Lectures on Fractal Geometry and Dynamical Systems
Lectures on Fractal Geometry and Dynamical Systems

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Foreword: MASS and REU at Penn State University

This book is part of a collection published jointly by the American Mathematical Society and the MASS (Mathematics Advanced Study Semesters) program as a part of the Student Mathematical Library series. The books in the collection are based on lecture notes for advanced undergraduate topics courses taught at the MASS and/or Penn State summer REU (Research Experiences for Undergraduates). Each book presents a self-contained exposition of a non-standard mathematical topic, often related to current research areas, which is accessible to undergraduate students familiar with an equivalent of two years of standard college mathematics, and is suitable as a text for an upper division undergraduate course.

Started in 1996, MASS is a semester-long program for advanced undergraduate students from across the USA. The program’s curriculum amounts to sixteen credit hours. It includes three core courses from the general areas of algebra/number theory, geometry/topology, and analysis/dynamical systems, custom designed every year; an interdisciplinary seminar; and a special colloquium. In addition, every participant completes three research projects, one for each core course. The participants are fully immersed into mathematics, and
this, as well as intensive interaction among the students, usually leads to a dramatic increase in their mathematical enthusiasm and achievement. The program is unique for its kind in the United States.

The summer mathematical REU program is formally independent of MASS, but there is a significant interaction between the two: about half of the REU participants stay for the MASS semester in the fall. This makes it possible to offer research projects that require more than seven weeks (the length of the REU program) for completion. The summer program includes the MASS Fest, a two- to three-day conference at the end of the REU at which the participants present their research and that also serves as a MASS alumni reunion. A non-standard feature of the Penn State REU is that, along with research projects, the participants are taught one or two intense topics courses.

Detailed information about the MASS and REU programs at Penn State can be found on the website www.math.psu.edu/mass.
Preface

This book emerged from the course in fractal geometry and dynamical systems, with emphasis on chaotic dynamics, that I taught in the fall semester of 2008 as part of the MASS program at Penn State University.

Both fractal geometry and dynamical systems have a long history of development that is associated with many great names: Poincaré, Kolmogorov, Smale (in dynamical systems), and Cantor, Hausdorff, Besicovitch (in fractal geometry), to name a few. These two areas interact with each other since many dynamical systems (even some very simple ones) often produce fractal sets that are a source of irregular “chaotic” motions in the system.

A unifying factor for merging dynamical systems with fractal geometry is self-similarity. On the one hand, self-similarity, along with complicated geometric structure, is a crucial feature of fractal sets. On the other hand, it is related to various symmetries in dynamical systems (e.g., rescaling of time or space). This is extremely important in applications, as symmetry is an attribute of many physical laws which govern the processes described by dynamical systems.

Numerous examples of scaling and self-similarity resulting in appearance of fractals and chaotic motions are explored in the fascinating book by Schroeder [Sch91]. Motivated in part by this book, I designed and taught a course—for a group of undergraduate and
graduate students majoring in various areas of science—whose goal was to describe the necessary mathematical tools to study many of the examples in Schroeder’s book. An expanded and modified version of this previous course has become the course for MASS students that I mentioned above.

This book is aimed at undergraduate students, and requires only standard knowledge in analysis and differential equations, but the topics covered do not fall into the traditional undergraduate curriculum and may be demanding. To help the reader cope with this, we give formal definitions of notions that are not part of the standard undergraduate curriculum (e.g., of topology, metric space and measure) and we briefly discuss them. Furthermore, many crucial new concepts are introduced through examples so that the reader can get some motivation for their necessity as well as some intuition of their meaning and role.

The focus of the book is on ideas rather than on complicated techniques. Consequently, the proofs of some statements, which require rather technical arguments, are restricted to some particular cases that, while allowing for simpler methods, still capture all the essential elements of the general case. Moreover, to help the reader get a broader view of the subject, we included some results whose proofs go far beyond the scope of the book. Naturally, these proofs are omitted.

Currently, there are some textbooks for undergraduate students that introduce the reader to the dynamical system theory (see for example, [Dev92] and [HK03]) and to fractal geometry (see for example, [Fal03]), but none of them presents a systematic study of their interplay and connections to the theory of chaos. This book is meant to cover this gap.

Chapter 1 of this book starts with a discussion of the principal threefold cord of dynamics, fractals, and chaos. Here our core example is introduced—a one-dimensional linear Markov map whose biggest invariant set is a fractal and whose “typical” trajectories are chaotic. Although this map is governed by a very simple rule, it exhibits all the principal features of dynamics that are important for our purposes.
After being immersed into the interplay between dynamics and fractal geometry, the reader is invited to a more systematic study of dimension theory and its connections to dynamical systems, which are presented in Chapters 2, 3, and 4. Here the reader finds, among other things, rigorous definitions of various dimensions and descriptions of their basic properties; various methods for computing dimensions of sets, most importantly of Cantor sets; and relations between dimension and some other characteristics of dynamics.

Chapters 5 through 9 are dedicated to two “real-life” examples of dynamical systems—the FitzHugh–Nagumo model and the Lorenz model, where the former describes the propagation of a signal through the axon of a neuron cell and the latter models the behavior of fluid between two plates heated to different temperatures. While the underlying mechanism in the FitzHugh–Nagumo model is a map of the plane, the Lorenz model is a system of differential equations in three-dimensional space. This allows the reader to observe various phenomena naturally arising in dynamical systems with discrete time (maps) as well as with continuous time (flows).

An important feature of these two examples is that each system depends on some parameters (of which one is naturally selected to be the leading parameter) so that the behavior of the system varies (bifurcates) when the leading parameter changes. Thus, the reader becomes familiar in a somewhat natural way with various types of behavior emerging when the parameter changes, including homoclinic orbits, Smale’s horseshoes, and “strange” (or “chaotic”) attractors.

Let me say a few words on how the book was written. My coauthor, Vaughn Climenhaga (who at the time of writing the book was a fourth-year graduate student) was the TA for the MASS course that I taught. He was responsible for taking and writing up notes. He did this amazingly fast (usually within one or two days after the lecture) so that the students could have them in “real time”. The notes, embellished with many interesting details, examples, and some stories that he added on his own, were so professionally written that, with few exceptions, I had to do only some minor editing before they were posted on the web. These notes have become the ground material for the book. Turning them into the book required adding some new
material, restructuring, and editing. Vaughn’s participation in this process was at least an equal share, but he also produced all the pictures, the TeX source of the book, etc. I do not think that without him this book would have ever been written.

Yakov Pesin
Suggested Reading

There are many books which cover topics in fractal geometry and/or dynamical systems (although few which consider their interaction in much detail). We mention a few titles which ought to be accessible to the reader of the present volume, as well as some more advanced works which are suitable for further in-depth study of the material. Finally, we mention some background references for basic material, and some popular, less technical, accounts of the subject. Complete references may be found in the bibliography.

Concurrent reading

An introduction to dimension theory, with many aspects of modern fractal geometry, may be found in


The book includes a more detailed discussion of the Hausdorff measure of various sets than we have given here, as well as some elements of the multifractal analysis which we do not consider here.

On the dynamical side of things, a very accessible treatment of the basic concepts in dynamics is given in
Here the discussion focuses on some of the topological aspects of one-dimensional dynamics (with emphasis on bifurcation theory) as well as complex dynamics (that is, dynamical systems in which the variables are complex numbers, rather than real numbers), which we have not had space to consider here.

The reader with an interest in a more complete theoretical development of the topics in dynamical systems introduced here is encouraged to have a look at


Applications of dynamical systems to various areas of science are presented, along with the basic theory, in


A good account of the theory of chaos, with many examples of chaotic dynamical systems, can be found in


### Further reading

A more advanced treatment of the rich and variegated connections between fractal geometry and dynamical systems, including the core results in multifractal analysis of dynamics, may be found in


A concise introduction to dynamical systems, written at the graduate level, is

For the reader seeking a more encyclopedic treatment, the most comprehensive reference and text in dynamical systems available at the present time is


Our discussion of invariant measures and entropy is only the tip of the iceberg in the theory of measure-preserving transformations. A more complete discussion is given in


This includes, among other things, the traditional introduction of Kolmogorov–Sinai entropy, and a proof of the variational principle. For a discussion of entropy in its various guises, the reader is referred to


We have also only scratched the surface of bifurcation theory. A more complete account of that theory, along with many other topics, may be found in


The FitzHugh–Nagumo model is just one of many important models in mathematical biology which may profitably be studied using techniques from dynamical systems. A good overview of the field is


The Lorenz equations discussed in Chapter 9 carry within them a much richer panoply of behaviours than we have had occasion to unveil. A more complete story is told in

Background reading

The basic concepts of point set topology, which we introduced briefly in Lecture 4, along with a more complete exposition of Lebesgue measure (and a host of other basic results and techniques) may be found in either of the following:


Our discussions of measure theory (in Chapter 3), Jordan normal form and other concepts of linear algebra (in the Appendix), and the basic theory of ordinary differential equations (in Chapter 9) are all rather brief. Full details and proofs may be found in


Popular references

There are also a number of books which touch on various subjects covered in this book at a less technical level, and which are targeted at a broader audience, either within the scientific community or beyond it. One such book, which helped to motivate the course in which the present work had its genesis, is


This requires some mathematics to follow, but covers an impressively broad range of topics, and is accessible to scientists from other fields.

For historical impact, it is hard to surpass

This requires a background similar to that required by Schroeder’s book.

Finally, a very readable account of the historical development of chaos theory is given in


This book can be appreciated by specialists and laypersons alike.
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Both fractal geometry and dynamical systems have a long history of development and have provided fertile ground for many great mathematicians and much deep and important mathematics. These two areas interact with each other and with the theory of chaos in a fundamental way: many dynamical systems (even some very simple ones) produce fractal sets, which are in turn a source of irregular “chaotic” motions in the system. This book is an introduction to these two fields, with an emphasis on the relationship between them.

The first half of the book introduces some of the key ideas in fractal geometry and dimension theory—Cantor sets, Hausdorff dimension, box dimension—using dynamical notions whenever possible, particularly one-dimensional Markov maps and symbolic dynamics. Various techniques for computing Hausdorff dimension are shown, leading to a discussion of Bernoulli and Markov measures, and of the relationship between dimension, entropy, and Lyapunov exponents.

In the second half of the book some examples of dynamical systems are considered and various phenomena of chaotic behaviour are discussed, including bifurcations, hyperbolicity, attractors, horseshoes, and intermittent and persistent chaos. These phenomena are naturally revealed in the course of our study of two real models from science—the FitzHugh-Nagumo model and the Lorenz system of differential equations.

This book is accessible to undergraduate students, and requires only standard knowledge in calculus, linear algebra, and differential equations. Elements of point set topology and measure theory are introduced as needed.

This book is a result of the MASS course in analysis at Penn State University in the fall semester of 2008.