This book is an elementary introduction to knot theory. Unlike many other books on knot theory, this book has practically no prerequisites; it requires only basic plane and spatial Euclidean geometry but no knowledge of topology or group theory. It contains the first elementary proof of the existence of the Alexander polynomial of a knot or a link based on the Conway axioms, particularly the Conway skein relation. The book also contains an elementary exposition of the Jones polynomial, HOMFLY polynomial and Vassiliev knot invariants constructed using the Kontsevich integral. Additionally, there is a lecture introducing the braid group and shows its connection with knots and links.

Other important features of the book are the large number of original illustrations, numerous exercises and the absence of any references in the first eleven lectures. The last two lectures differ from the first eleven: they comprise a sketch of non-elementary topics and a brief history of the subject, including many references.
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Foreword

The present book consists of lecture notes for a one-semester introductory course in knot theory, but can also be used as a first textbook on the subject.

The book differs from other textbooks and monographs on knot theory in that it presupposes very little knowledge of the traditional prerequisites for the course. Only a few basic facts of elementary Euclidean geometry, of 2-dimensional and 3-dimensional topology, and of group theory are required. We do not use such notions from topology as the fundamental group, homology theory, coverings, properties of homeomorphisms of $\mathbb{R}^3$. From group theory, we need only the basic definitions (group, homomorphism, subgroup, quotient group).

As the result, the book does not contain such traditional topics of knot theory as the Wirtinger presentation of the fundamental group of knot complements, J.W. Alexander’s original definition of his polynomial invariant (via homomorphisms of 1-homology), Jones’s definition of his polynomial (involving representation theory, operator algebras and Markov’s theorem on the closure of braids), Victor Vassiliev’s original definition of finite type invariants (based on a cohomology spectral sequence of a filtered infinite-dimensional linear space).

But it does contain a rigorous exposition of the most important results of knot theory obtained in the last forty years: the Alexander–Conway knot polynomial (defined on the basis of the Conway skein relation),
the Jones polynomial (defined via the Kauffman bracket) and other knot polynomials (also obtained without the use of any advanced mathematics), the Vassiliev invariants (defined by axioms whose consistency is proved via the Kontsevich integrals). The set of invariants presented in this course is more powerful and aesthetically more satisfying that those coming from the fundamental group of knot complements, so that I only mildly regret the absence of Wirtinger’s algorithm in this course.

The rigorous simplified exposition of knot theory presented in the course was made possible by the work of several researchers: by Louis Kauffman (his elementary construction of the bracket named after him led to a simple definition of the Jones polynomial), by John Conway and the author of these lectures (leading to the elementary definition of the Alexander polynomial given in this course), by Joan Birman, Maxim Kontsevich, Dror Bar Natan, Pierre Vogel, Sergey Duzhin, and others (leading to the axiomatization of Vassiliev’s theory of finite-type knot invariants and to a simpler proof of their existence). The final step in making this book elementary was made by using the elementary (although long, detailed, and delicate) proof of the existence of the Conway polynomial due to Roman Garaev.

Another unusual aspect of the present course is the absence, in the first eleven lectures, of references to any books or articles, except to the books by D. Rolfsen [CD] and by S. Duzhin et. al., [DR], which are mostly mentioned to refer to various tables that they contain. To make the exposition really self contained, the course includes some preliminary material (outside of knot theory) that does not appear in the (truly very minimal!) list of prerequisites. These excursions are few and I call them “digressions”. They deal with

(i) classification of surfaces and the Euler characteristic;
(ii) graded algebras;
(iii) algorithmic problems in algebra and topology.

Lectures 12 (“Other Important Topics”) and 13 (“Brief History of Knot Theory”) were planned, but were not actually delivered to the students — semesters at the Independent University are thirteen weeks long and the midterm and final exam left no time for more than eleven lectures. Hence, the additional lectures are brief surveys rather than detailed expositions, they require mathematical knowledge beyond the declared prerequisites, and they contain numerous references.
To conclude my comments about the contents of the course, let me stress the importance of the exercises, which appear at the end of each lecture. In order to learn knot theory (and most other branches of mathematics), it is more important to be able to solve problems than to memorize the theory. This is especially important when the book is used for individual study (say in a reading course); the reader should always try to solve a good part of the exercises after reading each chapter. If it turns out that such attempts are mostly unsuccessful, the reader should return to the material of the lecture, reread it, and try to figure out which parts of the material can be used to solve the elusive exercises.

∗ ∗ ∗

This course was given online in the framework of the Math in Moscow program in the fall semester of 2020 at the Independent University of Moscow. The course consisted of an hour-and-a-half lecture via Zoom and exercise classes of the same length each week. Students were of different nationalities and worked at home in Beijing, Singapore, London, Moscow, and San Francisco. All the students turned out to be unusually bright and motivated; almost all of them submitted (by e-mail) correct solutions of practically all the problems. I am grateful to them for pointing out misprints and other errors in my original handouts (or should I say “e-mail sendouts”?) and complaining when the exposition was not clear enough.

Because of the time differences, only part of the students participated in the lecture zoom sessions (the others would look at videos of the lectures on YouTube at a later hour), and there were two exercise classes. I conducted one, the other was done by my colleague Vladimir Medvedev, to whom I am grateful for helping to find original solutions to the exercises and pointing out some mistakes in the handouts.

I am grateful to Yuri Thorkhov of MCNMO Publishers for the suggestion to publish a small printrun of the lecture notes of the original course sent to the students, to Sergei Gelfand of the AMS for proposing the publication of a book based on a seriously revised version of the lecture notes, to Sergei Lvovsky for his highly qualified editorial work, and am especially grateful to Victor Shuvalov for authoring the illustrations, reformatting the text, and correcting a few errors.
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