The Classification of the Finite Simple Groups, Number 6

Daniel Gorenstein
Richard Lyons
Ronald Solomon

American Mathematical Society
The Classification
of the Finite Simple Groups,
Number 6

Part IV: The Special Odd Case

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ABSTRACT. The Special Odd Case of the proof of the classification of finite simple groups is given,
as outlined in the first number of this series.

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To the memory of
Walter Feit (1930–2004)
and
Leonard D. Solomon (1916–2000)
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Preface

This volume contains the proofs of Theorems $\mathcal{C}_2$ and $\mathcal{C}_3$, as stated in the first volume of this series [I$_2$].

Theorems $\mathcal{C}_2$ and $\mathcal{C}_3$ constitute the classification of finite simple groups $G$ of special odd type. This condition requires that no 2-component $K$ of an involution centralizer of $G$ is of generic type in the sense that $K/O_2^+(K) \in S_2$, but $G$ is not of restricted even type. The latter condition is rather technical, but primarily it entails that either $G$ has 2-rank 2 or, for some 2-component $K$ of an involution centralizer of $G$, $K/Z^*(K) \cong L_2(q)$ for some odd $q$ or $K/O_2^+(K)$ is a member of a small finite set of additional quasisimple groups. In fact, we prove a strengthened version of Theorem $\mathcal{C}_2$, which we call Theorem $\mathcal{C}_2^*$, in which the ban on 2-components of generic type is relaxed. Theorem $\mathcal{C}_2^*$ permits us to classify all $\mathcal{K}$-proper simple groups $G$ having a 2-Thin Configuration in the sense of [III$_1$; 2.1]. This extension is an essential ingredient in the proof of Theorem $\mathcal{C}_7$, the classification of groups of generic type (begun in the previous volume and to be completed in the next volume). The simple groups arising as conclusions to our theorems are the finite simple groups of Lie type in odd characteristic of $BN$ rank 1 or 2 (with some exceptions) together with $L_4(q)$, $(q$ odd, $q \equiv 1$ (mod 8)), $A_n$, $n \in \{7,9,10,11\}$, and the five sporadic groups $M_{11}$, $M_{12}$, $Mc$, $Ly$ and $O'N$.

The special odd condition represents our measure of smallness for simple groups which are not of even type. Other measures have been used in the past, namely 2-rank, normal 2-rank or sectional 2-rank. Indeed, our list of conclusions differs little from the conclusions of the Sectional 2-Rank 4 Theorem of Gorenstein and Harada [GH1]. The Gorenstein-Harada Memoir depends on a long list of prior results. Some of these likewise form part of our Background Results, most notably the Feit-Thompson Theorem [FT1] on the solvability of groups of odd order, and the body of results yielding recognition theorems for the split $(B,N)$-pairs of rank 1. Others have been incorporated into our proof of Theorem $\mathcal{C}_2$, notably the classification of finite simple groups of 2-rank 2 by Gorenstein-Walter [GW1], Alperin-Brauer-Gorenstein [ABG1] and Lyons [L1] and much of the classification of finite simple groups with an abelian Sylow 2-subgroup by Walter [Wa1], together with involution centralizer recognition theorems for finite simple groups of Lie type in odd characteristic of $BN$-rank 2 by Brauer [Br5], Fong and W. J. Wong [FW1], [Fo1].

The classification of finite simple groups of 2-rank at most 2 by Brauer and Suzuki [BrSu1], Feit and Thompson, Gorenstein and Walter, Alperin and Brauer and Gorenstein, and Lyons was a major accomplishment of the 1960’s. During the late 1960’s, the Signalizer Functor Method was developed, primarily by Gorenstein and Walter, and with particular emphasis on the prime 2. The importance of 2-connectivity for this method (cf. [L$_G$; Section 22]) again focussed attention on
the dichotomy between groups \( G \) in which a Sylow 2-subgroup \( S \) has a normal abelian subgroup of rank 3 \((\text{SCN}_3(2) \neq \emptyset)\) and those having no such subgroup \((\text{SCN}_3(2) = \emptyset)\). This dichotomy had already proved critical in the Odd Order paper and in Thompson’s \( N \)-Group paper \([T2]\), and it seemed essential to classify simple groups with \( \text{SCN}_3(2) = \emptyset \) before the Signalizer Functor Method could proceed smoothly. However, the so-called “Scan-3 empty” condition is poorly behaved with respect to subgroups and quotient groups. An inductive condition was needed and this was provided when MacWilliams proved \([\text{MacW1}]\) that if \( S \) is a 2-group with \( \text{SCN}_3(S) = \emptyset \), then every section of \( S \) has 2-rank at most 4. This condition, denoted sectional 2-rank at most 4, clearly inherits to all sections of \( S \), and Gorenstein and Harada provided a classification of all finite simple groups having a Sylow 2-subgroup of sectional 2-rank at most 4 in the sizable memoir \([\text{GH1}]\).

In spite of the serious technical difficulties associated with the use of the Signalizer Functor Method in groups \( G \) of small 2-rank, Gorenstein-Harada and others, in a series of papers leading up to their memoir, managed to apply signalizer functor methods in an ad hoc manner to yield precise involution centralizers for the simple groups of sectional 2-rank 4, after which the characterization theorems of Fong-Wong \([FW1], [Fo1]\) and others could be invoked. This is not the approach we take. Instead we employ the so-called Bender Method (cf. \([\text{IG}; \text{ Section 19}]\)), when feasible.

The Bender Method originated in Bender’s revision \([\text{Be2}]\) of Walter’s classification of simple groups with an abelian Sylow 2-subgroup, though it was foreshadowed in Bender’s revision of the Feit-Thompson Uniqueness Theorem \([\text{Be6}]\). This method was extended by Goldschmidt in his classification of groups with a strongly closed abelian Sylow 2-subgroup \([\text{Go5}]\). It was also employed by Bender and Glauberman in their revision \([\text{Be5}]\), \([\text{BeG12}]\) of the Gorenstein-Walter classification of simple groups with a dihedral Sylow 2-subgroup. This work inspired the third author of this memoir to undertake, beginning in 1980, an analogous revision of the Alperin-Brauer-Gorenstein classification of simple groups with a semidihedral or wreathed Sylow 2-subgroup. We are indebted to Bob Gilman for several valuable conversations in the early stages of this work, and to Michael Aschbacher, who provided a local argument treating the case in which \( G \) contains a weakly 2-embedded \( p \)-local subgroup for some odd prime \( p \). In the end, we have not used Aschbacher’s argument, choosing instead to reuse a modular character-theoretic argument of Richard Brauer, since we were in any case unable to avoid Brauer’s results at other points in the analysis.

Beginning in 1982, the authors undertook to extend the Bender Method to as much of the Special Odd Case signalizer analysis as possible. This led to the subdivision of the Special Odd Case into two subcases:

1. \( G \) is of \( \mathcal{L} \mathcal{B}_2 \)-type, or
2. \( G \) is of \( \mathcal{L} \mathcal{T}_2 \)-type,

the former to be handled by the Bender Method (hence the \( \mathcal{B} \)) and the latter by the Signalizer Functor Method. It is not clear that the ideal subdivision was chosen. Again, the definitions are technical, but, roughly speaking, \( G \) is of \( \mathcal{L} \mathcal{B}_2 \)-type if either \( G \) has 2-rank at most 3 or some 2-component \( K \) of an involution centralizer satisfies \( K/O_2(K) \cong \text{SL}_2(q) \), for some odd \( q \). (Of course \( G \) is of special odd type.) We are indebted to Richard Foote for the suggestion that 2-fusion arguments of Aschbacher could be adapted to the \( \mathcal{L} \mathcal{B}_2 \)-type case and could lead quickly to tight
control over the structure of $O^2(C_G(z))/O^2(C_G(z))$ for some 2-central involution $z$. This suggestion is indeed implemented in the proof of Theorem PU$_5$ in [I13], and the analysis is completed in Chapter 3 of this volume, incorporating also some arguments of Harada.

At this point the stage is set for the Bender Method. A technical difficulty not encountered by Bender in [Be2] or [Be5] arises from the failure of $p$-stability and hence the impossibility of invoking Glauberman’s ZJ-Theorem. This problem was confronted originally by Alperin, Brauer and Gorenstein, who conceived an “Extended ZJ-Theorem” tailored to the Semidihedral Theorem. Our approach is inspired by Goldschmidt’s elegant method [Go5] of applying the ZJ-Theorem, not to a Sylow $p$-subgroup of $G$ but to a maximal $A$-invariant $p$-subgroup of $G$, where $A$ is a strongly closed abelian 2-subgroup of $G$. We likewise prove an equivariant ZJ-Theorem, applied to an element of $W^*_G(T, p)$ for a suitably chosen 2-subgroup $T$ of $G$, though in fact it is more convenient in general to use Glauberman’s $K^\infty$-subgroup in place of $Z(J(P))$.

By our use of the Bender Method, we achieve a more uniform treatment of the Signalizer Analysis, in which only the 2-rank 2 case stands out as presenting serious additional difficulties. The reduction in the dihedral case to the hypotheses of our background recognition theorem for $L_2(q)$ follows rather slavishly the work of Bender and Glauberman in [Be5], [Be9], and [BeGl2]. The identifications of $A_7$ and $M_{11}$, however, use arguments of S. K. Wong, which appeared first in [SW1].

The remainder of the proof of Theorem C$_2$, for the identification of $^2G_2(q)$, $G_2(q)$, $^3D_4(q)$ and $PSp_4(q)$, follows fairly closely the original arguments of Walter [Wa3] and Enguehard [E1] for the Ree groups, and Fong-Wong ([FW1], [Fo1]) for the $BN$-rank 2 groups. We do achieve some simplifications of the Fong-Wong argument in the $G_2(q)$ case. In particular, we are indebted to Korchagina [Ko1] for a short argument crucial to the identification of $G_2(9)$. The final identification of $L_4(q)$ and $U_4(q)$ is made not through their $BN$-structure but by use of a Curtis-Tits-Phan presentation.

Theorem C$_3$ treats the case in which $G$ is of Special Odd Type, but not of $L_2$-type. In particular, $G$ has 2-rank at least 4 and some involution centralizer has a 2-component $K$ with $K/O_{2^r}(K)\cong A_7$ or $L_2(q)$, $q$ odd, or a covering group of $L_3(4)$ with center of exponent 1 or 2. The only groups arising in this case are $A_9$, $A_{10}$ and $A_{11}$. Historically this case was treated in work of Gilman-Solomon [GiS1], Griess-Solomon [GrS1], Foote [Fo03], Harris-Solomon [HrS1] and Harris [Hr1], [Hr2]. The principal difficulties to be overcome in this case are to establish that $K$ may be chosen to be 2-terminal and to apply the Signalizer Functor Method when $m_2(C_G(K)/O_{2^r}(K)) \leq 2$. The former difficulty was handled in the Gilman-Solomon paper and we proceed similarly, achieving their result and somewhat more in Stage 1 of our proof. Importantly, we also establish that we may assume $m_2(C_G(K)/O_{2^r}(K)) \geq 2$, thereby bypassing the delicate fusion analysis in the work of Harris and Harris-Solomon.

Finally, we arrive at the cases treated by Foote and Griess-Solomon. They used Solomon’s Maximal 2-Component Theorem [S1] along with results and methods of Goldschmidt and Gorenstein-Harada. We instead, in Stage 2, follow a signalizer approach more in the spirit of our proof of Theorem C$_7$ in [I13]. Stage 3 provides the final identification of $A_n$ for $9 \leq n \leq 11$. 


We continue the notational conventions established in Volume 2 of this series $[I_G]$. We refer to the chapters of the current volume as $[IV_i]$, $1 \leq i \leq 9$, and $[IV_K]$, the tenth chapter, our collection of $X$-group lemmas. As in previous volumes, the chapter $[IV_K]$ is used in all the main chapters and logically precedes them.

As noted above, our work has benefitted immensely over the decades of its germination from the advice and encouragement of many friends. The inspired work of Bender, Glauberman and Goldschmidt is the strong framework on which much of the proof of Theorem $C_2$ is constructed, along with the brilliant 2-fusion analyses of Harada. Valuable conversations and collaborations with many colleagues have contributed greatly to the final product. In addition to those already mentioned, Bob Gilman, Richard Foote, S. K. Wong, Inna Korchagina, and Michael Aschbacher, we offer thanks to Curt Bennett, Ralf Gramlich, Koichiro Harada, Corneliu Hoffman and Sergei Shpectorov.

We gratefully acknowledge many years of support by the National Science Foundation. Much of this work was achieved during numerous visits by the third author to Rutgers University, most notably from January 1983 to June 1984. At all times, he has appreciated the university’s generosity and cherished the warm hospitality of the faculty, whom he regards as a second family.

Little did the third author suspect in 1980 that work begun at the time his eldest son was born would not see the light of day until that son was 24 years old. It took the vision and coercive power of Danny Gorenstein during the decade 1982–92 to inflate a modest exercise into a major endeavor. Most of this volume is the fruit of a close collaborative effort of the three authors throughout much of the 1980’s, with help from our friends, as noted above. In particular, Danny’s inspiration and energy is evident throughout this volume, and we miss him as much as ever.

As we were putting the finishing touches on this volume, we were shocked and saddened to learn of the death of Walter Feit. When one thinks of Walter’s contributions to the Classification Theorem, of course one thinks first of the pioneering Odd Order Paper of Feit and Thompson, which changed the entire landscape of the subject. But, especially in the context of this volume, it should be remembered that Walter was one of the architects of the classification of Zassenhaus groups, historically the first step toward the classification of split $BN$ pairs of rank 1. Also the Feit-Higman Theorem on finite generalized $n$-gons has a significant bearing on the classification of split $BN$ pairs of rank 2. All this is true. But Walter’s impact on the authors of this volume is far deeper and more personal. Much of Danny Gorenstein’s zest for finite simple groups was acquired during a pivotal year spent at Cornell University in Walter’s company. And the second author, as a Gibbs instructor, and the third author, as a graduate student, gained so much from Walter and the little group theory paradise – shared with David Goldschmidt and Leonard Scott – which he created in New Haven in the years 1969–1972. Walter loved life, loved mathematics, loved good food and good company. To be with him was to see his eyes twinkle and to share those joys.

Richard Lyons and Ronald Solomon
July, 2004
Background References

NOTE. The chapters of the current number are referenced as [IV$_1$], [IV$_2$], . . . , [IV$_9$] and [IV$_K$]=[IV$_{10}$].

The previous numbers of this series are referenced as follows.


NOTE. The full list of Background References appears in the first book of this series. The list below contains all Background References to which we refer in this book. The numbering of the Background and the Expository References is consistent with that in the earlier books.


[FT1] W. Feit and J. G. Thompson, Solvability of groups of odd order (Chapter V, and the supporting material from Chapters II and III only), Pacific J. Math. 13 (1963), 775–1029.


**Note.** The following reference is cited for a few facts from elementary number theory. It is thus to be added as Background References, but strictly limited to these citations.

Expository References


[Fo1] P. Fong, A characterization of the finite simple groups $\text{PSp}(4, q)$, $G_2(q)$, $D_4^2(q)$, II, Nagoya Math. J. 39 (1970), 30–79.


## Glossary

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