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Essentials of Brownian Motion and Diffusion

Frank B. Knight

American Mathematical Society

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To the memory of MY FATHER

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PREFACE

This work was first-drafted five years ago at the invitation of the editors of the Encyclopedia of Mathematics and its Applications. However, it was found to contain insufficient physical applications for that series, hence it has finally come to rest at the doorstep of the American Mathematical Society. The first half of the work is little changed from the original, a fact which may partly explain both the allusions to applications and the elementary approach. It was written to be understood by a reader having minimal familiarity with continuous time stochastic processes. The most advanced prerequisite is a discrete parameter martingale convergence theorem.

In the first half (Chapters 1 to 4) some of the details are glossed over slightly in the interest of brevity. We are confident that they will be filled in quite easily, as we have filled them in while using the material as the basis for a course on stochastic processes. It may come as a surprise here that there are no stochastic integrals. However disappointing this may be to the applied student of the subject, at least it has the advantage of allowing the treatment to be carried out path by path, so to speak, without any intrinsic use of the term "almost surely".

In the second half, by contrast, it may be found that too many details have been included. In fact, we at least sketch complete proofs of all the significant results. Our rationale for this, if it requires any, would have two bases. First, the material concerns local time, which is a more difficult topic than those treated earlier and in our view is essential to a real understanding of diffusion. Second, many of the basic concepts of current research in Markov processes find their prototypes in diffusion (as also, to some extent, in the theory of Markov chains). Therefore, it seems worthwhile to treat matters pertaining to the excursion measures and the inverse local times in some detail, since the intuition gained here may go far toward giving an understanding of more general situations.

We may now give a rapid chapter-by-chapter summary of what is covered. In Chapter 1 we construct the Brownian motion in three ways and prove a uniqueness assertion. In Chapter 2, we use these constructions to obtain some of the most familiar results, such as the law of the iterated logarithm and the nonrecurrence in two dimensions. In Chapter 3, we place the process in the general setting of Markov semigroups and strong Markov properties. The machinery of infinitesimal generators and stopping times is developed, which is indispensable to all that follows. In Chapter 4, we begin with the probabilistic solutions of the Dirichlet problem and the heat equation, which are classical except for the results concerning the Dynkin generators. Then we develop the method of "killing" the process by a continuous additive functional, and apply this to obtain a form of the Kac-Rosenblatt method of finding the distributions of functionals (Wiener integrals). Finally, we define the time-changed processes of Brownian motion leading to generators $a(x)(d^2/dx^2) + b(x)(d/dx)$, and present a number of basic examples (Bessel processes, continuous state branching processes, etc.) which play a basic role in the sequel.

In Chapter 5, we first obtain the local time processes by a random walk approximation. This is not easy, but neither are the other known methods, and the present one seems most intuitive. We then develop the general theory of excursions and diffusion local times, and end by proving the basic 0-or-1 results on Brownian motion not included in Chapter 2. §§5.1–5.3 may be considered the key to Chapters 6 and 7. These last have undergone an evolution in which Chapter 6 became shorter as it was incorporated partly in Chapter 7. At present, Chapter 6 serves as an explicit example of the general results of Chapter 7, which are done abstractly. Thus it provides an introduction to the latter, and in a few instances it provides the proofs.

In Chapter 7 we characterize and construct all diffusions on an interval which are nonsingular in the interior. It should be observed that many of the methods and results obtained earlier for Brownian motion now extend directly to the general case. The problem of sifting out those diffusions of particular interest for which explicit formulas can be given for the various relevant probabilities is not attempted here. Presumably it is to be viewed as a topic for research. At present, relatively little is known beyond the examples at the end of Chapter 4, and our bibliography may be reasonably complete.

Confining to the one-dimensional case, one can still extend the scope in two directions. The theory of interior singular points, as outlined in [**I.1**, 4.8 and 4.9], is the most immediate extension, but perhaps not the most interesting. The other is the theory of discontinuous diffusion, including birth-and-death processes. This has a literature of its own and is not considered here. For the reader interested in extending in this direction, the general additive functionals of Chapter 7 could provide the basis. The outcome of the extension is surveyed very briefly in the papers of D. Ray [**R.2**] and S. Watanabe [**W. 3**].

Because of the many and various contributions to the present work by individuals other than the author, we make no attempt to list contributors or to acknowledge help from individuals. As to institutions, the first thanks must go to the University of Strasbourg, where the work was begun in the fall of 1974. Second thanks is to the Institute for Advanced Study of Princeton, New Jersey, where the work was continued during the spring of 1975. And final thanks is to the University of Illinois at Champaign-Urbana where the work was revised and completed in the following years.

Frank B. Knight
Urbana, Illinois
December 1979

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Errata

Page	Line Number	Currently	Should be
52	-6	...whenever we have continuous paths. For general...	whenever we have continuous paths and $X_O \notin D$. For general...
56	1	$M = \prod_{k=1}^n f_k(t_k)$	$M = \prod_{k=1}^n f_k(B(t_k))$
66	25–26	in such a way that $E^y u(B_r(T_{D^c})) = u(y)$ for $y \in D_n$	in such a way that $E^y u(B_r(T_{D_n^c})) = u(y)$ for $y \in D$
66	-14	martingale for each $y \in D$	p^y-martingale for each $y \in D_n$
67	-8	if $\partial 0 \cap \partial D$ is a finite set	if $\partial \underline{0} \cap \partial D$ is a finite set
82	19 (end of equation)	$I_{\{t<T_D\}}dt$	$I_{\{t<T_{D^c}\}}dt$
89	16	[W.1, p. 79, (4)]	[W.4, p. 79, (4)]
147	7–8	upper class for $k > 2$ but not for $k \leq 2$	upper class for $k > r$ but not for $k \leq r$
169	-6	Remark. We have... ...likewise $T(x-) \geq T(x)$...	Remark. When $X(0) \leq x$, we have... ...likewise for $T(x-) \geq T(x)$...

The book aims to develop the topic of what is loosely called Brownian motion and diffusion theory in such a way as to make the fundamentals accessible to a nonspecialist in the field and to provide a sound basic grasp of the subject without going into the most refined of the technicalities. The intent has been to select and emphasize those results which either have an immediate observational meaning or which seem to contribute most to a general understanding of the subject.

The first part of the book presents general properties of the Brownian motion, including the definition, probabilistic and analytic properties, general Markov methods, generalizations, and applications. The second part contains the study of local times (in particular, the Trotter theorem) and various types of boundary conditions for Brownian motion.

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