IAS/PARK CITY
MATHEMATICS SERIES
Volume 27

Harmonic Analysis and Applications

Carlos E. Kenig
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Editors

American Mathematical Society
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10.1090/pcms/027

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Ian Morrison, Series Editor
Carlos E. Kenig, Fang Hua Lin, Svitlana Mayboroda, and Tatiana Toro, Volume Editors.

IAS/Park City Mathematics Institute runs mathematics education programs that bring together high school mathematics teachers, researchers in mathematics and mathematics education, undergraduate mathematics faculty, graduate students, and undergraduates to participate in distinct but overlapping programs of research and education. This volume contains the lecture notes from the Graduate Summer School program

2010 *Mathematics Subject Classification.* Primary 42-06, 53-06, 35-06, 28-06.

Library of Congress Cataloging-in-Publication Data

Names: Kenig, Carlos E., 1953– editor. | Lin, Fanghua, 1959– editor. | Mayboroda, Svitlana, 1981– editor. | Toro, Tatiana, 1964– editor. | Institute for Advanced Study (Princeton, N.J.) | Society for Industrial and Applied Mathematics.

Title: Harmonic analysis and applications / Carlos E. Kenig, Fang Hua Lin, Svitlana Mayboroda, Tatiana Toro, editors.

Description: Providence : American Mathematical Society, [2020] | Series: IAS/Park City mathematics series, 1079-5634 ; volume 27 | Includes bibliographical references.

Identifiers: LCCN 2020025415 | ISBN 9781470461270 (hardcover) | ISBN 9781470462819 (ebook)

Subjects: LCSH: Harmonic analysis. | Differential equations. | AMS: Harmonic analysis on Euclidean spaces – Proceedings, conferences, collections, etc.. | Differential geometry – Proceedings, conferences, collections, etc.. | Partial differential equations – Proceedings, conferences, collections, etc.. | Measure and integration – Proceedings, conferences, collections, etc.

Classification: LCC QA403 .H215 2020 | DDC 515/.2433–dc23

LC record available at https://lccn.loc.gov/2020025415

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10 9 8 7 6 5 4 3 2 1 25 24 23 22 21 20

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IAS/Park City Mathematics Series
S 1079-5634(XX)0000-0

Preface

The IAS/Park City Mathematics Institute (PCMI) was founded in 1991 as part of the Regional Geometry Institute initiative of the National Science Foundation. In mid-1993 the program found an institutional home at the Institute for Advanced Study (IAS) in Princeton, New Jersey.

The IAS/Park City Mathematics Institute encourages both research and education in mathematics and fosters interaction between the two. The three-week summer institute offers programs for researchers and postdoctoral scholars, graduate students, undergraduate students, high school students, undergraduate faculty, K-12 teachers, and international teachers and education researchers. The Teacher Leadership Program also includes weekend workshops and other activities during the academic year.

One of PCMI's main goals is to make all of the participants aware of the full range of activities that occur in research, mathematics training and mathematics education: the intention is to involve professional mathematicians in education and to bring current concepts in mathematics to the attention of educators. To that end, late afternoons during the summer institute are devoted to seminars and discussions of common interest to all participants, meant to encourage interaction among the various groups. Many deal with current issues in education: others treat mathematical topics at a level which encourages broad participation.

Each year the Research Program and Graduate Summer School focuses on a different mathematical area, chosen to represent some major thread of current mathematical interest. Activities in the Undergraduate Summer School and Undergraduate Faculty Program are also linked to this topic, the better to encourage interaction between participants at all levels. Lecture notes from the Graduate Summer School are published each year in this series. The prior volumes are:

- Volume 1: *Geometry and Quantum Field Theory* (1991)
- Volume 2: *Nonlinear Partial Differential Equations in Differential Geometry* (1992)
- Volume 3: *Complex Algebraic Geometry* (1993)
- Volume 4: *Gauge Theory and the Topology of Four-Manifolds* (1994)
- Volume 5: *Hyperbolic Equations and Frequency Interactions* (1995)
- Volume 6: *Probability Theory and Applications* (1996)
- Volume 7: *Symplectic Geometry and Topology* (1997)
- Volume 8: *Representation Theory of Lie Groups* (1998)
- Volume 9: *Arithmetic Algebraic Geometry* (1999)

- Volume 10: *Computational Complexity Theory* (2000)
- Volume 11: *Quantum Field Theory, Supersymmetry, and Enumerative Geometry* (2001)
- Volume 12: *Automorphic Forms and their Applications* (2002)
- Volume 13: *Geometric Combinatorics* (2004)
- Volume 14: *Mathematical Biology* (2005)
- Volume 15: *Low Dimensional Topology* (2006)
- Volume 16: *Statistical Mechanics* (2007)
- Volume 17: *Analytical and Algebraic Geometry* (2008)
- Volume 18: *Arithmetic of L-functions* (2009)
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- Volume 21: *Geometric Group Theory* (2012)
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- Volume 25: *The Mathematics of Data* (2016)
- Volume 26: *Random Matrices* (2017)

The American Mathematical Society publishes material from the Undergraduate Summer School in their Student Mathematical Library and from the Teacher Leadership Program in the series IAS/PCMI—The Teacher Program.

After more than 25 years, PCMI retains its intellectual vitality and continues to draw a remarkable group of participants each year from across the entire spectrum of mathematics, from Fields Medalists to elementary school teachers.

Rafe Mazzeo
PCMI Director
March 2017

IAS/Park City Mathematics Series
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Introduction

Carlos Kenig, Fanghua Lin, Svitlana Mayboroda, and Tatiana Toro

The origins of the harmonic analysis go back to the ingenious idea of Fourier that any reasonable function can be represented as an infinite linear combination of sines and cosines. His initial interest stemmed from the study of the heat equation, but shortly after, the Fourier series approach became a major tool in the study of diffusion processes and of propagation of waves. Today's harmonic analysis incorporates elements of geometric measure theory, number theory and probability, and has countless applications in mathematics from data analysis to image recognition, and in physics, from the classical study of sound and vibrations to contemporary problems like properties of matter waves. Due in part to an array of compelling new problems from many sciences and in part to an influx of methods and results from other areas of mathematics, harmonic analysis has seen incredible breakthroughs in the beginning of the 21st century. Hence, the time was ripe for a comprehensive school devoted to these new topics, techniques and results.

The most straightforward generalization of Fourier's decomposition into plane waves is the representation of thermal diffusion processes or wave propagation on a manifold as linear combinations of its Laplace-Beltrami eigenfunctions. It is surprising that some of the most fundamental properties of these eigenfunctions remain imperfectly understood. The first lectures in this book, by Eugenia Malinnikova, give an approachable exposition of spectacular recent progress concerning Hausdorff dimensions of nodal sets, doubling properties of solutions, and the mechanism underlying quantitative unique continuation. The latter, in a sense, describes the limitations on the rate of growth (or decay) of the eigenfunctions of second order elliptic PDEs in a way somewhat analogous to the three sphere theorem. It also leads to an array of results pertaining to Yau's conjecture and to questions of Nadirashvili regarding different aspects of the structure of the nodal domains.

In the context of the Schrödinger operator with a random or otherwise disordered potential, the situation is even more mysterious. The eigenfunctions can exhibit localization or confinement, where away from a small portion of the initial domain, they decay exponentially. In simple terms, the waves that, in a homogeneous environment, look like the familiar sines and cosines in the plane can behave like rapidly decaying exponentials in a disordered medium. This effect, still rather poorly understood, is one of the cornerstones of condensed matter physics, recognized in one of its manifestations by Anderson's Nobel Prize in Physics.

The lectures by Svetlana Jitomirskaya give an exposition of the spectral properties, localization, and structure of eigenfunctions for almost Mathieu operators and for 1D-Schrödinger operators with quasi-periodic potential, as well as their analogues and generalizations. In this context, a delicate mix of PDEs, analysis, and number theory yields beautiful and explicit results, rather rare in the world of Anderson localization, leading to the Cantor structure of the spectrum and detailed patterns of decay of the eigenfunctions.

A different direction, just as important for applications, of recent developments in harmonic analysis replaces the random potential discussed above with a random matrix of coefficients of the divergence form elliptic operator, $-\mathrm{div}\, A\nabla$. Properties of their solutions can be obtained by techniques of stochastic homogenization. Two sets of lectures treat these themes.

The first, by Zhongwei Shen, deals with fundamentals of the periodic homogenization, when the coefficients are actually deterministic, periodic and rapidly oscillating. Here, roughly speaking, are the questions. Does the system behave at large scales like a constant, non-oscillating one? If so, what is the relevant constant coefficient limit? How can the convergence and errors be described? What is the ultimate behavior of eigenvalues, eigenfunctions, and solutions to the boundary value problems?

The second set of lectures, by Charles Smart, passes to the case of random coefficients. Here again, the idea is that if A is sufficiently random in a suitable sense then the solutions at large scales behave like harmonic functions. The notes present some cutting-edge techniques in the theory, such as construction of correctors, qualitative and quantitative aspects of approximation, and proof of regularity of solutions.

Finally, we pass to the case of a rough geometric environment and the properties of PDEs in the presence of irregularities and structural defects of the boundary and of the environment itself. A lot of effort in the 20th century was devoted to understanding the properties of the solutions of boundary value problems in increasingly complicated geometric scenarios: smooth domains, domains with isolated singularities, Lipschitz domains, and eventually even rougher settings. Simultaneously, the development of the calculus of variations and related techniques led to great advances in the context of free boundary problems, which aim to address the converse question: How do properties of the systems of PDEs affect smoothness of the boundary? For instance, what are the natural domains formed by the minimizers? This has been a very active area of research, which has only recently culminated in optimal, "if and only if" results, identifying the precise structural and regularity assumptions on domain needed for key properties of the solutions to linear and nonlinear PDEs and the exact character of minimizing surfaces.

The first lectures exploring this direction, by Steve Hofmann, cover so-called T1 and Tb techniques. One of the major ways harmonic analysis enters questions of the solvability of boundary value problems is through singular integral representation formulas. The Calderón-Zygmund theory says that any reasonable singular integral operator bounded in L^2 is also bounded in all L^p spaces, $1 < p < \infty$. It is an extremely powerful and perhaps somewhat surprising result that to test L^2 boundedness of a singular integral operator it is sufficient to apply it to the identity, or a suitable local substitute, and to show that the value has bounded mean oscillation. This theorem became known as the "T of 1", and of course a detailed statement, especially in contexts more general than that of classical singular integral operators, ends up being quite intricate. The development of this circle of ideas, however, brought a resolution of an array of long-standing conjectures, including the Kato square root problem, and most recently, the proof that absolute continuity of harmonic measure with respect to the Hausdorff measure is equivalent, with some mild topological assumptions, to the geometric property of rectifiability of the boundary.

Two sets of lectures attack these questions from a different end, the geometry of minimizers. One can think, for instance, of the Plateau problem, asking what is the geometry of the area-minimizing set spanned by a given curve, or in other words, of a soap film on a given wire frame. The problem can be described as completely resolved or wide open depending on how "spanned" and "area" are defined, what properties of the initial curve are required, and how detailed a description of the minimizing surface is desired. Even problems motivated by actual applications to the real world remain open.

Guy David's lectures present recent progress in the context of Almgren's notion of minimality. They touch on weaker and more general results (local Ahlfors regularity, rectifiability, limits, monotonicity of density) in the interior of the domain, but ultimately concentrate on regularity near the boundary.

The lectures of Camillo De Lellis address area minimizing graphs, both in co-dimension 1 and in higher co-dimensions, and focus on de Giorgi's celebrated ε-regularity theorem and a new approach to Almgren's center manifold due to De Lellis and Spadaro.

As the reader has probably gleaned from some of the terminology used above, the beautiful synergy of harmonic analysis, PDEs, and geometric measure theory that leads to the preceding results has brought to the fore some structural, scale-invariant (rather than pointwise) characterizations of regularity. These roughly speaking require a set, or a measure, to be reasonably close to some other nice flat one instead of imposing conditions, typically more restrictive, such as pointwise smoothness or Lipschitz regularity. One example of such a notion is rectifiability, but there are others.

Concentrating on the geometric side of these questions, Aaron Naber's lectures present two theorems pertaining to the structure of measures and sets. One is the classical Reifenberg theorem which says that sets in Euclidean space which are well approximated by affine subspaces on all scales must be homeomorphic to balls. The other is its recent celebrated analogue, the Rectifiable Reifenberg theorem, which says that if a measure μ is summably close on all scales to affine subspaces L^k of dimension k, then μ may be decomposed as $\mu = \mu^+ + \mu^k$ where μ^k is k-rectifiable with uniform Hausdorff measure estimates, and μ^+ has uniform bounds on its mass. Circling back to the questions discussed above, we mention that these results have direct applications to the singular parts of solutions of nonlinear equations and to minimizing surfaces.

In concluding this Introduction, let us express our sincere gratitude to Rafe Mazzeo who had the initial idea for the Summer School in Harmonic Analysis and helped us carry out its planning and organization every step of the way. We are also indebted to Michelle Wachs for her help through the years leading to the event and during the program in Park City. We are very thankful to the PCMI staff—Beth Brainard and Dena Vigil—for all their efforts to make the school a successful one. We feel that that this PCMI program had a huge impact on the field, establishing and strengthening collaborative projects at all levels, and seeding a generation of successful young researchers. We are immensely thankful for this opportunity.

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The present volume is based on lectures presented at the summer school on Harmonic Analysis. These notes give fresh, concise, and high-level introductions to recent developments in the field, often with new arguments not found elsewhere. The volume will be of use both to graduate students seeking to enter the field and to senior researchers wishing to keep up with current developments.

ISBN 978-1-4704-6127-0

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