the circle S with center B' that is tangent to MN; consequently, the point X is the intersection of the line MN and the tangent from A to the circle S.

Second solution. Again, assume X has been found. Let A' be the image of A in the line $B'X$ (we are using the same notation as in the first solution). $B'X$ bisects the angle AXM; therefore A' lies on the line XM and $B'A = B'A'$ (Figure 83b). Thus A' can be found as the intersection of the line MN with the arc of a circle with center B' and radius $B'A$. The point X is now obtained as the intersection of the line MN with the perpendicular dropped from B' onto AA'.



a b

Figure 83

26. (a) Assume that the triangle ABC has been constructed, with l_2 bisecting angle B and l_3 bisecting angle C (Figure 84a). Then the lines BA and BC are images of each other in l_2, and the lines BC and AC are images of each other in l_3, and therefore the points A' and A'' obtained from A by reflection in the lines l_2 and l_3 lie on the line BC.

Thus we have the following construction: Reflect the point A in the lines l_2 and l_3 to obtain the points A' and A''. The vertices B and C are the points of intersection of the line $A'A''$ with the lines l_2 and l_3.

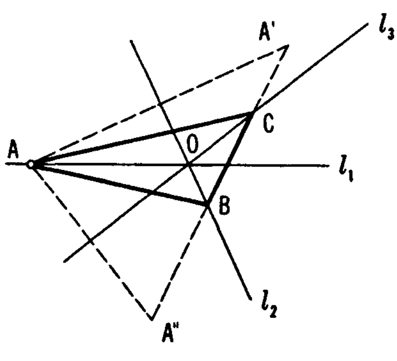


Figure 84a

If l_2 and l_3 are perpendicular, then the line $A'A''$ passes through the point of intersection of the three given lines and the problem has no solution; if l_1 is perpendicular to one of the lines l_2 and l_3, then $A'A''$ will be parallel to the other line and again the problem will have no solution. In case no two of the three given lines are perpendicular, the problem has a unique solution; however only in case each of the three given lines is included in the obtuse angle formed by the other two will the three lines bisect the *interior* angles of the triangle ABC; if, for example, l_1 is included in the acute angle formed by l_2 and l_3, then these last two lines bisect the exterior angles of the triangle (Figure 84b). We leave the proof of this statement to the reader.

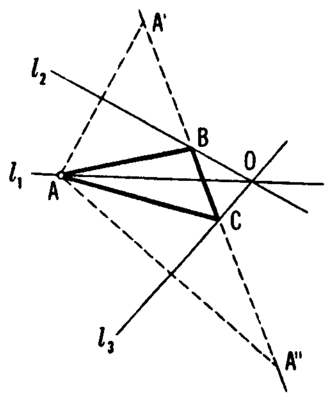


Figure 84b

(b) Choose an arbitrary point A' on one of the lines and construct the triangle $A'B'C'$ having the lines l_1, l_2, and l_3 as bisectors of its interior angles [see part (a) of this problem]. Construct tangents to S parallel to the sides of triangle $A'B'C'$ (Figure 85). The triangle thus obtained is the solution to the problem. The problem has a unique solution if each of the three lines l_1, l_2, l_3 is included in the obtuse angle formed by the other two; if one of them is included in the acute angle formed by the other two then the given circle will be an *escribed circle* or *excircle*[T] of the triangle.

[T] Every triangle has an inscribed circle or incircle and three excircles. Each excircle is tangent to the extensions of two of the sides of the triangle and to the third side (externally). The center of each excircle is the point of intersection of an internal angle bisector and the bisectors of the exterior angles at the other two vertices.

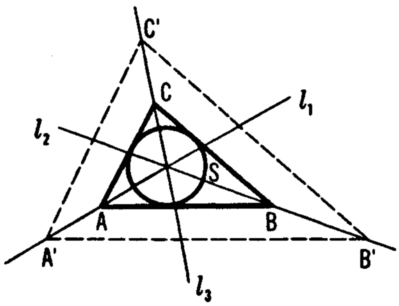


Figure 85

(c) Let us assume that the triangle ABC has been found (Figure 86). Since the point A is the image of the point B in the line l_2, it must lie on the line that is the image of BC in l_2; and since A is the image of C in l_3, it must also lie on the line that is the image of BC in l_3.

Thus we have the following construction: Pass a line m through A_1 perpendicular to l_1. Then construct the lines m' and m'' obtained from m by reflection in the lines l_2 and l_3. The point of intersection of m' and m'' will be the vertex A of the desired triangle; the vertices B and C are the images of this vertex in the lines l_2 and l_3 (Figure 86).

If the lines l_2 and l_3 are perpendicular, then either the lines m' and m'', obtained from m by reflection in l_2 and l_3, will be parallel (provided that the point A_1 does not coincide with the point O of intersection of the three lines l_1, l_2, and l_3) or they will coincide (if A_1 coincides with O). In the first case the problem has no solution, while in the second the solution is not determined uniquely. In all other cases the solution is unique.

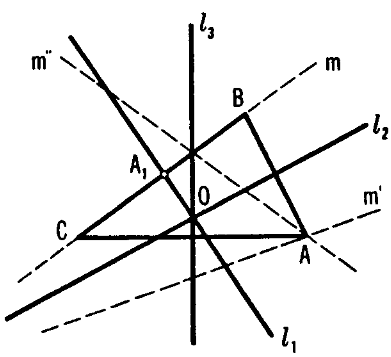


Figure 86

27. (a) Assume that the problem has been solved. Pass a line MN through the vertex C parallel to AB, and let B' be the image of B in the line MN (Figure 87). Let α and β be the angles at the base AB (we shall assume that $\alpha > \beta$). Then

$$\measuredangle ACN = 180° - \alpha, \qquad \measuredangle B'CN = \measuredangle BCN = \beta;$$

$$\measuredangle ACB' = (180° - \alpha) + \beta = 180° - (\alpha - \beta) = 180° - \gamma.$$

Thus we have the following construction: Lay off the segment $AB = a$, and construct a parallel line MN at a distance h from AB. Let B' be the image of B in the line MN. On the segment AB' construct the arc that subtends an angle of $180° - \gamma$. The point of intersection of this arc with the line MN is the vertex C of the triangle. The problem has a unique solution.

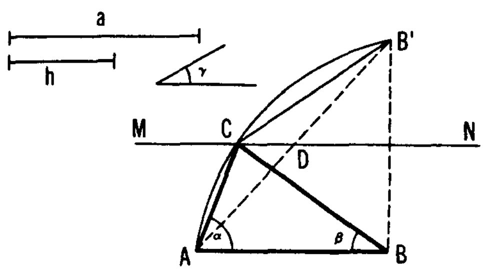


Figure 87

(b) Assume that the problem has been solved and determine the line MN and the point B' as in part (a) (Figure 87).

Since

$$\measuredangle ACB' = 180° - \gamma,$$

we can construct the triangle ACB' from the two sides AC and $CB' = BC$ and their included angle $180° - \gamma$. MN coincides with the median CD of this triangle (because MN is a "midline" of triangle ABB', that is, MN is parallel to the base AB and midway between this base and the opposite vertex B'). Finally, the vertex B is obtained as the image of B' in the line MN. The problem has a unique solution.

28. Assume that the problem has been solved and let B' be the image of B in OM (Figure 88). We have:

$$\measuredangle B'XA = \measuredangle B'XB + \measuredangle YXZ;$$

but

$$\sphericalangle B'XB = 2\sphericalangle OXZ = 2(\sphericalangle XZY - \sphericalangle MON)$$

(because $\sphericalangle XZY$ is an exterior angle of triangle XOZ). Consequently

$$\begin{aligned}\sphericalangle B'XA &= 2\sphericalangle XZY - 2\sphericalangle MON + \sphericalangle YXZ \\ &= \sphericalangle XZY + \sphericalangle XYZ + \sphericalangle YXZ - 2\sphericalangle MON \\ &= 180° - 2\sphericalangle MON.\end{aligned}$$

Thus $\sphericalangle B'XA$ is known. Now X can be found as the point of intersection of the ray OM with the arc, constructed on the chord AB', that subtends an angle equal to $180° - 2\sphericalangle MON$. The problem has a unique solution.

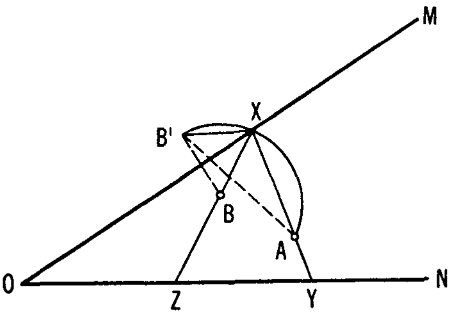


Figure 88

29. (a) Assume that the quadrilateral $ABCD$ has been constructed and let B' be the image of B in the diagonal AC (Figure 89). Since $\sphericalangle BAC = \sphericalangle DAC$ the point B' lies on the line AD. The three sides of the triangle $B'DC$ are known:

$$DC, \quad B'C = BC, \quad \text{and} \quad DB' = AD - AB' = AD - AB.$$

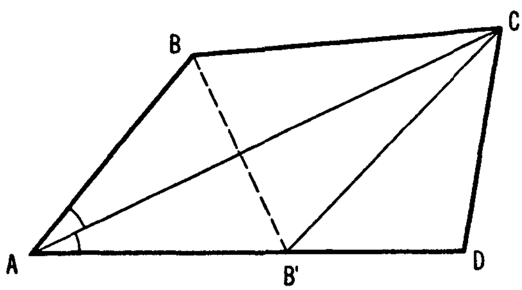


Figure 89

Construct this triangle, and locate the vertex A (this can be done since the distance AD is known). The vertex B is then obtained as the image of B' in the line AC. The problem has a unique solution if $AD \neq AB$; it has no solution whatsoever if $AD = AB$ and $CD \neq CB$; it has more than one solution if $AD = AB$ and $CD = CB$.

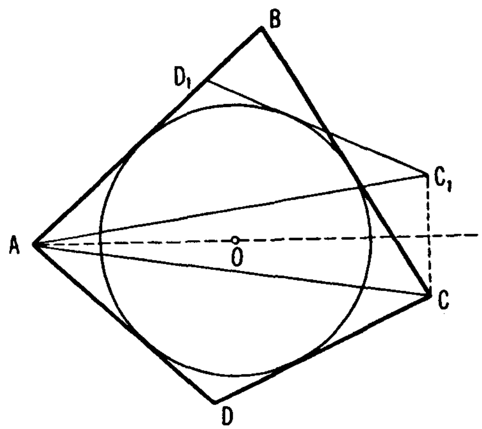


Figure 90

(b) Assume that the problem has been solved (Figure 90), and let triangle AD_1C_1 be the image of triangle ADC in the line AO (O is the center of the circle inscribed in the quadrilateral). Clearly the point D_1 lies on the line AB, and the side D_1C_1 is tangent to the circle inscribed in the quadrilateral $ABCD$.

Thus we have the following construction: On an arbitrary line lay off the segments AB and $AD_1 = AD$. Since $\angle ABC$ and $\angle AD_1C_1 = \angle ADC$ are known, we can find the lines BC and D_1C_1 (although we do not yet know the positions of the points C and C_1 on these lines). Now we can construct the inscribed circle since it is tangent to the three lines AB, BC, and D_1C_1. Finally, the side AD and the line DC are obtained as the images of AD_1 and D_1C_1 by reflection in the line AO. (The point C is the intersection of line BC with the image of line D_1C_1.)

The problem has a unique solution if $\angle ADC \neq \angle ABC$; it has no solution at all if $\angle ADC = \angle ABC$, $AD \neq AB$; it has more than one solution if $\angle ADC = \angle ABC$, $AD = AB$.

30. (a) Assume that the problem has been solved, that is, that points $X_1, X_2, \cdots, X_n$ have been found on the lines $l_1, l_2, \cdots, l_n$ such that

$$AX_1X_2\cdots X_nB$$

is the path of a billiard ball (in Figure 91 the case $n = 3$ is represented). It is easy to see that the point X_n is the point of intersection of the line l_n with the line $X_{n-1}B_n$, where B_n is the image of B in l_n [see the solution to Problem 25(a)], that is, the points B_n, X_n, X_{n-1} lie on a line. But then the point X_{n-1} is the point of intersection of the line l_{n-1} with the line $X_{n-2}B_{n-1}$, where B_{n-1} is the image of B_n in l_{n-1}. Similarly one shows that the point X_{n-2} is the intersection of the lines l_{n-2} and $X_{n-3}B_{n-2}$, where B_{n-2} is the image of B_{n-1} in l_{n-2}; the point X_{n-3} is the intersection of the lines l_{n-3} and $X_{n-4}B_{n-3}$, where B_{n-3} is the image of B_{n-2} in l_{n-3}, and so forth.

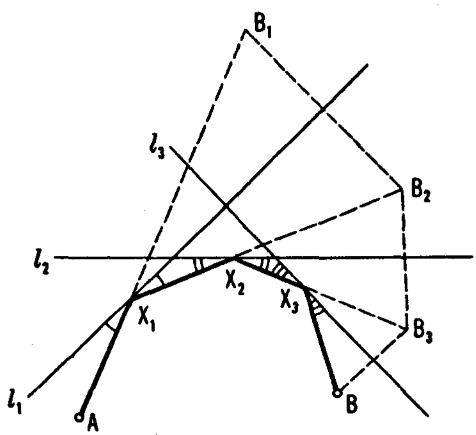


Figure 91

Thus we have the following construction: Reflect the point B in l_n, obtaining the point B_n; next reflect B_n in l_{n-1} to obtain B_{n-1}, and so forth, until the image B_1 of the point B_2 in line l_1 is obtained. The point X_1, that determines the direction in which the billiard ball at A must be hit, is obtained as the point of intersection of the line l_1 with the line AB_1. It is then easy to find the points $X_2, X_3, \cdots, X_n$ with the aid of the points $B_2, B_3, \cdots, B_n$ and X_1.

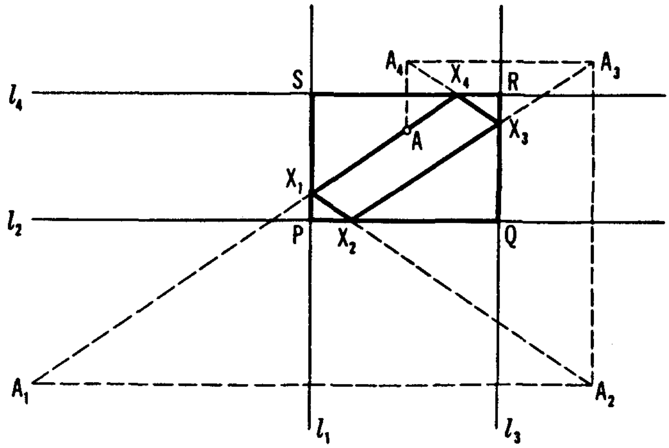


Figure 92

(b)[T] Following the procedure of part (a), we first reflect the point A in l_4 to obtain A_4, then reflect A_4 in l_3 to obtain A_3, and so forth until we reach A_1 (see Figure 92). It is easily verified that reflection in l_4 followed by reflection in l_3 is equivalent to a half turn about the point of intersection, R, of these two lines.[TT] Similarly, the next two reflections are equivalent to a half turn about the point P. Hence the four reflections are equivalent to the sum of two half turns, about R and P. But as we know (see Figure 17), this is equivalent to a translation in the direction PR through a distance of twice PR.

Thus AA_1 is parallel to, and twice as long as, the diagonal PR. By considering angles it is easy to see that the path $AX_1X_2X_3X_4A$ is a parallelogram (the opposite sides are parallel) with sides parallel to the diagonals. Thus if the ball is not stopped when it returns to the point A, it will describe exactly the same path a second time.

Finally, it can be seen from the figure that the total length of the path is equal to AA_1, that is, to twice the length of a diagonal.

31. (a) Let us assume that the problem has been solved. Draw the circle S_1 of center A and radius a, and the circle S_2 of center X and radius XB (Figure 93a). Clearly these two circles are tangent at a point lying on the line AX. Since S_2 passes through the point B, it must also

[T] This solution was inserted by the translator in place of the original solution.

[TT] See page 50.

pass through the point B', the image of B in the line l. Thus the problem has been reduced to the construction of a circle S_2, passing through two known points B and B' and tangent to a given circle S_1, that is, to Problem 49(b) of Vol. 2.[T] The center X of the circle S_2 is the desired point. This problem has at most two solutions; there may only be one or there may be none at all.

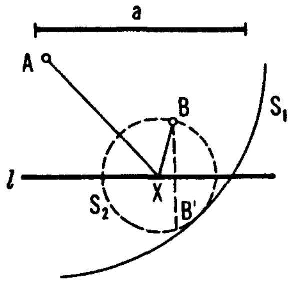


Figure 93a

(b) Assume that the problem has been solved, let S_1 be the circle of center A and radius a, and let S_2 be the circle of center X and radius BX (Figure 93b). The circles S_1 and S_2 are tangent at a point that lies on the line AX. In addition S_2 passes through the point B' that is the image of B in the line l. Therefore this problem is also reduced to Problem 49(b) of Vol. 2.[T] There are at most two solutions.

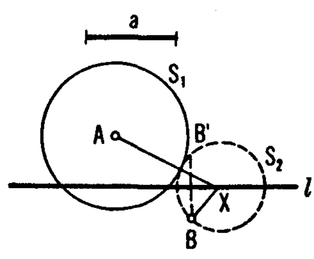


Figure 93b

[T] Since at this time Volume 2 is not available in English, we refer the reader to p. 175, Problem V of *College Geometry* by Nathan Altschiller-Court, Johnson Publishing Co., 1925, Richmond.

32. (a) Let H_1 be the image of H in the side BC (Figure 94). Let P, Q, R be the feet of the altitudes. We have

$$\sphericalangle BH_1C = \sphericalangle BHC \qquad (\text{because} \qquad \Delta BH_1C \cong \Delta BHC).$$

But

$$\sphericalangle BHC = \sphericalangle RHQ,$$

and

$$\sphericalangle RHQ + \sphericalangle RAQ = \sphericalangle BH_1C + \sphericalangle RAQ = 180°;$$

therefore $\sphericalangle BH_1C + \sphericalangle BAC = 180°$, and from this it follows that H_1 lies on the circle through the points A, B, C. The images of H in sides AB and AC can be treated in the same way.

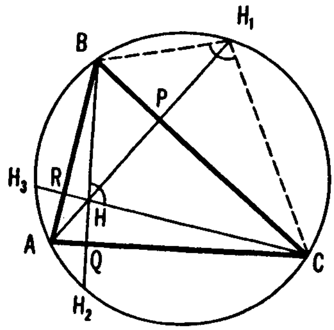


Figure 94

(b) Let us assume that triangle ABC has been constructed. The points H_1, H_2, and H_3 lie on the circumscribed circle [see Problem (a)]. Since

$$\sphericalangle BRC = \sphericalangle BQC (= 90°)$$

and $\sphericalangle BHR = \sphericalangle CHQ$, it follows that $\sphericalangle RBH = \sphericalangle QCH$, that is, arc AH_3 is equal to arc AH_2. Similarly one shows that arcs BH_1 and BH_3 are equal, and that arcs CH_1 and CH_2 are equal. From this it follows that the vertices A, B, and C of the triangle are the midpoints of the arcs H_2H_3, H_3H_1, and H_1H_2 of the circle through the three points H_1, H_2, and H_3. The problem has a unique solution unless the points H_1, H_2, and H_3 lie on a straight line, in which case there is no solution at all.

33. (a) Clearly; for example, the altitudes of triangle $A_2A_3A_4$ are the lines

$$A_1A_4 \perp A_2A_3, \qquad A_1A_3 \perp A_2A_4, \qquad \text{and} \qquad A_1A_2 \perp A_3A_4;$$

the point of intersection of these altitudes is the point A_1.

(b) Let A_4' be the image of A_4 in the line A_2A_3 (Figure 95). This point lies on the circle S_4, circumscribed about triangle $A_1A_2A_3$ [see Problem 32(a)]. Thus the circle circumscribed about triangle $A_2A_4'A_3$ coincides with S_4; from this it follows that the circle S_1, circumscribed about triangle $A_2A_3A_4$, is congruent to S_4 (S_1 and S_4 are images of each other in the line A_2A_3). Similarly one shows that the circles S_2 and S_3 are also congruent to S_4.

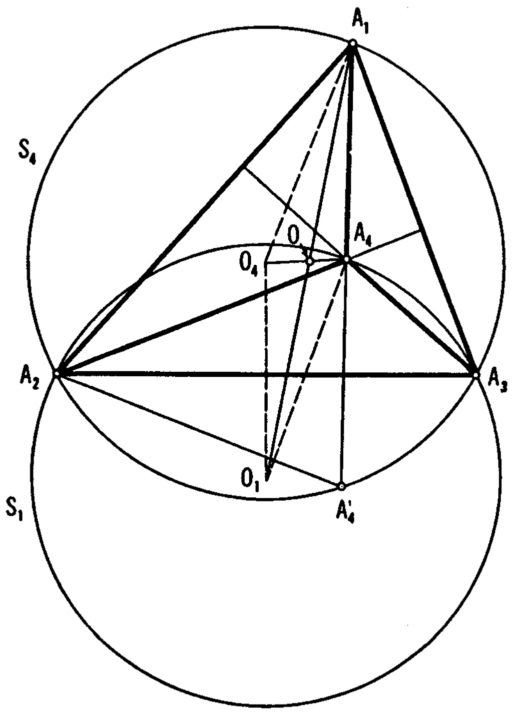


Figure 95

(c) At least one of the triangles $A_1A_2A_3$, $A_1A_2A_4$, $A_1A_3A_4$, and $A_2A_3A_4$ must be acute angled; indeed, if triangle $A_2A_3A_4$ has an obtuse angle at A_4, then triangle $A_2A_3A_1$ (where A_1 is the point of intersection of the altitudes of triangle $A_2A_3A_4$) will be acute. Thus we shall assume that triangle $A_1A_2A_3$ is acute and that the point A_4 lies inside it.

Consider the quadrilateral $A_1A_4O_1O_4$. The points O_1 and O_4 are centers of circles S_1 and S_4 that are images of each other in the line A_2A_3 [see Figure 95 and the solution to part (b) of this problem]. Therefore O_1 and O_4 are images of each other in A_2A_3, and so $O_1O_4 \perp A_2A_3$. In the quadrilateral $A_1A_4O_1O_4$ we thus have

$$O_4O_1 \parallel A_1A_4 \qquad \text{and} \qquad O_1A_4 = O_4A_1 = R$$

(where R is the radius of the circles S_1, S_2, S_3, and S_4). Therefore this quadrilateral is either a parallelogram or an isosceles trapezoid. But it cannot be an isosceles trapezoid because the perpendicular bisector A_2A_3 of side O_4O_1 does not meet side A_1A_4. Hence $A_1A_4O_1O_4$ is a parallelogram and its diagonals A_1O_1, A_4O_4 meet in a point O that is the midpoint of each of them. In the same way one shows that O is the midpoint of A_2O_2 and of A_3O_3.

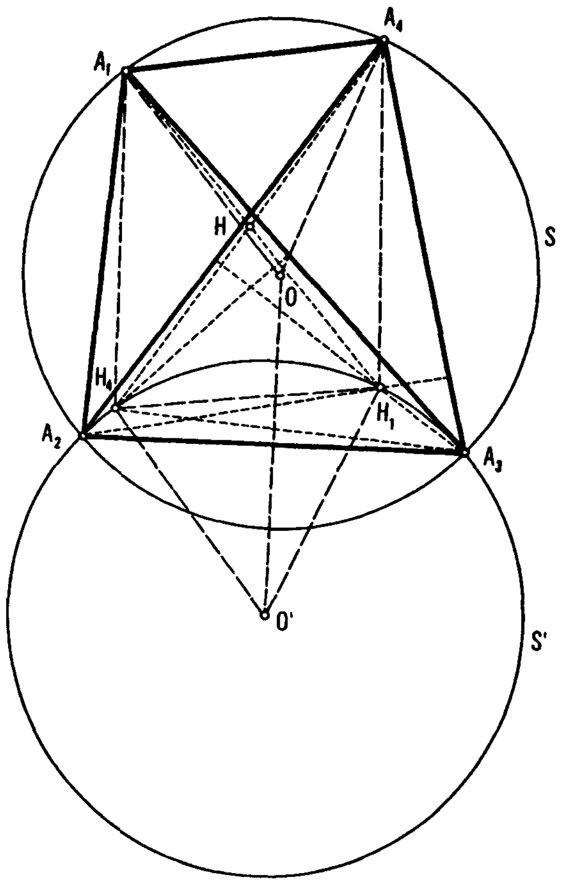


Figure 96

34. (a) Let O' be the image of the center O of the circle S in the line A_2A_3 (Figure 96). The quadrilaterals $OO'H_4A_1$ and $OO'H_1A_4$ are parallelograms [see the solution to Problem 33(c)]. Therefore

$$A_1H_4 = OO' = A_4H_1, \qquad A_1H_4 \parallel OO' \parallel A_4H_1,$$

and so $A_1H_4H_1A_4$ is a parallelogram. From this it follows that the segments A_1H_1 and A_4H_4 have a common midpoint H. In the same way one shows that H is also the midpoint of A_2H_2 and A_3H_3.

(b) By comparing Figure 96 and Figure 95 one sees that, for example, H_4 lies on the circle S', the image of S in the line A_2A_3; H_1 also lies on this circle. Thus A_2, A_3, H_1, and H_4 all lie on a circle congruent to S. The remaining assertions of the theorem are proved similarly.

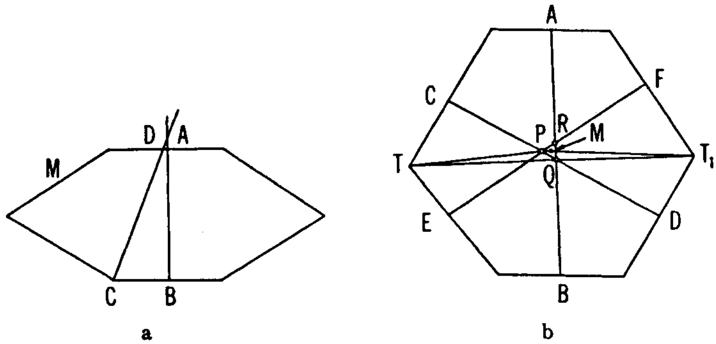


Figure 97

35. First of all it is clear that any two axes of symmetry AB and CD of the polygon M must intersect inside M; indeed, if this were not the case (Figure 97a), then they could not both divide the figure into two parts of equal area. Now let us show that if there is a third axis of symmetry EF, then it must pass through the point of intersection of the first two. Assume that this were not the case; then the three axes of symmetry AB, CD, and EF would form a triangle PQR (Figure 97b). Let M be a point inside this triangle. It is easy to see that each point in the plane lies on the same side of at least one of these three axes of symmetry as does M. Let T be the vertex of the polygon that is farthest from M (if there is more than one such vertex, let T be any one of them), and let T and M lie on the same side of the axis of symmetry AB. Thus, if T_1 is the image of T in AB (T_1 is therefore a vertex of the polygon), then $MT_1 > MT$ (since the projection of MT_1 onto TT_1 is larger than the projection of MT on TT_1; see Figure 97b). This contradiction proves the theorem.

[In a similar way it can be shown that if any bounded figure (not necessarily a polygon) has several axes of symmetry, then they must all pass through a common point. For unbounded figures this is not so: Thus, the strip between two parallel lines l_1 and l_2 has infinitely many axes of symmetry, perpendicular to l_1 and l_2 and all parallel to each other.]

Remark: The assertion of this problem is evident from mechanical considerations. The center of gravity of a homogeneous, polygonal-shaped body, having an axis of symmetry, must lie on that axis. Consequently, if there are several axes of symmetry they must all pass through the center of gravity.

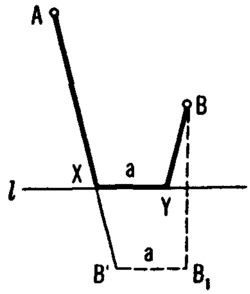


Figure 98

36. Since the segment XY has length a, we are required to minimize the sum $AX + BY$. Let us assume that the segment XY has been found. A glide reflection in the axis l through a distance a carries B into a new point B', and carries Y into X (Figure 98); therefore $BY = B'X$, and so

$$AX + BY = AX + B'X.$$

Thus it is required that the path AXB' should have minimum length. From this it follows that X is the point of intersection of l with AB'.

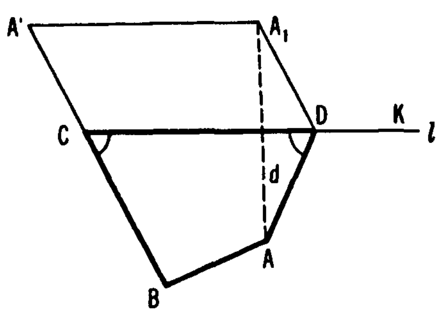


Figure 99

37. (a) Assume that the quadrilateral $ABCD$ has been constructed. Let A' be the image of A under a glide reflection in the axis DC through a distance DC (Figure 99); then $\measuredangle A'CD = \measuredangle ADK$ (where DK is the extension of side DC past the point D) because if A_1 is the image of A in DC, then

$$\measuredangle A'CD = \measuredangle A_1DK = \measuredangle ADK.$$

But

$$\measuredangle ADK = 180° - \measuredangle D = 180° - \measuredangle C;$$

consequently, $\measuredangle A'CD = 180° - \measuredangle C$, that is, $A'CB$ is a straight line. In addition we know that

$$A'B = A'C + CB = AD + CB,$$

and we know the distance d from A to CD.

Thus we have the following construction: Let l be any line, let A be a point at a distance d from l, and let A' be the image of A under a glide reflection in the line l through a distance CD. The vertex B of the quadrilateral can now be found, since we know the distances AB and

$$A'B = AD + BC.$$

The vertex C is the point of intersection of the segment $A'B$ with the line l, and the vertex D which lies on l is found by laying off the known distance CD from the point C. The problem can have two, one, or no solutions.

(b) Draw the segment AB; the line l can now be found as the common tangent to the two circles of radii d_1 and d_2, with centers at the points A and B respectively (Figure 100). It remains to put the segment DC on the line l in such a position that the sum of the lengths $AD + BC$ has the given value [compare with Problem 31(a)].

Assume that the points D and C have been found and let A' and A'' be the images of A under a translation in the direction of the line l through a distance DC, and under a glide reflection with axis l through a distance DC. Clearly the circle of center C and radius AD passes through the points A' and A'' $(A'C = A''C = AD)$ and is tangent to the circle S with center B and radius

$$BC + CA'' = BC + AD.$$

But the circle S can be constructed from the given data, and thus it only remains to find the circle passing through the two known points A'

and A'' and tangent to S [see Problem 49(b) of Vol. 2].[T] The center of this circle is the vertex C.

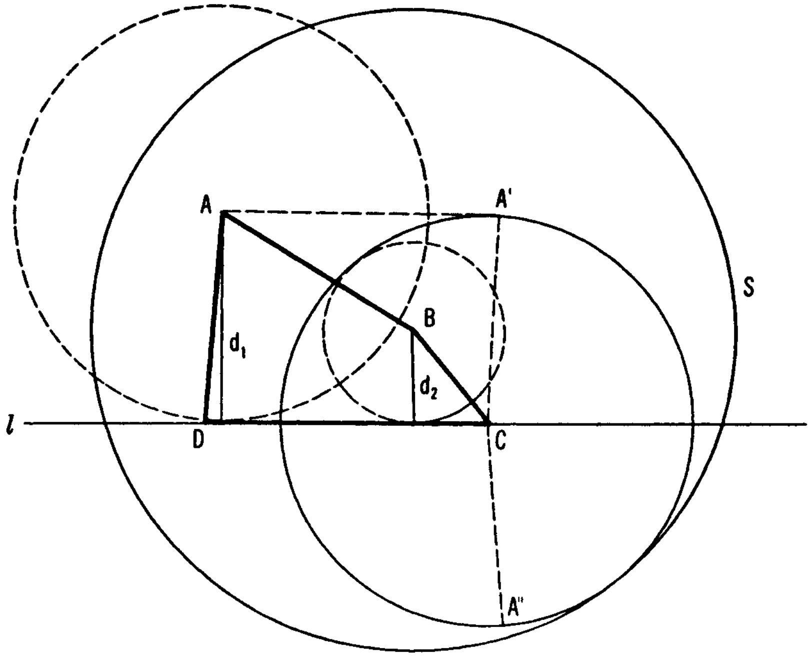


Figure 100

38. *First solution.* Clearly the only time a ray of light will be reflected from a mirror in a direction exactly opposite to the direction of incidence is when the path is perpendicular to the mirror. From now on we shall assume that the ray of light does not strike the first side of the angle at right angles. Let us now consider the case when the ray MN, after two reflections in the angle ABC, leaves along a path PQ exactly opposite to MN (Figure 101a). In this case we have:

$$\sphericalangle PNB + \sphericalangle NPB = 180° - \sphericalangle NBP = 180° - \alpha;$$

$$\begin{aligned} 2(180° - \alpha) &= 2\sphericalangle PNB + 2\sphericalangle NPB \\ &= \sphericalangle ANM + \sphericalangle PNB + \sphericalangle NPB + \sphericalangle CPQ \\ &= 180° - \sphericalangle MNP + 180° - \sphericalangle NPQ \\ &= 360° - (\sphericalangle MNP + \sphericalangle NPQ). \end{aligned}$$

[T] See translator's note on page 107.

Since the rays MN and PQ are parallel and oppositely directed,

$$\sphericalangle MNP + \sphericalangle NPQ = 180°,$$

so

$$2(180° - \alpha) = 360° - 180°, \qquad \text{and} \qquad \alpha = 90°.$$

Conversely, if $\alpha = 90°$ then $\sphericalangle MNP + \sphericalangle QPN = 180°$, that is, the direction of the departing ray PQ is opposite to MN.

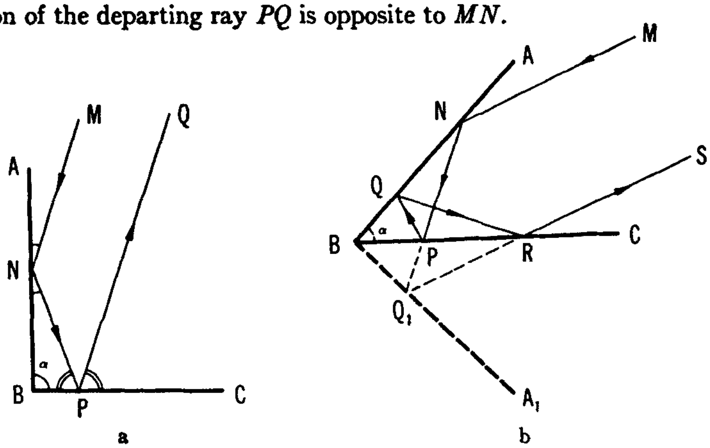


Figure 101

Next consider the case when the incoming ray MN, after four reflections in the sides of the angle, leaves in a direction RS opposite to MN (Figure 101b; the only way in which a light ray can leave in the opposite direction to the direction of incidence after exactly three reflections is if it hits the second side of the angle at right angles; this cannot happen for every incoming light ray—in fact, for a given angle α there is only one angle of incidence for which this will happen). Reflect the line AB and the path PQR in the line BC; the line BA_1 is the image of BA, and the point Q_1 is the image of Q in BC. Then

$$\sphericalangle ABA_1 = 2\sphericalangle ABC = 2\alpha.$$

Further

$$\sphericalangle QPB = \sphericalangle Q_1PB = \sphericalangle NPC;$$

therefore, NPQ_1 is a straight line. In the same way it can be shown that Q_1RS is a straight line (since $\sphericalangle QRB = \sphericalangle Q_1RB = \sphericalangle SRC$). Finally, $\sphericalangle BQ_1P = \sphericalangle A_1Q_1R$, since these angles are equal respectively to the angles BQP and AQR, which are equal. Thus we see that the ray MN, reflected from the points N and Q_1 of the angle $ABA_1 = 2\alpha$, leaves in a direction Q_1S, opposite to the incoming direction. But then by what was shown previously $2\alpha = 90°$ and therefore $\alpha = 90°/2$. Conversely, if $\alpha = 90°/2$ then $\measuredangle ABA_1 = 90°$ and so the ray MN, after four reflections in the sides of the angle ABC, leaves in the opposite direction to the direction of incidence.

Now consider the case when the incoming ray MN is reflected six times in the sides of the angle, and then leaves along a path TU opposite to the incoming path (Figure 101c; in general a light ray cannot leave along a path opposite to the incoming path after exactly five reflections). Reflect the line AB and the path $PQRST$ in the line BC; let BA_1 be the image of BA and let Q_1 and S_1 be the images of Q and S in the line BC. Just as before we can conclude that NPQ_1 is a straight line ($\measuredangle Q_1PB = \measuredangle QPB = \measuredangle NPC$), that S_1TX is a straight line ($\measuredangle S_1TB = \measuredangle STB = \measuredangle UTC$) and that

$$\measuredangle Q_1RB = \measuredangle S_1RC, \qquad \measuredangle RQ_1B = \measuredangle PQ_1A_1, \qquad \measuredangle RS_1B = \measuredangle TS_1A_1.$$

Thus we find that the ray MN, reflected successively from the lines AB, BA_1, BC, and again from BA_1 at the points N, Q_1, R, and S_1 leaves in the direction S_1U, opposite to the incoming direction MN.

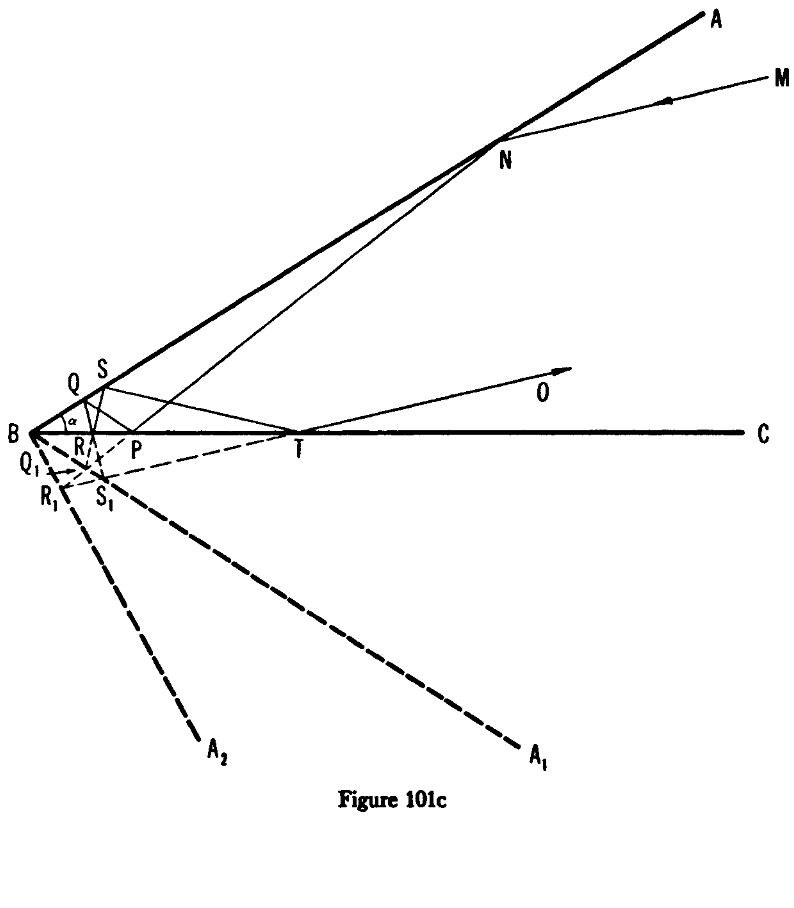


Figure 101c

Now reflect the line BC and the path Q_1RS_1 in the line BA_1; let BA_2 be the image of BC and let R_1 be the image of R in the line BA_1. Then NPQ_1R_1 is a straight line (because $\sphericalangle R_1Q_1B = \sphericalangle RQ_1B = \sphericalangle PQ_1A_1$) and R_1S_1TU is a straight line (because $\sphericalangle R_1S_1B = \sphericalangle RS_1B = \sphericalangle TS_1A_1$), and $\sphericalangle Q_1R_1B = \sphericalangle S_1R_1A_2$ (because they are equal respectively to the angles Q_1RB and S_1RC, which are equal). Thus we find that the ray MN after being reflected in the sides of the angle ABA_2 ($= 3\alpha$) at the points N and R_1 leaves in the direction R_1U, opposite to the incoming direction MN. But then by what was proved earlier we must have $3\alpha = 90°$, that is, $\alpha = 90°/3$. Conversely, if $\alpha = 90°/3$, then $\sphericalangle ABA_2 = 90°$ and the ray MN, after being reflected six times in the sides of angle ABC, leaves in the direction opposite to the direction of incidence.

Finally, suppose that after $2n$ reflections in the sides of an angle $ABC = \alpha$ the ray leaves in the direction opposite to the direction of the incoming ray [in general a light ray cannot leave in a direction opposite to the direction of incidence after $(2n - 1)$ reflections in the sides of an angle].

Proceed as in the previous cases,[T] that is, if the incoming ray strikes AB, reflect the path of the ray in line BC; let BA_1 be the image of AB after this reflection. Next, reflect BC in line BA_1 to obtain BA_2, then reflect BA_1 in BA_2 to obtain BA_3, and so forth, until, after $n - 1$ reflections, we have BA_{n-1}. The angle $ABA_{n-1} = n\alpha$.

Next, establish that the incoming ray, when continued by the proper reflections, forms a straight line which hits $A_{n-1}B$, is reflected there, then hits BA so that it leaves in the direction opposite to that of its entry. Then, by what was proved earlier, conclude that $n\alpha = 90°$, and hence, that

$$\alpha = \frac{90°}{n}.$$

Second solution. Let ABC be the given angle, and let $MNPQ\cdots$ be the path of the light ray (see Figure 102a, where the case $n = 2$, $\alpha = 45°$ is shown). We are only interested in the directions of the path, and it will be convenient to have all these directions emanate from a single point O (in the figure

$$O1 \parallel MN, \qquad O2 \parallel NP, \qquad O3 \parallel PQ,$$

and so forth). Since $\sphericalangle MNA = \sphericalangle PNB$, it follows that the ray $O2$ is the image of $O1$ in the line $OU \parallel AB$ (to prove this it is sufficient to note

[T] In the Russian version of this book, the details of this proof were carried out. We have omitted them here in order to save space and to avoid the somewhat complicated notation.

that in Figure 102a, NM' is the image of NP in NB). Similarly, the ray $O3$ is the image of $O2$ in the line $OV \parallel BC$. Therefore by Proposition 3 on page 50, the ray $O3$ is obtained from the ray $O1$ by a rotation through an angle $2\sphericalangle UOV = 2\alpha$. Similarly the ray $O5$ is obtained from the ray $O3$ by a rotation through an angle 2α in the same direction; consequently the ray $O5$ is obtained from the ray $O1$ by a rotation through an angle 4α, and so forth. Therefore, if $\alpha = 90°/n$ then the ray $O(2n + 1)$, which has the same direction as that of a light ray after n reflections from each of the two faces of the angle, will form an angle $n \cdot 2\alpha = 180°$ with the ray $O1$, which establishes the assertion of the problem. [Here we are assuming that $0 < \sphericalangle MNA < \alpha$; if $\sphericalangle MNA > \alpha$, then MN will intersect BC, which means that the incoming light ray has to be reflected from side BC before it can hit side BA. This fact guarantees that the rays in the directions $O1$, $O3$, $O5$, $\cdots$, etc. will all hit the mirror BA, while the rays in the directions $O2$, $O4$, $\cdots$, etc. will hit the mirror BC. If $\sphericalangle MNA = \alpha$, that is, if the incoming ray MN is parallel to side BC, then the ray $O(2n)$ will already be opposite in direction to $O1$: In this case the final ray leaves along a path opposite to the path of the original incoming ray; however the number of reflections is one fewer than in the general case; see Figure 102b, where $\sphericalangle ABC = 45°$, $\sphericalangle MNA = 45°$.]

These considerations show that if $\alpha \neq 90°/n$, then not every incoming light ray will, after successive reflections in the sides, leave in a direction opposite to the direction of approach of the original ray.

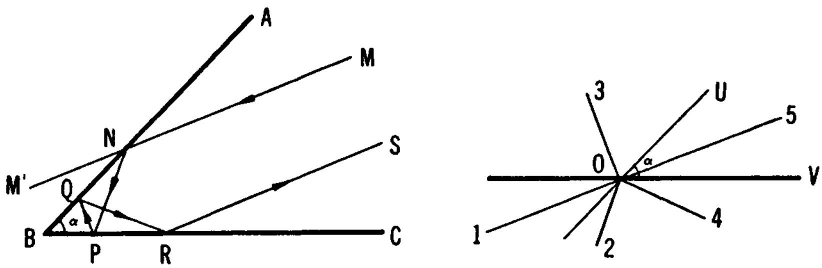


Figure 102a

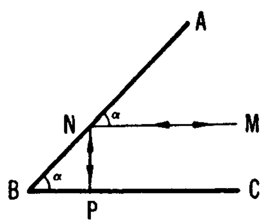


Figure 102b

39. (a) *First solution* (see also the first solutions to Problems 15 and 21). Let $A_1, A_2, \cdots, A_n$ be the desired n-gon and let B_1 be any point in the plane. Reflect the segment A_1B_1 successively in the lines

$$l_1, \ l_2, \ \cdots, \ l_{n-1}, \ l_n;$$

we obtain segments $A_2B_2, A_3B_3, \cdots, A_nB_n, A_1B_{n+1}$. Since these segments are all congruent to each other, it follows that $A_1B_1 = A_1B_{n+1}$, that is, the point A_1 is equidistant from B_1 and B_{n+1}, and lies therefore on the perpendicular bisector of the segment B_1B_{n+1}.

Now choose another point C_1 in the plane and let $C_2, C_3, \cdots, C_n, C_{n+1}$ be the points obtained, starting from C_1, by successive reflections in the lines $l_1, l_2, \cdots, l_{n-1}, l_n$. Clearly the vertex A_1 of the n-gon is also equidistant from C_1 and C_{n+1}, and therefore lies on the perpendicular bisector to C_1C_{n+1}. Therefore A_1 can be found as the intersection of the perpendicular bisectors to the segments B_1B_{n+1} and C_1C_{n+1} (the segments B_1B_{n+1} and C_1C_{n+1} can be constructed, once we have chosen any two distinct points for B_1 and C_1). By reflecting A_1 successively in the n given lines we obtain the remaining vertices of the n-gon.

The problem has a unique solution provided that the segments B_1B_{n+1} and C_1C_{n+1} are not parallel (i.e., provided that the perpendicular bisectors p and q intersect in one point); if $B_1B_{n+1} \parallel C_1C_{n+1}$ then the problem has no solution when p and q are distinct, and has infinitely many solutions (the problem is undetermined) when p and q coincide.

The n-gon obtained as the solution to the problem may intersect itself.

One drawback to this solution is that it gives no indication of the essential difference between the cases when n is even and when n is odd (see the second solution to the problem).

Second solution (see also the second solutions to Problems 15 and 21). Let $A_1A_2\cdots A_n$ be the desired n-gon (see Figure 50a). If we reflect the vertex A_1 successively in the lines $l_1, l_2, \cdots, l_{n-1}, l_n$ we obtain the points $A_2, A_3, \cdots, A_n$ and, finally, A_1 again. Thus, A_1 is a *fixed point* of the sum of the reflections in the lines $l_1, l_2, \cdots, l_n$.

We now consider separately two cases.

First case: n even. In this case the sum of the reflections in the lines $l_1, l_2, \cdots, l_n$ is, in general, a rotation about some point O (see page 55), which can be found by the construction used in the addition of reflections. The point O is the only fixed point of the rotation, and so A_1 must coincide with O. Having found A_1, one has no difficulty in finding all the remaining vertices of the n-gon. The problem has a unique solution in this case.

In the exceptional case, when the sum of the reflections in the lines $l_1, l_2, \cdots, l_n$ is a translation or is the identity transformation (a rotation through an angle of zero degrees, or a translation through zero distance), the problem either has no solution at all (a translation has no fixed points) or has more than one solution—any point in the plane can be taken for the vertex A_1 (every point is a fixed point of the identity transformation).

Second case: n odd. In this case the sum of the reflections in the lines $l_1, l_2, \cdots, l_n$ will, in general, be a glide reflection (see pages 55–56). Since a glide reflection has no fixed points, there will in general be no solution when n is odd. In the exceptional case, when the sum of the reflections in the lines $l_1, l_2, \cdots, l_n$ is a reflection in a line l (this line can be constructed), the solution will not be uniquely determined; any point of the line l can be taken for the vertex A_1 of the n-gon (every point of the axis of symmetry is a fixed point under reflection in this line).

(Thus, for $n = 3$, the problem has, in general, no solutions; the only exceptions are the cases when the lines l_1, l_2, l_3 meet in one point [see Problem 26(c)] or are parallel; in these cases the problem has more than one solution [see Proposition 4 on page 53]).

(b) This problem is similar to Problem (a). If $A_1A_2\cdots A_n$ is the desired n-gon (see Figure 50b), then the line A_nA_1 is taken by successive reflections in the lines $l_1, l_2, \cdots, l_{n-1}, l_n$ into the lines

$$A_1A_2, \quad A_2A_3, \quad \cdots, \quad A_{n-1}A_n$$

and finally back into A_nA_1. Thus A_nA_1 is a *fixed line* of the sum of the reflections in the lines $l_1, l_2, \cdots, l_n$. We consider two cases.

First case: n even. In this case the sum of the reflections in the lines $l_1, l_2, \cdots, l_n$ is, in general, a reflection about some point O and, therefore, has in general no fixed lines. Thus for n even our problem has, in general, no solution. In the exceptional cases when the sum of the reflections is a half turn about the point O (a rotation through an angle of 180°), or is a translation, or is the identity transformation, the problem has more than one solution. In the first case one can take any line through the center of symmetry to be the line A_nA_1; in the second case one can take any line parallel to the direction of translation; in the third case one can take any line whatsoever in the plane.

Second case: n odd. In this case the sum of the reflections in the lines $l_1, l_2, \cdots, l_n$ is, in general, a glide reflection with an axis l (that can be constructed). Since l is the only fixed line of a glide reflection, it follows that the side A_nA_1 of the desired n-gon must lie on l; by reflecting l successively in the lines $l_1, l_2, \cdots, l_{n-1}$, we obtain all the remaining sides of the n-gon. Thus for odd n the problem has, in general, a unique solution. An exception occurs when the sum of the reflections in the given lines is a reflection in a line l; in this case the problem has more than one solution. For the side A_nA_1 one can take the line l itself, or any line perpendicular to it.

(Thus, for $n = 3$, the problem has in general a unique solution; the lines l_1, l_2, l_3 will either all be bisectors of the exterior angles of the triangle, or two of them will bisect interior angles and the third will bisect the exterior angle. The only exception is when the three lines l_1, l_2, and l_3 meet in a point; in this case the problem has more than one solution [see Problem 26(a)]; the lines l_1, l_2, l_3 will all bisect interior angles, or two or them will bisect exterior angles and the third will bisect the interior angle.)

We leave it to the reader to find a solution to part (b) similar to the first solution to part (a).

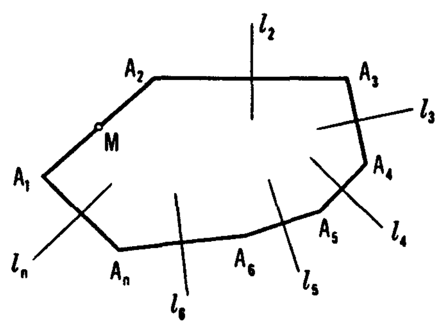


Figure 103

40. (a) Assume that the problem has been solved (Figure 103). A half turn about the point M will carry the vertex A_1 into A_2, a reflection in the line l_2 will carry the vertex A_2 into A_3, a reflection in l_3 will carry A_3 into A_4, and so forth. Finally, a reflection in l_n carries A_n into A_1. Thus, A_1 is a fixed point of the sum of a half turn about M followed by reflections in the lines $l_2, l_3, \cdots, l_n$. A half turn about the point M is equivalent to a pair of reflections in lines. We shall consider separately two cases.

First case: n odd. In this case the problem reduces to finding fixed points of the sum of an even number of reflections in lines. This sum is, in general, a rotation about some point O (which can be constructed from the point M and the lines $l_2, l_3, \cdots, l_n$). Therefore for odd n the problem has in general a unique solution [compare this with the first case in the solution to Problem 39(a)]. The only exceptional cases are when the sum of the even number of reflections in the lines is a translation—then the problem has no solution at all; or is the identity transformation—then the problem has many solutions.

Second case: n even. In this case the problem reduces to finding the fixed points of an odd number of reflections in lines. In general this sum is a glide reflection and the problem has no solution (a glide reflection has no fixed points). In the special case when the sum of the reflections is itself a reflection in some line l, the problem will have many solutions (reflection in a line has an infinite number of fixed points, namely all the points on the line l).

The construction can also be carried out in a similar manner to the construction in the first solution to Problem 39(a). The polygon obtained as the solution may intersect itself.

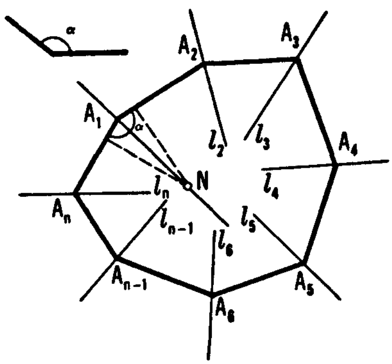


Figure 104

(b) Assume that the problem has been solved (Figure 104). A rotation of $180° - \alpha$ about the point M carries the line A_nA_1 into A_1A_2. A reflection in l_2 carries A_1A_2 into A_2A_3, a reflection in l_3 carries A_2A_3 into A_3A_4, and so forth. Finally, a reflection in l_n carries $A_{n-1}A_n$ into A_nA_1. Thus, A_nA_1 is a fixed line of the transformation consisting of the sum of a rotation through $180° - \alpha$ about the point M (which can be replaced by two reflections in lines) and $n - 1$ reflections in the lines $l_2, l_3, \cdots, l_n$.

We consider separately two cases.

First case: n even. The sum of an odd number of reflections in lines is in general a glide reflection; it has a unique fixed line, the axis of symmetry l (that can be constructed), and therefore the problem has a unique solution. In the special case when the sum of the reflections is a reflection in some line, the problem will have infinitely many solutions (because reflection in a line has infinitely many fixed lines).

Second case: n odd. In this case the transformation we are considering will be the sum of an even number of reflections in lines which, in general, is a rotation. In this case the problem will have no solution. In special cases, however, this sum of reflections may be a half turn about some point, a translation, or the identity transformation; in each of these cases the problem will have more than one solution.

The polygon that was constructed to solve the problem may intersect itself; the lines $l_2, l_3, \cdots, l_n$ will bisect either the exterior or the interior angles.

The construction can also be carried out in a manner similar to that in the first solution to Problem 39(a).

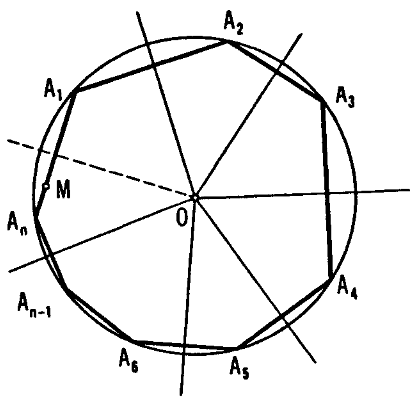


Figure 105

41. (a) Let $A_1A_2A_3\cdots A_n$ be the desired n-gon (Figure 105). Reflect the vertex A_1 successively in lines drawn from the center O of the circle and perpendicular to the sides $A_1A_2, A_2A_3, \cdots, A_{n-1}A_n, A_nA_1$ of the n-gon (these lines are known, since we are given the directions of the sides of the n-gon); the vertex A_1 is first taken into A_2, then A_2 is taken into $A_3, \cdots$, then A_{n-1} is taken into A_n, and finally A_n is taken back into A_1. Thus A_1 is a fixed point of the sum of n reflections in known lines. Let us consider two cases separately.

First case: n odd. Since the sum of three reflections in lines meeting in a point is again a reflection in some line through this point (See Proposition 4 on page 53), it is not difficult to see that the sum of any odd number of reflections in lines that all pass through a common point is again a reflection in some line through this point. (First replace the first three reflections by a single reflection, then consider the sum of this reflection and the next two, etc.) Therefore the sum of our n reflections is a reflection in some line passing through the center O of the circle. There are exactly two points on the circle that are left fixed by reflection in l—they are the points of intersection of the circle with l. Taking one of these points for the vertex A_1 of the desired polygon, we find the other vertices by successive reflections of this one in the n lines. The problem has two solutions.

Second case: n even. The sum of any two reflections in lines passing through the point O is a rotation about O through some angle. From this it follows that the sum of an even number, n, of reflections in lines passing through O may be replaced by the sum of $\frac{1}{2}n$ rotations about O; from this it is clear that the sum is itself a rotation about O. Since a rotation about O has, in general, no fixed points on a circle with center O, our problem has no solutions in general. An exception is the case when the sum of the n reflections is the identity transformation; in this case the problem has infinitely many solutions—any point on the circle can be chosen for the vertex A_1 of the desired n-gon.

(b) Assume that the n-gon has been constructed (see Figure 105). Reflect the vertex A_1 successively in the $(n - 1)$ lines perpendicular to the sides A_1A_2, A_2A_3, $\cdots$, $A_{n-1}A_n$ and passing through the center O of the circle (these lines are known, since we know the point O and the directions of the sides of the polygon); this process takes A_1 into A_n. We consider separately two cases.

First case: n odd. In this case the sum of $(n - 1)$ reflections in lines passing through the point O is a rotation about O through an angle α (that can be found). Thus, angle $A_1OA_n = \alpha$ is a known angle, and so we know the length of the chord A_1A_n and its distance to the center. Since A_1A_n must pass through a given point M, it only remains to pass tangents from the point M to the circle with center O and radius equal to the distance from the chord A_1A_n to the center O. The problem can have two, one, or no solutions.

Second case: n even. In this case the sum of $(n - 1)$ reflections in lines passing through a common point is a reflection in some line l through this point. Therefore A_1 and A_n are images of each other in l. Since A_1A_n must pass through a known point M, it can be found by simply dropping the perpendicular from M onto l. The problem always has a unique solution.

42. (a) Since the sum of the reflections in the three lines l_1, l_2, and l_3 meeting in the point O is a reflection in some line l (also passing through the point O), it follows that the point A_3 is obtained from A by a reflection in l. But A_6 is obtained from A_3 by a reflection in l, and so A_6 coincides with A.

This result is valid for any *odd* number of lines meeting in a point (compare Problem 13). If we have an even number n of lines meeting in a point O, then the sum of the n reflections in these lines is a rotation about O through some angle α, and so the point A_{2n} obtained after $2n$ rotations will coincide with the original point A only in case α is a multiple of 180°.

Remark. The point A_6 obtained from an arbitrary point A of the plane by six successive reflections in lines l_1, l_2, l_3, l_1, l_2, l_3 will coincide with the initial point A if and only if l_1, l_2, and l_3 meet in a point or are parallel [if $l_1 \parallel l_2 \parallel l_3$, then the sum of the reflections in l_1, l_2, and l_3 is a reflection in some line l, and the reasoning used in the solution to Problem 42(a) can be applied]. In all other cases the sum of the reflections in l_1, l_2, and l_3 is a glide reflection, and thus the point A_6 is obtained from A by two successive glide reflections along some axis l, that is, by a translation in the direction of l; therefore A_6 cannot coincide with A. [The sum of two (identical) glide reflections along an axis l can be written as the sum of the following four transformations: translation along l, reflection in l, reflection in l, and translation along l (see page 48), that is, as the sum of two (identical) translations along l.]

(b) This problem is essentially the same as part (a) [see also Problem 14(b)].

(c) The sum of the reflections in l_1 and l_2 is a rotation about their point of intersection O through some angle α; the sum of the reflections in l_3 and l_4 is a rotation about O through some angle β. From this it follows that (no matter in which order these reflections are performed!) the point A_4 is obtained from A by a rotation about O through an angle of $\alpha + \beta$, which was to be proved [compare with Problem 14(a)].

43. (a) Since the three lines CM, AN, BP meet in a point, it follows that the sum of the reflections in the lines CM, AN, BP, CM, AN, BP is the identity transformation [see Problem 42(a)]. To show that the lines CM', AN', BP' meet in a point it is sufficient to show that the sum of the reflections in the lines $CM', AN', BP', CM', AN', BP'$ is also the identity transformation [see the remark following the solution to Problem 42(a)]. However reflection in the line CM' is the same as the sum of the reflections in the three lines CB, CM and CA all meeting in the point C—this follows from the fact that rotation through angle BCM' about the point C carries line CM into CA, and carries CB into line CM', which is the image of CM in the bisector of angle BCA (compare Figure 106a with Figure 47b, and see the proof of the second half of Proposition 4, page 53). Similarly, reflection in AN' is the same as the sum of the reflections in the three lines AC, AN, and AB, and reflection in BP' is the sum of the reflections in the lines BA, BP, and BC. From this it follows that the sum of the reflections in CM', AN', and BP' is the same as the sum of the reflections in the following nine lines: CB, CM, CA, $AC(= CA)$, AN, AB, $BA(= AB)$, BP, and BC. Since two consecutive reflections in the same line cancel each other, this is the same as the sum of the reflections in the following five lines: CB, CM, AN, BP, and BC. Now perform this transformation twice; we obtain the sum of the reflections in the following ten lines: CB, CM, AN, BP, BC, CB $(= BC)$, CM, AN, BP, and BC, which is the same as the sum of the reflections in the eight lines CB, CM, AN, BP, CM, AN, BP, and BC. But if the sum of the reflections in the six "inner" lines is the identity transformation, then the sum of our eight reflections in the eight lines reduces to the sum of the two reflections in CB and BC $(= CB)$, that is, to the identity transormation!

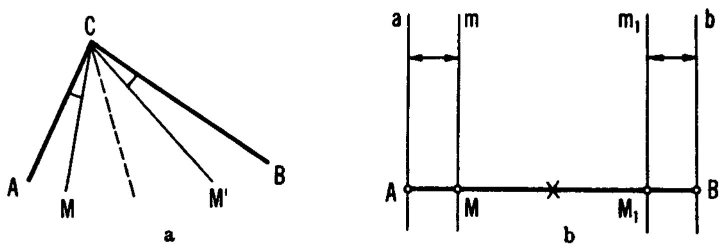


Figure 106

(b) Let the perpendiculars to the sides AB, BC and CA of the triangle ABC, erected at the points M and M_1, N and N_1, P and P_1 be denoted by m and m_1, n and n_1, p and p_1; let a and b denote the perpendiculars to side AB erected at the endpoints A and B. We must show that *if the sum of the reflections in the lines m, n, p, m, n, p is the identity transformation, then the sum of the reflections in the lines m_1, n_1, p_1, m_1, n_1, p_1 is also the identity transformation* [compare the solution to Problem (a)]; clearly the perpendiculars to two different sides of a triangle cannot be parallel to one another. But the reflection in m_1 is identical with the sum of the reflections in the point A, in the line m and in the point B; similarly, the reflection in n_1 is the sum of the reflections in B, n and C, and the reflection in p_1 is the sum of the reflections in C, p and A. To prove the first of these assertions, note that the reflection in A is the sum of the reflections in AB and a, and the reflection in B is the sum of the reflections in b and AB; thus, the sum of the reflections in A, m and B is equal to the sum of the reflections in the following five lines: AB, a, m, b, and AB. But the sum of the three "inner" reflections is equal to the reflection in m_1 alone—this follows from the fact that the translation of the two lines a and m, carrying m into b, carries a into m_1 (since m_1 is the reflection of m in the midpoint of the segment AB; compare Figure 106b with Figure 47a). Therefore the sum of the five reflections is equivalent to the sum of the reflections in the three lines: AB, m_1, and AB, or to the sum of the reflections in M_1 and AB. The reflection in M_1 is also equal to the sum of the reflections in m_1 and AB taken in that order; therefore the sum of the reflections in M_1 and AB is equal to the sum of the reflections in m_1, AB, and AB, and this is clearly the same as a single reflection in m_1 alone.

It is now clear that the sum of the reflections in the six lines m_1, n_1, p_1, m_1, n_1, p_1 is equal to the sum of the reflections in the following points and lines: A, m, B; B, n, C; C, p, A; A, m, B; B, n, C; C, p, A, or, what is the same thing, to the reflections in A, m, n, p, m, n, p, A. Therefore, if the sum of the six "inner" reflections is the identity transformation, then the sum of all the reflections (which reduces in this case to two reflections in the point A) is also the identity transformation [compare the solution to part (a)].

44. If we take the sum of the reflections in three lines in the plane twice, then we obtain either the identity transformation or a translation [see the solution to Problem 42(a), and in particular the remark following the solution]. Thus the "first" point A_{12} is obtained from A by the

sum of two translations (one or even both of them may be "translations through zero distance"—that is, the identity transformation); the "second" point (which we shall call A'_{12}) is obtained from A by the sum of the same two translations taken in the opposite order. The assertion of the problem follows from this [compare the solution to Problem 14(a)].

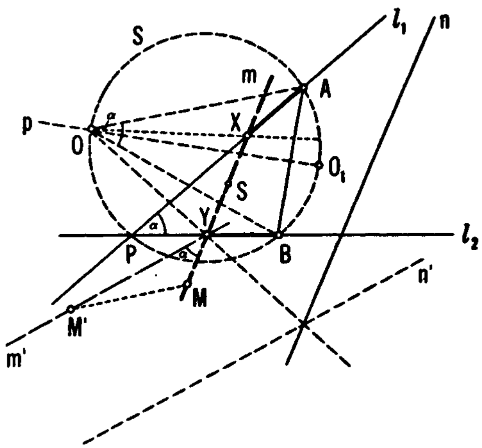


Figure 107a

45. *First solution* (based on Theorem 1, page 51). Suppose first that the lines l_1 and l_2 are not parallel (Figure 107a). Assume that the problem has been solved. By Theorem 1 the segment AX can be taken by a rotation into the congruent segment BY, so that A is taken into B and X into Y (since l_1 and l_2 are not parallel, AX cannot be taken into BY by a translation). The angle of rotation α is equal to the angle between l_1 and l_2; therefore the center of rotation O can be found as the point of intersection of the perpendicular bisector p of the segment AB with the circular arc constructed on AB and subtending an angle α (this arc lies on the circle S circumscribed about triangle ABP, where P is the point of intersection of l_1 and l_2).† Let this rotation take the desired line m into a line m', also passing through Y. We shall now consider Problems (a), (b), (c), and (d) separately.

† The circle S and the perpendicular bisector p intersect in two points O and O_1; they correspond to the cases when X and Y are situated on the same, or on opposite sides of the line through AB.

(a) Rotate the line n through an angle α about the center O that was found above, and let n' be the line thus obtained. The line OY will bisect the angles between m and m', and between n and n'; hence Y can be found as the point of intersection of l_2 with the line joining O to the point of intersection of n and n'. The problem can have two solutions (see the note†).

(b) m' passes through the point M' that is the image of M under a rotation through an angle α about the point O; the angle between m and m' is equal to α. Therefore Y can be found as the point of intersection of the line l_2 with the circular arc on MM' that subtends the angle α. The problem can have two solutions.

(c) In the isosceles triangle OXY we know the vertex angle α and the base $XY = a$; this enables us to find the distance OX from O to the unknown point X. The problem can have up to four solutions.

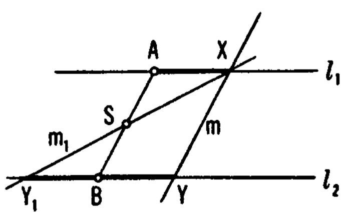


Figure 107b

(d) Let S be the midpoint of XY. Since the angles of the isosceles triangle OXY are known, we also know the ratio

$$\frac{OS}{OX} = k \qquad \text{and the angle} \qquad XOS = \frac{1}{2}\alpha.$$

Therefore the point S is obtained from X by a known spiral similarity (see Vol. 2, Chapter 1, Section 2).† The point S is found as the intersection of the line r and the line l_1' obtained from l_1 by this spiral similarity. The desired line m is perpendicular to OS. The problem has, in general, two solutions; if l_1' coincides with r then the solution is undetermined.

If $l_1 \parallel l_2$ then the desired line m either passes through the midpoint S of the segment AB or is parallel to AB (Figure 107b). In these cases the

† Here the second solution is preferable, as it does not use material from Vol. 2.

problem becomes much simpler. We shall merely indicate the number of solutions:

(a) One solution if n is not parallel to l_1 or to AB; no solutions if $n \parallel l_1 \parallel l_2$; infinitely many solutions if $n \parallel AB$.

(b) Two solutions if M does not lie on the line AB or on the line l_0 midway between l_1 and l_2 and parallel to them; one solution if M lies on AB or on l_0 but does not coincide with S; infinitely many solutions if M coincides with S.

(c) Two solutions if $a \neq AB$, and $a > d$ (where d is the distance between l_1 and l_2); one solution if $a = d$ but $AB \neq d$; no solutions if $a < d$; infinitely many solutions if $a = AB$ $(\geq d)$.

(d) One solution if r is not parallel to $l_1 \parallel l_2$ and does not pass through S; no solutions if $r \parallel l_1$ but does not pass through S; infinitely many solutions if r passes through S.

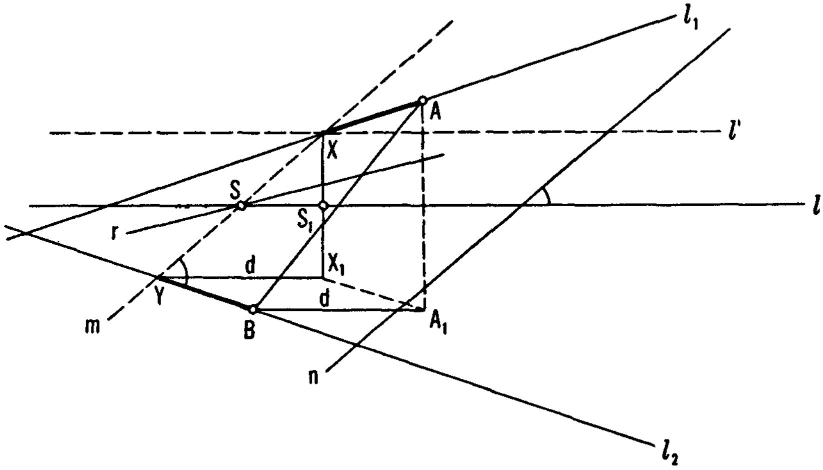


Figure 108

Second solution of parts (a), (c), (d) (based on Theorem 2, page 64). By Theorem 2 the segment AX can be taken by a glide reflection (or by an ordinary reflection in a line, which may be regarded as a special case of a glide reflection) into the congruent segment BY so that A goes into B and X into Y. Also, the axis l of the glide reflection is parallel to the bisector of the angle between l_1 and l_2 and passes through the midpoint of segment AB;† the distance d of the translation is equal to A_1B where A_1 is the image of A in l (Figure 108). Also, let X_1 be the image of X in l; in this case

$$X_1Y \parallel l \quad \text{and} \quad X_1Y = d.$$

We now consider the three cases (a), (c) and (d) separately.

(a) In triangle XX_1Y the side $X_1Y = d$ is known, as is $\measuredangle XYX_1$ (it is equal to the angle between m and l); hence the length of side XX_1 can be found. Now X can be found as the point of intersection of the line l_1 and the line l', parallel to l at a distance of $\frac{1}{2}XX_1$. In the general case, when l_1 is not parallel to l_2, the problem has two solutions.

(c) In triangle XX_1Y the hypotenuse $XY = a$ and the side $X_1Y = d$ are known; hence the other side XX_1 can be found. The remainder of the construction is similar to that in part (a); in general the problem has two solutions.

(d) The midpoint S of the segment XY must lie on the midline l of triangle XX_1Y. Therefore S is the point of intersection of l and r. X can now be found as the intersection of l_1 with the perpendicular p to l at the point S_1 (where $SS_1 = \frac{1}{2}d$). In general the problem has two solutions.

46. Suppose that the lines l_1, l_2, and l_3 are not all parallel to each other for example l_3 is not parallel to l_1 or to l_2. Assume that the problem has been solved (Figure 109). By Theorem 1 there is a rotation carrying AX into CZ and there is a rotation carrying BY into CZ; the angles of rotation α_1 and α_2 are equal respectively to the angles between l_1 and l_3, and between l_2 and l_3. The centers of rotation O_1 and O_2 are found just as in the first solution to Problem 45(a) – (d). From the isosceles triangles O_1XZ and O_2YZ with angles at O_1 and O_2 equal respectively to α_1 and α_2, one can find

$$\measuredangle O_1ZX = 90° - \tfrac{1}{2}\alpha_1, \qquad \measuredangle O_2ZY = 90° - \tfrac{1}{2}\alpha_2.$$

† Since there are two angle bisectors of the angles formed by l_1 and l_2, the glide reflection carrying AX into BY can be chosen in two different ways (corresponding to the cases when X and Y are situated on the same, or on opposite sides of the line AB). If $l_1 \parallel l_2$ then the axis of one of these glide reflections is parallel to l_1 and l_2 while the other axis is perpendicular to them; this explains the special role played by the case when l_1 and l_2 are parallel in the solution of parts (a), (c), (d).

From this it follows that

$$\measuredangle O_1ZO_2 = \tfrac{1}{2}(\alpha_1 \pm \alpha_2),$$

and, therefore, Z can be found as the point of intersection of l_3 with the arc of a circle constructed on the segment O_1O_2 and subtending the known angle $\tfrac{1}{2}(\alpha_1 + \alpha_2)$ or $\tfrac{1}{2}(\alpha_1 - \alpha_2)$.

Each of the angles α_1 and α_2, and each of the centers of rotation O_1 and O_2, can be determined in two different ways (compare the solution of the preceding problem). Hence there are at most 16 solutions to the problem.

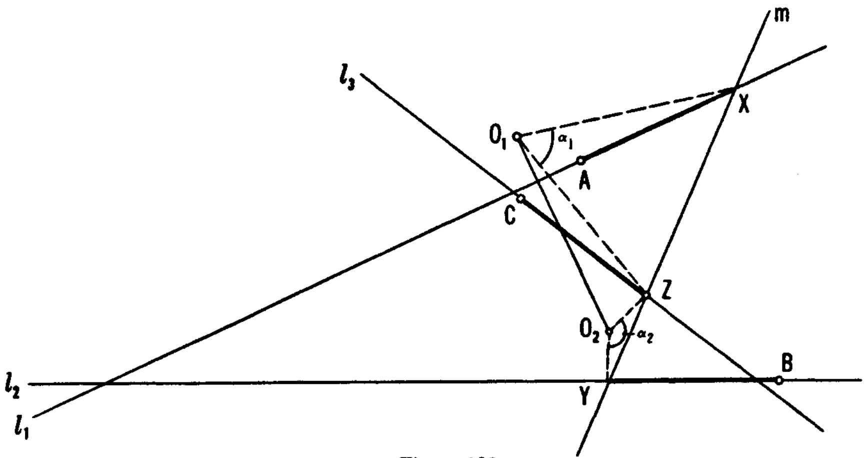


Figure 109

47. Assume that the problem has been solved (Figure 110). By Theorem 1 there is a rotation carrying BP into CQ; the angle of rotation α is equal to the angle between AB and AC, and the center of rotation O is found just as in the first solution to Problem 45(a) — (d). Since in the isosceles triangle OPQ we know the angle α at the vertex O, we also know the ratio

$$\frac{OP}{PQ} = k.$$

But by the conditions of the problem, $PQ = BP$; therefore

$$\frac{OP}{BP} = k,$$

which enables us to find P as the point of intersection of side AB with the circle that is the locus of points the ratio of whose distances to O

and B is equal to k. This geometric locus is a circle, as can be seen, for example, from the fact that the bisectors of the interior and exterior angles of ΔOPB from P (see Figure 111, where P is any point for which $OP/BP = k$) intersect the base OB in constant (independent of P) points M and N determined by the conditions

$$\frac{OM}{MB} = \frac{ON}{BN} = k = \frac{OP}{BP}$$

Since the two bisectors are perpendicular to each other, P belongs to the circle with diameter MN.[T]

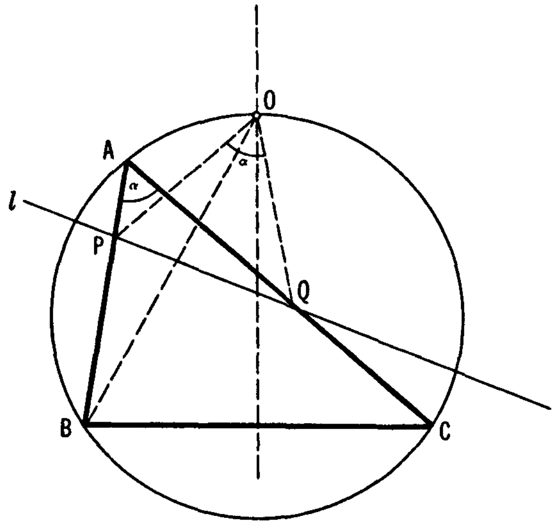


Figure 110

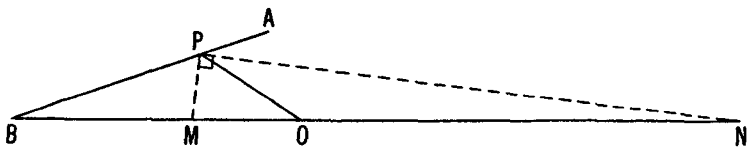


Figure 111

[T] See also page 14, Locus 11, of *College Geometry* by Nathan Altschiller-Court, Johnson Publishing Co., 1925, Richmond.

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Geometric Transformations I

I. M. Yaglom
translated by **Allen Shields**

Almost everyone is acquainted with plane Euclidean geometry as it is usually taught in high school. This book introduces the reader to a completely different way of looking at familiar geometrical facts. It is concerned with transformations of the plane that do not alter the shapes and sizes of geometric figures. Such transformations (isometries) play a fundamental role in the group-theoretic approach to geometry.

The treatment is direct and simple. The reader is introduced to new ideas and then is urged to solve problems using these ideas. The problems form an essential part of this book and the solutions are given in detail in the second half of the book.

Isaac Moisevitch Yaglom was born on March 6, 1921 in the city of Kharkov. He graduated from Sverdlovsk University in 1942 and received his Candidate's Degree (the equivalent of an American PhD) from Moscow State University in 1945. He received the DSc degree in 1965. An influential figure in mathematics education in the Soviet Union, he was the author of many scientific and expository publications. In addition to *Geometric Transformations*, English translations of his books include *Convex Figures* (Holt, Rinehart and Winston, 1961, written jointly with V. G. Boltyanskii), *Challenging Mathematical Problems with Elementary Solutions* (Holden-Day, 1964, written jointly with his twin brother Akiva M. Yaglom), *Complex Numbers in Geometry* (Academic Press, 1968), *A Simple Non-Euclidean Geometry and Its Physical Basis* (Springer, 1979), *Probability and Information* (Reidel, 1983, written jointly with Akiva), *Mathematical Structures and Mathematical Modelling* (Gordon and Breach, 1986), and *Felix Klein and Sophus Lie* (Birkhäuser, 1988). Professor Yaglom died April 17, 1988 in Moscow.

Allen Shields (1927–1989), the translator of this volume, was professor of mathematics at the University of Michigan for most of his career. He worked on a wide range of mathematical topics including measure theory, complex functions, functional analysis and operator theory.

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