THE MARKOFF AND LAGRANGE SPECTRA

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MATHEMATICAL SURVEYS AND MONOGRAPHS

NUMBER 30

Published by the American Mathematical Society http://dx.doi.org/10.1090/surv/030

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Providence, Rhode Island

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11J06, 11J04; Secondary 11D09, 11H50, 11J70, 11K55.

Library of Congress Cataloging-in-Publication Data

Cusick, Thomas W., 1943-
The Markoff and Lagrange spectra/Thomas W. Cusick, Mary E. Flahive.
p. cm. -- (Mathematical surveys and monographs; no. 30)
Bibliography: p. 93
ISBN 0-8218-1531-8
1. Markov spectrum. 2. Lagrange spectrum. I. Flahive, Mary E., 1948- . II. Title. III. Series.
QA242.C84 1989 89-14867
512′.72--dc20 CIP

The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability. ♾
This publication was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX,
the American Mathematical Society's TEX macro system.

Contents

Preface

Our purpose is to present a comprehensive overview of the theory of the Markoff and Lagrange spectra. The origins of the subject lie in two papers of A. Markoff from 1897–1880. Developments after that time were sporadic until the last twenty years or so, when there has been a resurgence of interest in the spectra.

We first give an account of all of the older work and then move on to the recent developments. Many of our proofs are new and we have corrected various errors in the literature. We are only concerned with the Markoff and Lagrange spectra themselves; there is hardly any mention of the diverse analogous spectra which have been defined in various contexts.

The list of references is intended to be thorough but not encyclopedic. In the text, references are identified by the author's name followed by the year in square brackets. Works in the same year are distinguished by **a**, **b**, . . . ; for example, Perron **[1921b]**.

We want to thank Richard Bumby and Harvey Cohn for helpful conversations and correspondence on various parts of this work.

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Mary E. Flahive

APPENDIX 1

Alternative Definitions of the Spectra

In Chapter 1 the Markoff spectrum $\mathbf{M}$ and the Lagrange spectrum $\mathbf{L}$ were defined as follows: Given an indefinite binary quadratic form

$$f(x,y) = ax^2 + bxy + cy^2 \tag{1}$$

with real coefficients and positive discriminant $d(f) = b^2 - 4ac$, we define the minimum $m(f)$ by

$$m(f) = \inf |f(x,y)|,$$

where the infimum is taken over all pairs of integers x, y not both zero. We define $\mathbf{M}$ to be the set of all values of $\sqrt{d(f)}/m(f)$.

For any real number α, we define $\mu(\alpha)$ by

$$\mu(\alpha)^{-1} = \limsup_{q\to\infty} |q(q\alpha - p)|, \tag{2}$$

where p and q are arbitrary integers. We define $\mathbf{L}$ to be the set of all values of $\mu(\alpha)$.

For any doubly infinite sequence of positive integers $A = \dots, a_{-j}, \dots, a_{-1}, a_0, a_1, \dots, a_k, \dots$ we define

$$\lambda_i(A) = [a_i, a_{i+1}, \dots] + [0, a_{i-1}, a_{i-2}, \dots] \quad \text{for each integer } i.$$

We further define $L(A) = \limsup \lambda_i(A)$, where the lim sup is taken over all integers i. It is easy to see that the set $\mathbf{L}$ is the set of all values of $L(A)$. Indeed, we suppose

$$\alpha = [b_0, b_1, b_2, \dots]. \tag{3}$$

By a well-known theorem on continued fractions (see Theorem 11 in Perron [**1929**, p. 45]), the inequality $|q(q\alpha - p)| < \frac{1}{2}$ implies that p/q must equal one of the convergents

$$p_n/q_n = [b_0, b_1, \dots, b_n], \qquad n = 0, 1, 2, \dots,$$

of α. Therefore, in the lim sup in (2) we may assume that $p = p_n$ and $q = q_n$ for some n. From equation (7) in Perron [**1929**, p. 43] we have the formula

$$|q_n(q_n\alpha - p_n)| = ([b_{n+1}, b_{n+2}, \dots] + [0, b_n, b_{n-1}, \dots, b_1])^{-1}. \tag{4}$$

For any doubly infinite sequence A, either $L(A) = \limsup_{i\to\infty} \lambda_i(A)$ or $L(A) = \limsup_{i\to-\infty} \lambda_i(A)$. In the former case we define $\alpha = [a_0, a_1, a_2, \dots]$, and in the latter case we define $\alpha = [a_0, a_{-1}, a_{-2}, \dots]$. In either case (4) implies that $\mu(\alpha) = L(A)$, so $\mathbf{L}$ contains all values of $L(A)$. We next suppose we are given a real number α which has the continued fraction expansion given in (3). If we define the symmetric doubly infinite sequence A by

$$A = \dots, b_2, b_1, b_0, b_1, b_2, \dots,$$

then $L(A) = \mu(\alpha)$ follows from (4). Thus, the set of all values of $L(A)$ contains $\mathbf{L}$.

Defining $M(A) = \sup \lambda_i(A)$, where the supremum is taken over all integers i, Markoff [**1879**] proved that the Markoff spectrum $\mathbf{M}$ is the set of all values of $M(A)$. The following is a summary, without proofs, of what is used from the classical reduction theory of indefinite binary quadratic forms. More detailed accounts can be found in various textbooks, for example Chapter VII in Dickson [**1929**, pp. 99–111].

An indefinite binary quadratic form (1) is said to be reduced if

$$0 < \sqrt{d(f)} - b < 2|a| < \sqrt{d(f)} + b, \qquad ac < 0.$$

Equivalently, if for $d = d(f)$ we let

$$r = \frac{-b + \sqrt{d}}{2a} \quad \text{and} \quad s = \frac{-b - \sqrt{d}}{2a}$$

denote the roots of $ax^2 + bx + c = 0$, then the form (1) is reduced if

$$|r| < 1, \quad |s| > 1, \qquad rs < 0.$$

THEOREM 1. *There are only a finite number of reduced forms* (1) *with integer coefficients and given discriminant* $d > 0$.

The set of reduced quadratic forms of a fixed discriminant d can be partitioned into one or more *chains*, as follows: We define, for any integer i,

$$f_i(x, y) = (-1)^i a_i x^2 + b_i xy - (-1)^i c_i y^2, \tag{5}$$

where each discriminant $d(f_i) = d$ and a_i, b_i are positive real numbers. There is a unique integer c_i such that the substitution $x = Y$, $y = -X + c_i Y$ takes the reduced form f_i to the (equivalent) reduced form f_{i+1}. (We recall that two forms are said to be *equivalent* if we can obtain one from the other by a substitution of the form $x = rX + sY$, $y = tX + uY$ with $ru - st = \pm 1$.) In this manner a chain $\dots, f_{-1}, f_0, f_1, \dots$ of equivalent reduced forms is obtained.

THEOREM 2. *Any two equivalent reduced indefinite binary quadratic forms of the same discriminant belong to the same chain.*

Because of Theorem 1, the chains are periodic if the forms have integer coefficients. Defining $g_i = (-1)^i c_i$, $u_i = (\sqrt{d} + b_i)/2a_{i+1}$, and $v_i = (\sqrt{d} - b_i)/2a_{i+1}$, then $g_i > 0$, $u_i > 1$, $0 < v_i < 1$ and we have $u_i = g_i + u_{i+1}^{-1}$,

$v_i^{-1} = g_{i-1} + v_{i-1}$. Therefore, the continued fraction expansions of u_i and v_i are

$$u_i = [g_i, g_{i+1}, \dots], \qquad v_i = [0, g_{i-1}, g_{i-2}, \dots].$$

Furthermore, we have

$$u_i + v_i = \sqrt{d}/a_{i+1} = [g_i, g_{i+1}, \dots] + [0, g_{i-1}, g_{i-2}, \dots]. \tag{6}$$

The form (1) is said to *represent* the real number α *properly* if there exist relatively prime integers x_0, y_0 such that $f(x_0, y_0) = \alpha$. The following classical theorem of Lagrange connects the coefficients of members of a chain of reduced forms equivalent to $f(x, y)$ with the value of $m(f)$.

THEOREM 3. *If the forms in* (5) *form a chain of reduced quadratic forms with discriminant* $d > 0$, *then the absolute value of each number less than* $\sqrt{d}/2$ *which is represented properly by a form in the chain is an element of the set* $\{a_i\colon$ *integer* $i\}$.*Moreover, if* $f(x, y)$ *is any form of the chain, then* $M(f) = \inf_i a_i$, *where the infimum is taken over all integers* i.

From (6) and Theorem 3 it follows that $\mathbf{M}$ is the set of all values of $M(A)$.

APPENDIX 2

Facts about Continuants

For any positive integers $a_1, a_2, \dots, a_n$ the *continuant* $K(a_1, a_2, \dots, a_n)$ is defined to be the denominator of the continued fraction $[0, a_1, a_2, \dots, a_n]$; that is,

$$
\begin{aligned}
&K(a_1) = a_1,\\
&K(a_1, a_2) = a_1 a_2 + 1,\\
&K(a_1, a_2, a_3) = a_1 a_2 a_3 + a_1 + a_3, \dots,\\
&K(a_1, a_2, \dots, a_n) = a_n K(a_1, \dots, a_{n-1}) + K(a_1, \dots, a_{n-2}) \quad \text{for } n \geq 3.
\end{aligned}
$$

These continuants are also sometimes called *Euler polynomials*.

Any continuant $K(a_1, a_2, \dots, a_n)$ is clearly the sum of certain products of integers from the set $\{a_1, \dots, a_n\}$. Euler gave the following simple rule for determining which products occur: First take the product of the integers $a_1, \dots, a_n$. Then take every product that is obtained from omitting any pair of adjacent integers. Then take every product that is obtained from omitting any two separate pairs of adjacent integers. Proceed in this way until no more pairs can be omitted. If n is even, the empty product (equal to one) obtained by omitting all of the integers is included at the end. In this way the set of all products in the continuant $K(a_1, \dots, a_n)$ is obtained. This rule of Euler is easily proved by induction on the above recursion relation for $K(a_1, a_2, \dots, a_n)$.

From Euler's rule it follows at once that the value of a continuant is unchanged if its integers are written in the reverse order; that is,

$$K(a_1, a_2, \dots, a_n) = K(a_n, a_{n-1}, \dots, a_1).$$

Therefore, we also have the recursion relation

$$K(a_1, a_2, \dots, a_n) = a_1 K(a_2, \dots, a_n) + K(a_3, \dots, a_n) \quad \text{for } n \geq 3.$$

This can easily be extended to the more general formula

$$K(a_1, \dots, a_n) = K(a_1, \dots, a_m) K(a_{m+1}, \dots, a_n) + K(a_1, \dots, a_{m-1}) K(a_{m+2}, \dots, a_n),$$

for $1 \leq m < n$.

APPENDIX 3

Pell Equations and Automorphs of Indefinite Quadratic Forms

A linear transformation

$$x = \alpha X + \beta Y, \qquad y = \gamma X + \delta Y, \tag{1}$$

with integer coefficients and determinant $\alpha\delta - \beta\gamma = 1$, which leaves a quadratic form

$$f(x,y) = ax^2 + bxy + cy^2 \tag{2}$$

unaltered, is called an *automorph* of $f(x,y)$. If the indefinite form $f(x,y)$ has integer coefficients and is *primitive* (i.e., the coefficients have no common factor other than ± 1), then the following theorem shows that every automorph of $f(x,y)$ is associated with a solution of the Pell equation $r^2 - ds^2 = 4$, for $d = b^2 - 4ac$.

THEOREM 1. *If the indefinite binary quadratic form* (2) *has integer coefficients and is primitive, then the transformation in* (1) *is an automorph of* $f(x,y)$ *if and only if*

$$\alpha = (r - bs)/2, \quad \beta = -cs, \quad \gamma = as, \quad \delta = (r + bs)/2, \tag{3}$$

where r, s *is an integer solution of*

$$r^2 - ds^2 = 4, \qquad d = b^2 - 4ac. \tag{4}$$

PROOF. If the transformation in (1) is an automorph of $f(x,y)$ and ξ denotes either of the roots of $f(x,1) = 0$, then

$$\xi = (\alpha\xi + \beta)/(\gamma\xi + \delta);$$

that is,

$$\gamma\xi^2 + (\delta - \alpha)\xi - \beta = 0.$$

The coefficients in the left-hand side of the last equation must be proportional to those of $f(x,y)$, say

$$\gamma = as, \quad \delta - \alpha = bs, \quad \beta = -cs. \tag{5}$$

Also, since $f(x,y)$ is primitive, s must be an integer. Defining the integer r by

(6) $$r = \alpha + \delta$$

and combining (5) and (6), we obtain

$$r^2 = (\delta - \alpha)^2 + 4\alpha\delta = b^2 s^2 + 4(1 - acs^2) = 4 + ds^2,$$

where the second equality follows from

$$\alpha\delta = \beta\gamma + 1 = -acs^2 + 1,$$

and the last equality gives (4). This proves the "only if" part of the theorem.

To prove the converse, we observe that if $f(x,y)$ is transformed by (1) into $AX^2+BXY+CY^2$, then $A = a\alpha^2+b\alpha\gamma+c\gamma^2$, $B = 2a\alpha\beta+b(\alpha\delta+\beta\gamma)+2c\gamma\delta$, $C = a\beta^2+b\beta\delta+c\delta^2$. If we substitute the equations in (3) into these equations on A, B, C and then use (4), we obtain $A = a$, $B = b$, $C = c$. Thus, (1) and (3) define an automorph, and this completes the proof of the theorem.

If the form $f(x,y)$ in Theorem 1 is *reduced* (see, for example, either Appendix 1 or Dickson [**1929**, pp. 100–102]), then the following theorem shows that we can obtain an automorph by looking at the continued fraction expansion of one of the roots of $f(x,1) = 0$.

THEOREM 2. *Suppose the indefinite binary quadratic form* (2) *has integer coefficients with $a > 0$ and is primitive and reduced. Let ξ and η with $0 < \xi < 1$ and $\eta < -1$ denote the roots of $f(x,1) = 0$. Let the completely periodic continued fraction expansion of ξ be given by $\xi = [0, \overline{a_1, a_2, \ldots, a_n}]$, where n is taken to be even and $a_1, a_2, \ldots, a_n$ is the shortest period. Let $p_j/q_j = [a_1, a_2, \ldots, a_j]$ for all $j = 1, 2, \ldots$ denote the convergents to $1/\xi$. If we define*

(7) $$\alpha = q_{n-1}, \quad \beta = q_n, \quad \gamma = p_{n-1}, \quad \delta = p_n,$$

then the determinant of the transformation in (1) *is $\alpha\delta - \beta\gamma = 1$ and is an automorph of $f(x,y)$. Moreover, this automorph satisfies* (3), *where r, s is the least positive solution of* (4).

PROOF. Since $\xi = [0, a_1, a_2, \ldots, a_n, \xi^{-1}]$, by the theory of continued fractions we have

$$\xi^{-1} = (p_n\xi^{-1} + p_{n-1})/(q_n\xi^{-1} + q_{n-1});$$

using (7), $\gamma\xi^2+(\delta-\alpha)\xi-\beta = 0$. Thus, the form $\gamma x^2+(\delta-\alpha)xy-\beta y^2 = 0$ has integer coefficients which are multiples of the coefficients of $f(x,y)$. Since n is even, we have

$$\alpha\delta - \beta\gamma = p_n q_{n-1} - p_{n-1}q_n = 1,$$

so the transformation in (1) is an automorph of $f(x,y)$. By Theorem 1, the equations in (3) hold for some integer solution r, s of (4). Since this integer solution is associated with the shortest even period of ξ, it follows from the theory of Pell equations that r, s must be the least positive solution of (4).

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