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Morita Equivalence and Continuous-Trace C^*-Algebras

Iain Raeburn
Dana P. Williams

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Volume 60

Morita Equivalence and Continuous-Trace C^*-Algebras

Iain Raeburn
Dana P. Williams

American Mathematical Society

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ABSTRACT. We give a modern treatment of the classification of continuous-trace C^*-algebras up to Morita equivalence. This includes a detailed discussion of Morita equivalence of C^*-algebras, a review of the necessary sheaf cohomology, and an overview of recent developments in the area. The book should be accessible to anyone familiar with the basics of C^*-algebras up to the GNS-construction.

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To our parents:
Margaret and Fraser Raeburn
on their golden wedding anniversary,
and in memory of
Ruth and James Williams

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Introduction

The Gelfand-Naimark theorem says that a commutative C^*-algebra A with identity is determined up to isomorphism by its spectrum T, in the very strong sense that A is isomorphic to the algebra $C(T)$ of continuous functions on T. There are also non-commutative C^*-algebras with spectrum T: for example, the algebra $C(T, M_n(\mathbb{C}))$ of continuous functions from T into the matrix algebra $M_n(\mathbb{C})$. In this book we study theorems which classify algebras with spectrum T and automorphisms of these algebras in terms of topological invariants of the space T.

One of the goals of algebraic topology is to associate to a space T algebraic objects whose properties reflect the topological structure of T. The ones of interest to us are *cohomology groups* $H^n(T;\mathbb{Z})$: very roughly speaking, $H^n(T;\mathbb{Z})$ counts n-dimensional holes in the space T by chalking up a copy of $\mathbb{Z}$ for each such hole. Our C^*-algebras with spectrum T are classified by classes in the 3-dimensional cohomology group $H^3(T;\mathbb{Z})$, and their automorphisms by classes in $H^2(T;\mathbb{Z})$; these theorems were first proved, respectively, by Dixmier and Douady in 1963 and by J. Phillips and Raeburn in 1980. They have been used extensively in recent years: in the analysis of C^*-dynamical systems and their crossed products, in the K-theory of C^*-algebras [150], in differential geometry [14], and in mathematical physics [16]. We shall discuss all the background material needed to prove these classification theorems and apply them to C^*-dynamical systems, including the definitions and properties of the cohomology groups themselves.

There are several versions of the Dixmier-Douady Classification Theorem, involving different families of C^*-algebras and classifying up to different equivalence relations. The simplest concerns locally trivial bundles over T with fibres isomorphic to the algebra of compact operators on an infinite-dimensional Hilbert space, and identifies them up to bundle isomorphism; we shall prove this one first. For applications to operator algebra, it is more satisfactory to work with continuous-trace C^*-algebras, as Dixmier and Douady did. They used continuous fields of C^*-algebras, which had been developed by Fell to prove structure theorems for important classes of C^*-algebras. However, Fell's theory never became standard C^*-algebraic equipment, so anybody wanting to use the Dixmier-Douady classification faced a heavy start-up cost. We have used instead Rieffel's theory of Morita equivalence for C^*-algebras, which is now recognized to be a fundamental tool for C^*-algebraists. It has been folklore for the past 20 years that the Dixmier-Douady classification can be couched in terms of Morita equivalence (see, for example, [5] or [140, 20]), but this version has remained logically dependent on the special structure

theories developed by Fell and Dixmier-Douady. We aim to give a direct treatment, starting from the basic theory of C^*-algebras.

Morita equivalence was adapted to C^*-algebras by Rieffel in the 1970's in connection with a C^*-algebraic version of the Mackey machine [144, 147], and it has since become a standard tool for analyzing group C^*-algebras and crossed products [148, 62]. Although Rieffel's papers were carefully motivated and expounded, their primary concern was representation theory, and the general theory of Morita equivalence they contained only gradually emerged. We have therefore decided to give a detailed introduction to this general theory. To do this in modern language, we have to discuss Hilbert modules, and we have preferred to be self-contained: the recent expositions of Hilbert modules by Lance [94] and Wegge-Olsen [167] are tilted towards quantum groups and K-theory rather than Morita equivalence, and hence their motivating examples are different.

The classification theorems necessarily involve sheaf cohomology. It is slightly nonstandard sheaf cohomology: one has to deal with sheaves of nonabelian groups, and hence the techniques of homological algebra do not apply directly. (And many functional analysts would probably prefer to avoid these techniques anyway.) So we have chosen to give a concrete introduction to sheaf cohomology and fibre bundles based on cocycles. Thus our proof of the bundle-theoretic Dixmier-Douady classification is entirely self-contained, and should be accessible to readers from other areas as well as operator algebraists.

Our treatment of continuous-trace C^*-algebras reflects our emphasis on Morita equivalence: for us, a continuous-trace C^*-algebra A with spectrum T is almost by definition a C^*-algebra which is locally Morita equivalent to the commutative C^*-algebra $C(T)$, and the Dixmier-Douady class $\delta(A) \in H^3(T;\mathbb{Z})$ is the obstruction to the existence of a global Morita equivalence. To make all this precise, we need background material on the spectrum and primitive ideal space of a C^*-algebra, and we have included an introduction to the subject in Appendix A.

We originally intended to phrase all our results and calculations in terms of Morita equivalence. However, as we progressed we found this attitude restrictive, and noticed in particular that it is often easier to compute $\delta(A)$ using local rank-one projections and partial isometries, as in [127, §2]. We have come to believe that the power and enduring interest of the Dixmier-Douady theory lies in its many formulations, and we hope that our treatment will prove useful from several viewpoints. We have therefore been careful to tie up our main theorem with the bundle-theoretic version by including a discussion of the Stabilization Theorems of Kasparov and Brown-Green-Rieffel.

One punchline of our book is a version of the Dixmier-Douady Theorem in which the map $A \mapsto \delta(A)$ induces an isomorphism of a Brauer group of continuous-trace algebras with fixed spectrum T onto the cohomology group $H^3(T;\mathbb{Z})$. This lovely theorem is part of the folklore we are aiming to tidy up: it was mentioned almost in passing by Taylor in 1975 [161], and discussed in seminar notes by Green around 1978 [61]. However, even to formulate it, we need nontrivial facts about the spectra of tensor product C^*-algebras; again we have developed the necessary material in an Appendix.

When we started this book, one goal was to give a readable and complete intro-

duction to the research area which has been described by Rosenberg as the study of the "fine structure of the Mackey machine" [151]. Loosely speaking, this concerns actions of groups on continuous-trace algebras and the structure of the associated crossed products. We are still several books short of this goal, so we have compromised by providing an overview of the area and of the remaining background. This is the content of Chapter 7. We have used as a unifying theme the equivariant Brauer group introduced in [20]. The structure theory of this group involves a variety of cohomological invariants and principal bundles, and describes many fascinating relations between them; there are many more, we suspect, waiting to be found.

We have made a serious effort to keep required background to a minimum. There is by now a fairly standard first course in C^*-algebras, covering the Gelfand-Naimark Theorem, the continuous functional calculus, positivity, and the GNS-construction. We assume that readers have seen this; everything we need is in the first 3 chapters of Murphy's book [111], for example. We make the usual conventions of the subject: for example, homomorphisms of C^*-algebras are always $*$-preserving, and ideals are closed and two-sided unless otherwise stated. We have had to use some general topology and functional analysis, but we have tried not to overdo this, and to give detailed references. (Our preferred authority here is Pedersen's book [125], which covers both topology and functional analysis, is elegantly written, and has a good and practical selection of topics.) On the other hand, we have been careful to develop all the algebraic topology we need, and have arranged our material so that parts involving locally compact groups and integration can be avoided without loss of continuity.

We hope that students who have seen the basics of C^*-algebras at the Honours level in the UK or Australia, the Diploma level in Europe, or the beginning graduate level in North America can read this book and enjoy doing so. We have been careful to make the different sections of the book as self-contained as possible, so that those who want to know about Hilbert modules and Morita equivalence, the spectrum of a C^*-algebra, tensor products of C^*-algebras or basic sheaf cohomology should be able to turn to the relevant section and learn something useful without absurd difficulty.

Reader's Guide

Chapter 1 is about the algebra of compact operators and its properties. This algebra is fundamental both in the structure theory of nice C^*-algebras and as motivation for the theory of Morita equivalence. More experienced readers could probably skip or skim this chapter, but others may find it helpful to see our notation and viewpoint in a relatively elementary setting.

Chapter 2 is a self-contained introduction to Hilbert modules, multiplier algebras, and Rieffel's general theory of induced representations. Induced representations are defined using tensor products of Hilbert modules, but these are technically much easier than the tensor products of C^*-algebras discussed in Appendix B, and

we do not assume previous familiarity with tensor products of any sort. Rieffel's motivation came from the unitary representation theory of locally compact groups, and we have discussed this application in outline at the end of each section. However, this material can be safely and logically skipped by those who are not interested or not familiar with integration on locally compact groups. For those who are, we have included a detailed discussion in Appendix C.

Chapter 3 concerns imprimitivity bimodules and Morita equivalence of C^*-algebras, and builds on our previous treatment of Hilbert modules. We discuss in detail the Rieffel correspondence between ideals and quotients of Morita-equivalent C^*-algebras. From this and the definition of the topology on the primitive ideal space, we obtain the existence of the Rieffel homeomorphism between the primitive ideal spaces of Morita-equivalent C^*-algebras. Section 3.4, on the external tensor product of imprimitivity bimodules, uses the basic properties of the spatial tensor product discussed in Appendix B.1. Since this material is not used until Chapter 6, Section 3.4 can be skipped at this stage.

Chapter 4 is a self-contained introduction to sheaf theory covering the material necessary for basic applications to C^*-algebras, which should be accessible even to homologically-challenged functional analysts. The only slightly nonstandard algebraic background required is the direct-limit construction discussed in Appendix D.1. Those seeking instant gratification in the form of applications to C^*-algebras should see Proposition 4.27, which will be much more informative to those who have read Chapter 1. All the sheaf cohomology needed for Chapter 5 is contained in Section 4.1. In Section 4.2 we discuss fibre bundles and their classification in terms of sheaf cohomology: these ideas are used freely in the recent literature on crossed products of continuous-trace C^*-algebras, as outlined in Chapter 7. We can then give the Dixmier-Douady classification for locally trivial fibre bundles with fibre the algebra $\mathcal{K}(\mathcal{H})$ of compact operators. This version of the Dixmier-Douady classification theorem uses only the material in Chapters 1 and 4. In the last few pages of Chapter 4, we explain why this version of the theorem is not ideal for those working in C^*-algebras.

Chapter 5 is the heart of the book: the classification of continuous-trace C^*-algebras. To define this class, we need to consider C^*-algebras with Hausdorff spectrum. To appreciate the material in Section 5.1, one needs a working knowledge of the spectrum and primitive ideal space of a C^*-algebra; all the necessary details are in Appendix A. In Section 5.2, we define continuous-trace C^*-algebras, discuss the alternative characterizations, and give many nontrivial examples.

Our treatment of the Dixmier-Douady classification theorem and the analogous classification of automorphisms depends on all the earlier chapters. We give two versions of the Dixmier-Douady Theorem: the classification up to Morita equivalence, and the classification of separable stable continuous-trace C^*-algebras up to isomorphism. The latter depends on the Brown-Green-Rieffel Theorem, which says that Morita equivalence is the same as stable isomorphism, and which we prove in Section 5.5. This is a deep theorem, and the discussion in Section 5.5 is necessarily at a higher level than most of the main text. We conclude the chapter with a brief discussion of the pros and cons of the different versions of the Dixmier-Douady classification. At this stage we do not complete the classification

of automorphisms, since the most efficient route to this involves more sophisticated use of tensor products of C^*-algebras.

Chapter 6 begins with our punchline: the identification of the Brauer group $\mathrm{Br}(T)$ of continuous-trace C^*-algebras with $H^3(T;\mathbb{Z})$. The multiplication in $\mathrm{Br}(T)$ is a balanced C^*-algebraic tensor product; to understand it, we need the description of the spectrum of a tensor product from Appendices B.1–B.4. Once we have this balanced tensor product, we can quickly finish the classification of automorphisms of continuous-trace algebras. The other sections of Chapter 6 concern two constructions which yield interesting examples of continuous-trace C^*-algebras. The pull-back construction of Section 6.2, which is defined using the balanced tensor product, shows how to make the Brauer group into a functor. The inducing construction of Section 6.3 is a C^*-algebraic version of Mackey's original construction of the Hilbert space of an induced representation; it yields a class of nontrivial continuous-trace C^*-algebras whose Dixmier-Douady class we can explicitly compute.

The first two Appendices discuss aspects of the general theory of C^*-algebras which would not normally be covered in a first course on the subject, but which are required in the book. The first is a relatively gentle introduction to the spectrum of a C^*-algebra which highlights the examples of interest to us. The second is a self-contained treatment of tensor products of C^*-algebras whose main goal is to prove that the spectrum of the tensor product of two continuous-trace C^*-algebras is the product of their spectra. This result is certainly known, but it is hard to point to a detailed proof in the existing literature. Only the first four sections of Appendix B are used in the book proper; the fifth has been included to provide a detailed source for other known facts. The third Appendix contains a proof of Rieffel's formulation of Mackey's Imprimitivity Theorem; this material is logically independent of the rest of the book, and is included because we would be giving a lop-sided view of Morita equivalence if some applications to representation theory were not mentioned.

Hooptedoodle. This book occasionally breaks for a *Hooptedoodle.* These contain comments which go beyond the prevailing scope and level, but which might help put the material in context. The term Hooptedoodle comes from the prologue of John Steinbeck's *Sweet Thursday*, in which one of the main characters criticizes the previous book *Cannery Row*:

> Sometimes I want a book to break loose with a bunch of hooptedoodle. ... But I wish it was set aside so I don't have to read it. ... Then I can skip it if I want to, or maybe go back to it after I know how the story comes out.

Steinbeck then dutifully labels his digressions as hooptedoodles.

Material in our hooptedoodles is not used in the main text, and can easily be skipped. Indeed, they *should* be skipped if they don't seem to be helping!

Acknowledgments. We have tried to present a detailed account of the theory of continuous-trace C^*-algebras, along with the background necessary to read the current literature. Our account is logical rather than historical: we have not made any attempt to give detailed attributions, and we apologize to those we have not

mentioned*. We do acknowledge obvious debts to Rieffel and to Dixmier-Douady; where our subject matter overlaps, we have been unable to improve on Lance's treatment of Hilbert modules.

We thank those who helped us with this book. In addition to those who taught and introduced us to the subject, we thank the participants in the various seminars we gave at Newcastle based on parts of the book. We specifically thank Neal Fowler, Astrid an Huef, Wojciech Szymański, David Webb, and especially Paul Muhly for reading large parts of drafts and for their helpful suggestions. The remaining mistakes are the other author's fault.

*We did start out with better intentions, but soon realized we could cause even more offense this way. Our memories are not what we think they once were.

Appendix A

The Spectrum

The Gelfand theory identifies the maximal ideal space of a commutative Banach algebra as a crucial topological invariant — indeed, it determines a commutative C^*-algebra up to isomorphism (by the Gelfand-Naimark Theorem). Attempts to produce an analogue of the Gelfand-Naimark Theorem for noncommutative algebras have not been completely successful, but they have given us two important generalizations of the maximal ideal space: the spectrum and the primitive ideal space. Both sets carry natural topologies, and it is important in applications that we can describe the topology on the spectrum in several different ways. In this appendix we shall discuss two of these descriptions and some of their basic properties.

Although the GNS-construction of representations of C^*-algebras from positive functionals is really part of our stated prerequisites, its extension to algebras without identities raises a few subtleties which are not always fully discussed in elementary treatments. We therefore begin by reviewing this construction. In Section A.2, we discuss an ideal-theoretic approach to the topology on the spectrum, in which the topology is lifted from the *hull-kernel topology* on the set of primitive ideals. We use examples to illustrate the behavior of this topology and its implications for the structure theory of C^*-algebras, and then prove some basic properties of the spectrum, including a fundamental result called the Dauns-Hofmann Theorem. We can also define a topology by viewing the spectrum as the image under the GNS-construction of a collection of positive functionals, and pushing the weak-$*$ topology forward. In Section A.4 we explain how this works, and prove that it gives the same topology, so that one can switch freely between the two descriptions.

A.1 States and Representations

A *representation* of a C^*-algebra A is a homomorphism of A into the C^*-algebra $B(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$; just as we always assume that homomorphisms of C^*-algebras are $*$-preserving, so representations π of C^*-algebras automatically satisfy $\pi(a^*) = \pi(a)^*$. We often write $\mathcal{H}_\pi$ for the underlying Hilbert space $\mathcal{H}$ of a representation $\pi : A \to B(\mathcal{H})$. We are interested in sets

of representations which are large enough to describe A but small enough to be manageable, so we need to decide when two representations are essentially the same and to be able to decompose representations into smaller ones.

Two representations $\pi : A \to B(\mathcal{H}_\pi)$, $\rho : A \to B(\mathcal{H}_\rho)$ are *unitarily equivalent* (or just equivalent) if there is a unitary isomorphism U of $\mathcal{H}_\rho$ onto $\mathcal{H}_\pi$ such that

$$\pi(a) = U\rho(a)U^* \quad \text{for all } a \in A;$$

we then think of π and ρ as being the same representation, and write $\pi \sim \rho$. It is easy to check that unitary equivalence is an equivalence relation in the usual sense. If $\pi : A \to B(\mathcal{H}_\pi)$, $\rho : A \to B(\mathcal{H}_\rho)$ are representations, the formula

$$\pi \oplus \rho(a)(h,k) := \big(\pi(a)h, \rho(a)k\big)$$

defines a representation $\pi \oplus \rho$ of A on the Hilbert space direct sum $\mathcal{H}_\pi \oplus \mathcal{H}_\rho$, called the *direct sum* of π and ρ. Conversely, we think of an equivalence of a representation π with a direct sum $\rho_1 \oplus \rho_2$ as being a decomposition of π in terms of the smaller representations ρ_1 and ρ_2.

A closed subspace $\mathcal{K}$ of $\mathcal{H}_\pi$ is *invariant* under π if $\pi(a)k \in \mathcal{K}$ for every $a \in A$ and $k \in \mathcal{K}$. If $\mathcal{K}$ is invariant and y belongs to the orthogonal complement $\mathcal{K}^\perp$, then

$$(k \mid \pi(a)y) = (\pi(a)^*k \mid y) = (\pi(a^*)k \mid y) = 0 \quad \text{for all } k \in \mathcal{K},$$

so $\mathcal{K}^\perp$ is also invariant. For example, $\mathcal{H}_\pi = \{\,(h,0)\,\}$ and $\mathcal{H}_\rho = \{\,(0,k)\,\}$ are invariant for $\pi \oplus \rho$. Conversely, if $\mathcal{K}$ is invariant, both $\pi|_{\mathcal{K}} : a \mapsto \pi(a)|_{\mathcal{K}}$ and $\pi|_{\mathcal{K}^\perp}$ are representations of A, and we can recover π as the direct sum $\pi|_{\mathcal{K}} \oplus \pi|_{\mathcal{K}^\perp}$; more precisely, the canonical unitary isomorphism $U : (k,y) \mapsto k + y$ of $\mathcal{K} \oplus \mathcal{K}^\perp$ onto $\mathcal{H}_\pi$ implements an equivalence between $\pi|_{\mathcal{K}} \oplus \pi|_{\mathcal{K}^\perp}$ and π. A representation π is *irreducible* if there is no closed invariant subspace apart from $\{\,0\,\}$ and $\mathcal{H}_\pi$, and hence no nontrivial direct sum decomposition. Equivalently:

Lemma A.1. *A representation π of a C^*-algebra A is irreducible if and only if the only operators commuting with $\pi(A)$ are multiples of the identity operator $1_{\mathcal{H}_\pi}$.*

Proof. If π has $\mathcal{V}$ as a nontrivial invariant subspace, then the orthogonal projection P of $\mathcal{H}$ onto $\mathcal{V}$ commutes with every $\pi(a)$, and is not scalar. Conversely, if there is a non-scalar operator T commuting with $\pi(A)$, either the real or imaginary part of T is a non-scalar self-adjoint operator S commuting with $\pi(A)$. Some spectral projection P for S is not 0 or 1, and it too commutes with $\pi(A)$.* Then $P(\mathcal{H})$ is a nontrivial subspace of $\mathcal{H}$ which is invariant for π. □

Hooptedoodle A.2. For representations on infinite-dimensional Hilbert spaces, the notion of irreducibility we have defined above is sometimes called *topological irreducibility* to distinguish it from *algebraic irreducibility*: a representation is algebraically irreducible if it has no invariant subspaces, closed or not. Remarkably, these two notions coincide for representations of a C^*-algebra [28, Corollary 2.8.4].

*Alternatively, for those who have seen the continuous functional calculus rather than the classical spectral theorem: S non-scalar and selfadjoint implies that $\sigma(S)$ consists of more than one point. Thus there are nonzero f and g in $C(\sigma(S))$ such that $fg = 0$. Since $f(S), g(S) \in C^*(S) \subset \pi(A)'$, $\overline{f(S)\mathcal{H}}$ and $\overline{g(S)\mathcal{H}}$ are nonzero mutually orthogonal invariant subspaces, and π is reducible.

The *essential subspace* of a representation π of A is the closed subspace

$$\mathcal{K} := \overline{\text{span}}\{\, \pi(a)h : h \in \mathcal{H}_\pi, a \in A \,\}$$

of $\mathcal{H}_\pi$. The representation π is said to be *nondegenerate* if its essential subspace $\mathcal{K}$ is all of $\mathcal{H}_\pi$; this is equivalent to $\pi(1) = 1$ if A has an identity, and in general to the assertion that $\pi(e_i) \to 1$ strongly (i.e., $\pi(e_i)h \to h$ for all $h \in \mathcal{H}_\pi$) whenever $\{\, e_i \,\}$ is an approximate identity for A.

The essential subspace $\mathcal{K}$ of a representation is always invariant, and π is equivalent to $\pi|_{\mathcal{K}} \oplus \pi|_{\mathcal{K}^\perp} = \pi|_{\mathcal{K}} \oplus 0$. More generally, if I is an ideal in A, then $\mathcal{K} := \overline{\text{span}}\{\, \pi(a)h : h \in \mathcal{H}_\pi, a \in I \,\}$ is invariant, but π is not zero on $\mathcal{K}^\perp$ unless I is an essential ideal. Conversely, any nondegenerate representation of an ideal I extends canonically to a nondegenerate representation π of A on the same space (this a special case of of Proposition 2.50).

Remark A.3. That invariance of $\mathcal{K}$ implies invariance of $\mathcal{K}^\perp$ is a feature of $*$-representations of $*$-algebras; in fancy language, it says invariant subspaces are *reducing*. In other parts of mathematics, this is not automatic: for example, in dealing with representations of groups by invertible operators (rather than unitary ones), one has to distinguish between invariant and reducing subspaces.

The fundamental theorems about representations of C^*-algebras say that there are lots of them. Thus the Gelfand-Naimark Theorem[†] says that every C^*-algebra has a *faithful* representation — that is, a representation π with $\ker \pi = \{\, 0 \,\}$. We shall be particularly interested in irreducible representations, and here the first main result says that there are enough to separate points of A (Theorem A.14). All these representations are built using the GNS-construction, which starts from a positive functional on A. Here "GNS" stands for Gelfand-Naimark-Segal.

Recall that an element a of a C^*-algebra A is *positive* (written $a \geq 0$) if there is another element b of A with $a = b^*b$. Basic but highly nontrivial theorems assert that a is positive if and only if $\sigma(a) \subset [0, \infty)$ [111, 2.1.8 and 2.2.4], and that $\|a\|^2 b^*b - b^*a^*ab \geq 0$ for every $a, b \in A$. (To see the latter when A does not have an identity, note that because $\sigma(a)$ is by definition the spectrum in A^1, positivity is determined in A^1; since $\|a\|^2 1 - a^*a \geq 0$ in A^1, it follows from [111, 2.2.5] that $b^*(\|a\|^2 1 - a^*a)b \geq 0$ in A^1, and hence in A.) A linear functional ρ on A is *positive* if $a \geq 0 \implies \rho(a) \geq 0$; equivalently, ρ is positive if $\rho(b^*b) \geq 0$ for all $b \in A$. A positive linear functional of norm 1 is called a *state*. We need a basic lemma:

Lemma A.4. *Let f be a positive functional on a C^*-algebra A. Then for all $a, b \in A$ we have*

(a) $f(b^*a) = \overline{f(a^*b)}$;

(b) (The Cauchy-Schwarz inequality) $|f(b^*a)|^2 \leq f(b^*b)f(a^*a)$.

[†]There are two Gelfand-Naimark Theorems, and both are important: the first says that a *commutative* C^*-algebra is canonically isomorphic to an algebra of continuous functions, and the second that every C^*-algebra can be realized as a closed $*$-subalgebra of some $B(\mathcal{H})$. It is this second theorem which we shall prove in this section.

Proof. Fix $a, b \in A$. Then, for any $\lambda \in \mathbb{C}$,

$$0 \le f\big((\lambda a + b)^*(\lambda a + b)\big) = |\lambda|^2 f(a^*a) + \overline{\lambda} f(a^*b) + \lambda f(b^*a) + f(b^*b). \qquad \text{(A.1)}$$

First of all, since $|\lambda|^2 f(a^*a) + f(b^*b)$ is always real, we have

$$\operatorname{Im}\big(\overline{\lambda} f(a^*b) + \lambda f(b^*a)\big) = 0$$

for all λ. Taking $\lambda = 1$ and then $\lambda = i$ gives the equality of the imaginary and real parts of $f(a^*b)$ and $\overline{f(b^*a)}$. Next, we put $\lambda = x\overline{f(b^*a)}$ for $x \in \mathbb{R}$ in (A.1) to obtain

$$\begin{aligned} 0 &\le x^2 |f(b^*a)|^2 f(a^*a) + x|f(a^*b)|^2 + x|f(b^*a)|^2 + f(b^*b) \\ &= x^2 |f(b^*a)|^2 f(a^*a) + 2x|f(b^*a)|^2 + f(b^*b). \end{aligned}$$

Since the right-hand-side is a quadratic in x which is always ≥ 0, this implies

$$4|f(b^*a)|^4 - 4|f(b^*a)|^2 f(a^*a) f(b^*b) \le 0,$$

which gives (b). □

For the GNS-construction, we start with a state τ on a C^*-algebra A. We let

$$N_\tau := \{\, a \in A : \tau(a^*a) = 0 \,\},$$

and deduce from Lemma A.4 that $\tau(b^*a) = 0$ if either a or b lies in N_τ. Thus there is a well-defined inner product on A/N_τ such that

$$(a + N_\tau \mid b + N_\tau) = \tau(b^*a).$$

We let $\mathcal{H}_\tau$ be the Hilbert space completion of A/N_τ. Another application of Lemma A.4 shows that N_τ is a left ideal in A, so A acts by left multiplication on A/N_τ; since $\|a\|^2 b^*b - b^*a^*ab$ has the form c^*c, we have

$$\begin{aligned} \|a \cdot (b + N_\tau)\|^2 = \tau(b^*a^*ab) &= \|a\|^2 \tau(b^*b) - \tau(c^*c) \\ &\le \|a\|^2 \tau(b^*b) = \|a\|^2 \|b + N_\tau\|^2, \end{aligned}$$

so the elements of A act as bounded operators on A/N_τ, and extend to bounded operators $\pi_\tau(a)$ on the completion. We have now proved, at least in outline, the following proposition:

Proposition A.5. *For every state τ on a C^*-algebra A, the GNS-construction gives a nondegenerate representation π_τ of A on a Hilbert space $\mathcal{H}_\tau$.*

These GNS-representations π_τ are characterized by the existence of a *cyclic vector*: a vector $h \in \mathcal{H}_\pi$ is cyclic for π if the set $\{\, \pi(a)h : a \in A \,\}$ spans a dense subspace of $\mathcal{H}_\pi$. More precisely:

Proposition A.6. *If ρ is a state on a C^*-algebra A, then there is a unit vector h_ρ in $\mathcal{H}_\rho$ which is cyclic for π_ρ and satisfies $\rho(a) = (\pi_\rho(a)h_\rho \mid h_\rho)$ for all $a \in A$. Conversely, if h is a unit cyclic vector for a representation $\pi : A \to B(\mathcal{H}_\pi)$, then $\tau : a \mapsto (\pi(a)h \mid h)$ is a state on A, and the map $a \mapsto \pi(a)h$ induces a unitary isomorphism U of $\mathcal{H}_\tau$ onto $\mathcal{H}_\pi$ such that $\pi(a) = U\pi_\tau(a)U^*$ for all $a \in A$.*

This Proposition is quite easy if A has an identity: the vector $h_\rho := 1 + N_\rho$ is cyclic for π_ρ. For the general case, we need a lemma. To see where the first part comes from, note that if A has an identity and ρ is a positive functional on A, then the Cauchy-Schwarz inequality and the inequality $a^*a \leq \|a\|^2 1$ in A imply that

$$|\rho(a)|^2 = |\rho(1^*a)|^2 \leq \rho(1)\rho(a^*a) \leq \rho(1)\|a\|^2\rho(1),$$

and hence that $\|\rho\| = \rho(1)$.

Lemma A.7. *Suppose that A is a C^*-algebra which does not have an identity, and $\rho \in S(A)$. Then*

(a) *if $\{ e_i \}$ is an approximate identity for A, then $\rho(e_i) \to 1$, and*

(b) *the formula $\tau(a + \lambda 1) = \rho(a) + \lambda$ defines a state τ on the C^*-algebra A^1 obtained by adjoining 1 (Example 2.41).*

Proof. Since approximate identities in C^*-algebras are increasing, and ρ is positive, $\{ \rho(e_i) \}$ is an increasing net of positive real numbers bounded above by 1. Thus it converges to a real number L with $L \leq 1$. Note that we have

$$e_i^2 = (e_i^{\frac{1}{2}})^* e_i e_i^{\frac{1}{2}} \leq (e_i^{\frac{1}{2}})^* 1 e_i^{\frac{1}{2}} = e_i.$$

Thus if $a \in A$, then the Cauchy-Schwarz inequality gives

$$|\rho(e_i a)|^2 \leq \rho(e_i^2)\rho(a^*a) \leq \rho(e_i)\|a\|^2 \leq L\|a\|^2, \tag{A.2}$$

which is only compatible with $\|\rho\| = 1$ if $L \geq 1$. Thus $\rho(e_i) \to 1$.

Equation (A.2) also implies that $|\rho(a)|^2 \leq \rho(a^*a)$. Together with the identity $\rho(a^*) = \overline{\rho(a)}$ from Lemma A.4, this gives

$$\begin{aligned}
\tau\big((\lambda 1 + a)^*(\lambda 1 + a)\big) &= \tau\big(|\lambda|^2 1 + \overline{\lambda} a + \lambda a^* + a^*a\big) \\
&= |\lambda|^2 + 2\,\mathrm{Re}\big(\overline{\lambda}\rho(a)\big) + \rho(a^*a) \\
&\geq |\lambda|^2 - 2|\lambda|\,|\rho(a)| + \rho(a^*a) \\
&\geq |\lambda|^2 - 2|\lambda|\,|\rho(a)| + |\rho(a)|^2 \\
&\geq \big(|\lambda| - \rho(a)\big)^2 \geq 0.
\end{aligned}$$

Thus τ is positive and has norm $\tau(1) = 1$. □

Proof of Proposition A.6. As we mentioned above, $1 + N_\rho$ is cyclic for π_ρ if A has an identity. If A does not have an identity, we can use the previous lemma to extend ρ to a state τ on A^1. The inclusion of A in A^1 induces an isometry V of $\mathcal{H}_\rho$ into $\mathcal{H}_\tau$ which takes $a + N_\rho$ to $a + N_\tau$ and *intertwines* π_ρ and π_τ, in the sense that $V\pi_\rho(a) = \pi_\tau(a)V$ for all $a \in A$. Thus we can identify $\mathcal{H}_\rho$ with the subspace $V\mathcal{H}_\rho$ of $\mathcal{H}_\tau$. Indeed, $\mathcal{H}_\rho$ is then the essential subspace $\overline{\mathrm{span}}\{ \pi_\tau(a)\mathcal{H}_\tau \}$ of $\pi_\tau|_A$, and $\pi_\tau|_A = \pi_\rho \oplus 0$ on $\mathcal{H}_\rho \oplus \mathcal{H}_\rho^\perp$. The projection h_ρ of the vector $1 + N_\tau$ on $\mathcal{H}_\rho$ satisfies

$$\pi_\rho(a)h_\rho = \pi_\tau(a)(1 + N_\tau) = a + N_\tau,$$

so h_ρ is cyclic for π_ρ, and satisfies

$$(\pi_\rho(a)h_\rho \mid h_\rho) = \big(\pi_\tau(a)(1+N_\tau) \mid 1+N_\tau\big) = \tau(a) = \rho(a). \tag{A.3}$$

(Note that since ρ is a state, (A.3) implies that

$$1 = \|\rho\| \le \|h_\rho\|^2 \le \|1+N_\tau\|^2 = 1,$$

which is only possible if $h_\rho = 1 + N_\tau$. So we have proved also that $\mathcal{H}_\rho = \mathcal{H}_\tau$, or equivalently that A/N_τ is dense in $\mathcal{H}_\tau$.)

For the converse, note that τ is a positive functional of norm at most $\|h\|^2 = 1$; it has norm equal to 1 because $\pi(e_i)h \to h$ for any approximate identity $\{e_i\}$ of A. Next, note that

$$N_\tau = \{\, a \in A : (\pi(a^*a)h \mid h) = 0 \,\} = \{\, a \in A : \pi(a)h = 0 \,\},$$

so there is a well-defined linear map $U_0 : A/N_\tau \to \mathcal{H}_\pi$ such that $U_0(a+N_\tau) = \pi(a)h$. It is isometric:

$$\big(U_0(a+N_\tau) \mid U_0(b+N_\tau)\big) = \big(\pi(b^*a)h \mid h\big) = \tau(b^*a) = (a+N_\tau \mid b+N_\tau).$$

Thus U_0 extends to an isometric linear map U of the completion $\mathcal{H}_\tau$ onto $\overline{\operatorname{span}}\{\, \pi(a)h : a \in A \,\}$, which is all of $\mathcal{H}_\pi$ because h is cyclic. So U is unitary. The computation

$$U\pi_\tau(a)(b+N_\tau) = U(ab+N_\tau) = \pi(ab)h = \pi(a)(\pi(b)h) = \pi(a)U(b+N_\tau)$$

implies that $U\pi_\tau(a)U^* = \pi(a)$ because U is unitary. □

Corollary A.8. *If π is an irreducible representation of a C^*-algebra A and $h \in \mathcal{H}_\pi$ is a unit vector, then π is equivalent to the GNS-representation π_ω associated to the vector state $\omega : a \mapsto (\pi(a)h \mid h)$.*

Proof. Since $\mathcal{K} = \overline{\operatorname{span}}\{\, \pi(a)h : a \in A \,\}$ is invariant under π, the irreducibility of π implies that $\mathcal{K}$ is either $\{0\}$ or $\mathcal{H}_\pi$. Because π is nondegenerate (otherwise its essential subspace is a nonzero invariant subspace), we have $\pi(e_i)h \to h$; thus h is a non-zero vector in $\mathcal{K}$, and $\mathcal{K}$ must be all of $\mathcal{H}_\pi$. □

To produce representations of a C^*-algebra A with given properties, it is enough to produce appropriate states and apply the GNS-construction. In particular, to get a representation which is nonzero on a given element a, we look for a state ρ satisfying $\rho(a) \neq 0$. Since this problem only becomes harder if we make the algebra larger, we can suppose that A has an identity, and use the following characterization of states.

Lemma A.9. *Suppose that f is a linear functional of norm 1 on a C^*-algebra A with identity 1. Then f is a state if and only if $f(1) = 1$.*

Proof. We saw in the discussion preceding Proposition A.7 that every state f satisfies $f(1) = 1$. So suppose that $f(1) = 1$, and that there exists an $a \in A$ such that $f(a^*a) \notin [0, \infty)$. Since $\sigma(a^*a) \subset [0, \infty)$ it is geometrically clear (draw a picture!) that there exist $\lambda \in \mathbb{C}$ and $R > 0$ such that

$$\sigma(a^*a) \subset B(\lambda; R) \quad \text{but} \quad f(a^*a) \notin B(\lambda; R). \tag{A.4}$$

The first of these assertions implies $\sigma(a^*a - \lambda 1) \subset B(0; R)$, by the spectral mapping theorem. Then

$$\begin{aligned} r(a^*a - \lambda 1) \leq R &\Longrightarrow \|a^*a - \lambda 1\| \leq R \\ &\Longrightarrow |f(a^*a) - \lambda f(1)| \leq R\|f\| \\ &\Longrightarrow |f(a^*a) - \lambda| \leq R \\ &\Longrightarrow f(a^*a) \in B(\lambda; R), \end{aligned}$$

which contradicts the second assertion in (A.4). □

Lemma A.10. *Suppose A is a C^*-algebra and a is a self-adjoint element of A. Then there is a state ρ of A such that $|\rho(a)| = \|a\|$. In particular, for any $a \in A$, there is a state ρ such that $\rho(a^*a) = \|a\|^2$.*

Proof. As above, we may as well assume that A has an identity. Consider the C^*-subalgebra $C^*(a)$ of A generated by a and 1. This is a commutative C^*-algebra with identity, and hence the Gelfand transform is an isometric isomorphism of $C^*(a)$ onto the algebra $C(\Delta)$ of continuous functions on the maximal ideal space Δ. Since $\widehat{a}$ is a continuous map on a compact space, there exists $\phi \in \Delta$ such that

$$|\phi(a)| = |\widehat{a}(\phi)| = \|\widehat{a}\|_\infty = \|a\|.$$

By the Hahn-Banach theorem, there exists $\rho \in A^*$ such that $\rho|_{C^*(a)} = \phi$ and $\|\rho\| = \|\phi\| = 1$. We know that $\phi(1) = 1$, and hence $\rho(1) = \phi(1) = 1$ implies that ρ is a state on A by Lemma A.9. Since $|\rho(a)| = |\phi(a)| = \|a\|$, ρ is a state on A with the required property. □

Theorem A.11 (The Gelfand-Naimark Theorem). *Every C^*-algebra A has a faithful nondegenerate representation.*

Proof. For each non-zero $a \in A$, Lemma A.10 gives a state ρ_a such that $\rho_a(a^*a) = \|a\|^2$. Let $\pi_{\rho_a} : A \to B(\mathcal{H}_{\rho_a})$ be the corresponding GNS-representation, with cyclic vector h_{ρ_a}. Then $\pi_{\rho_a}(a) \neq 0$ because

$$0 < \|a\|^2 = \rho_a(a^*a) = \left(\pi_{\rho_a}(a^*a)h_{\rho_a} \mid h_{\rho_a}\right) = \|\pi_{\rho_a}(a)h_{\rho_a}\|^2,$$

and hence $\pi := \bigoplus_{a \in A, a \neq 0} \pi_{\rho_a}$ is a faithful representation of A, which is nondegenerate because it is a direct sum of such representations. □

The spectrum will consist of equivalence classes of irreducible representations, and for this to be a useful invariant, we need to know there are enough irreducible

representations to distinguish the points of A. To find them, we need to know when a GNS-representation is irreducible. The standard characterization uses the geometric structure of the space $S(A)$ of states on A.

The key point is that $S(A)$ is *convex*: for every $\phi, \psi \in S(A)$ and $\lambda \in [0,1]$, the *convex combination*

$$\lambda\phi + (1-\lambda)\psi : a \mapsto \lambda\phi(a) + (1-\lambda)\psi(a)$$

is also a state. In general, if S is a convex subset of a vector space V, a vector $v \in S$ is an *extreme point* of S if $v = \lambda w + (1-\lambda)z$ for some $w, z \in S$ and $\lambda \in (0,1)$ implies $v = w = z$. An extreme point of the convex subset $S(A)$ of A^* is called a *pure state* of A.

Lemma A.12. *Let ρ be a state of a C^*-algebra A. Then the GNS-representation π_ρ is irreducible if and only if ρ is a pure state.*

Proof. Suppose that $\pi := \pi_\rho$ is not irreducible: there is an invariant subspace $\mathcal{K}$ of $\mathcal{H}_\rho$ such that the orthogonal projection P of $\mathcal{H}_\rho$ onto $\mathcal{K}$ satisfies $P \neq 0 \neq 1 - P$. We aim to write ρ as a non-trivial convex combination of states. Since $\mathcal{K}$ is invariant, $\pi(a)P : \mathcal{H}_\rho \to \mathcal{K}$ and $\pi(a)P = P\pi(a)P$; taking adjoints shows that $P\pi(a) = P\pi(a)P = \pi(a)P$ for all $a \in A$. It follows from this and the cyclicity of h_ρ that $\|Ph_\rho\| \neq 0$:

$$Ph_\rho = 0 \Longrightarrow \pi(A)Ph_\rho = 0 \Longrightarrow P\pi(A)h_\rho = 0 \Longrightarrow P\mathcal{H}_\rho = 0 \Longrightarrow P = 0.$$

Similarly $\|(1-P)h_\rho\| \neq 0$. Thus $\lambda := \|Ph_\rho\|^2$ belongs to $(0,1)$, and

$$\phi(a) := \left(\pi(a)\frac{Ph_\rho}{\|Ph_\rho\|}\middle|\frac{Ph_\rho}{\|Ph_\rho\|}\right), \quad \psi(a) := \left(\pi(a)\frac{(1-P)h_\rho}{\|(1-P)h_\rho\|}\middle|\frac{(1-P)h_\rho}{\|(1-P)h_\rho\|}\right)$$

define states ϕ, ψ of A.

We now expand our original state ρ:

$$\begin{aligned}
\rho(a) &= \big(\pi(a)h_\rho \mid h_\rho\big) \\
&= \big(\pi(a)h_\rho \mid Ph_\rho\big) + \big(\pi(a)h_\rho \mid (1-P)h_\rho\big) \\
&= \big(\pi(a)Ph_\rho \mid Ph_\rho\big) + \big(\pi(a)(1-P)h_\rho \mid (1-P)h_\rho\big) \\
&= \lambda\left(\pi(a)\frac{Ph_\rho}{\|Ph_\rho\|}\middle|\frac{Ph_\rho}{\|Ph_\rho\|}\right) + (1-\lambda)\left(\pi(a)\frac{(1-P)h_\rho}{\|(1-P)h_\rho\|}\middle|\frac{(1-P)h_\rho}{\|(1-P)h_\rho\|}\right) \\
&= \lambda\phi(a) + (1-\lambda)\psi(a).
\end{aligned}$$

If we had $\rho = \phi$, then the equation $P\pi(a)P = \pi(a)P$ would imply

$$\big(\pi(a)h_\rho \mid h_\rho\big) = \|Ph_\rho\|^{-2}\big(\pi(a)h_\rho \mid Ph_\rho\big) \quad \text{for all } a \in A.$$

But this is only possible if $\|Ph_\rho\|^2 h_\rho = Ph_\rho$, which in turn implies $\|Ph_\rho\|^2 Ph_\rho = P^2 h_\rho = Ph_\rho$, and $\|Ph_\rho\|^2 = 1$, which contradicts $\lambda \in (0,1)$. Similarly we cannot have $\rho = \psi$, so we have a nontrivial convex decomposition of ρ, and ρ is not an extreme point of $S(A)$.

Conversely, suppose that $\pi := \pi_\rho$ is irreducible and that $\rho(a) = \big(\pi(a)h \mid h\big)$ for some unit vector $h \in \mathcal{H}_\pi$. We need to show that ρ is an extreme point; thus we suppose that $\rho = \lambda\phi + (1-\lambda)\psi$ for states ϕ and ψ and $\lambda \in (0,1)$, and aim to prove that $\phi = \rho$.

Since ψ and ϕ are positive

$$N_\rho = \{\, a \in A : \rho(a^*a) = 0 \,\} \subset N_\phi = \{\, a \in A : \phi(a^*a) = 0 \,\}.$$

Since $\pi(a)h = \pi(b)h$ exactly when $a - b \in N_\rho$, the Cauchy-Schwarz inequality (Lemma A.4(b)) implies that

$$\big[\pi(a)h, \pi(b)h\big] := \lambda\phi(b^*a)$$

is a well-defined sesquilinear form on the dense subspace $\pi(A)h := \{\, \pi(a)h : a \in A \,\}$ of $\mathcal{H}_\pi$. Furthermore,

$$\begin{aligned}\big[\pi(a)h, \pi(a)h\big] &= \lambda\phi(a^*a) = \rho(a^*a) - (1-\lambda)\psi(a^*a) \\ &\le \rho(a^*a) = \|\pi(a)h\|^2.\end{aligned}$$

It follows from the polarization identity (2.16) that $[\cdot,\cdot]$ is bounded on $\pi(A)h$ and can be extended to a bounded sesquilinear form q on all of $\mathcal{H}_\pi$. Therefore there is a bounded operator $T \in B(\mathcal{H}_\pi)$ such that $q(h,k) = (h \mid Tk)$ for all $h, k \in \mathcal{H}_\pi$ ([111, Theorem 2.3.6]); in particular,

$$\big(\pi(a)h \mid T\pi(b)h\big) = \big[\pi(a)h, \pi(b)h\big] = \lambda\phi(b^*a) \quad \text{for all } a, b \in A.$$

Since ϕ is positive and $\pi(A)h$ is dense, it follows that T is a positive operator of norm at most 1. Moreover we claim that T commutes with $\pi(A)$. For if $a, b, c \in A$, then

$$\begin{aligned}\big(\pi(a)h \mid T\pi(c)\pi(b)h\big) &= \lambda\phi\big((cb)^*a\big) = \lambda\phi\big(b^*(c^*a)\big) \\ &= \big(\pi(c^*a)h \mid T\pi(b)h\big) \\ &= \big(\pi(a)h \mid \pi(c)T\pi(b)h\big);\end{aligned}$$

since $\pi(A)h$ is dense, we have $\pi(c)T = T\pi(c)$ for all $c \in A$, as claimed. Since π is irreducible and T is positive, there is a nonnegative scalar z such that $T = z1$ (Lemma A.1). If $\{\, e_i \,\}_{i\in I}$ is an approximate identity for A, then for all $a \in A$,

$$\begin{aligned}\lambda\phi(a) &= \lim_i \lambda\phi(e_i a) = \lim_i \big(\pi(a)h \mid T\pi(e_i)h\big) = z\big(\pi(a)h \mid h\big) \\ &= z\rho(a).\end{aligned}$$

Then Lemma A.7(a) implies that $\lambda = \lim_i \lambda\phi(e_i) = \lim_i z\rho(e_i) = z$. Therefore $\phi = \rho$, which is what we set out to show. □

Lemma A.13. *For every element A of a C^*-algebra A, there is a pure state ρ on A such that $\rho(a^*a) = \|a\|^2$.*

For the proof of this lemma, we need a special case of the the *Krein-Milman Theorem*: every nonempty weak-$*$ compact convex set K in the dual X^* of a Banach space X is the closed convex hull of its extreme points — in other words, K is the intersection of all the weak-$*$ closed convex sets containing the extreme points of K. Here, we only need to know that every such K has extreme points, but this is a significant part of what the theorem says. For the details, see either [125, 7.5.4] or the Appendix in [111].

Proof. Let Σ denote the set of states ρ of A satisfying $\rho(a^*a) = \|a\|^2$, which is nonempty by the Lemma A.10 and is easily seen to be a weak-$*$ compact convex subset of A^*. The Krein-Milman theorem implies that Σ has an extreme point ρ. We claim that ρ is necessarily a pure state, i.e. that ρ is also an extreme point of the larger set $S(A)$. So suppose $\phi, \psi \in S(A)$, $\lambda \in (0,1)$ and $\rho = \lambda\phi + (1-\lambda)\psi$. Then the string of inequalities

$$\rho(a^*a) = \lambda\phi(a^*a) + (1-\lambda)\psi(a) \leq \lambda\|a\|^2 + (1-\lambda)\|a\|^2 = \|a\|^2 = \rho(a^*a)$$

forces equality in the middle, which is only possible if $\phi(a^*a) = \|a\|^2 = \psi(a^*a)$. But then ϕ and ψ are in Σ, and because ρ is extreme in Σ we deduce from $\rho = \lambda\phi + (1-\lambda)\psi$ that $\rho = \phi = \psi$. Thus ρ is a pure state, as required. □

Theorem A.14. *Let A be a C^*-algebra. Then for each $a \in A$ there is an irreducible representation π of A with $\|\pi(a)\| = \|a\|$.*

Proof. Given a, we choose ρ as in the previous Lemma. Then π_ρ is irreducible by Lemma A.12. The inequalities

$$\|a\|^2 = \rho(a^*a) = \big(\pi_\rho(a^*a)h_\rho \mid h_\rho\big) = \|\pi_\rho(a)h_\rho\|^2 \leq \|\pi_\rho(a)\|^2,$$

combine with the standard property $\|\pi(a)\| \leq \|a\|$ of all C^*-algebra homomorphisms to force $\|\pi_\rho(a)\|^2 = \|a\|^2$. □

A.2 The Spectrum of a C^*-Algebra

The *spectrum* $\hat{A}$ of a C^*-algebra A is the set* of unitary equivalence classes of irreducible representations of A. We begin our study of the spectrum with a couple of key examples.

Example A.15. We claim that every irreducible representation of $\mathcal{K}(\mathcal{H})$ is equivalent to the identity representation $\mathrm{id} : \mathcal{K}(\mathcal{H}) \to B(\mathcal{H})$, so that $\mathcal{K}(\mathcal{H})^\wedge = \{\mathrm{id}\}$. To establish the claim, we need to recall the description of $\mathcal{K}(\mathcal{H})$ as

$$\overline{\mathrm{span}}\{\, h \otimes \overline{k} : h, k \in \mathcal{H} \,\},$$

*For each irreducible representation π and each $h \in \mathcal{H}_\pi$, the set $\{\pi(a)h\}$ is dense in $\mathcal{H}_\pi$, so the dimension of $\mathcal{H}_\pi$ is bounded by the cardinality n of A. Since all Hilbert spaces of a given dimension are isomorphic, each irreducible representation of A is equivalent to a representation in one fixed Hilbert space of dimension n. Thus the equivalence classes of irreducible representations do form a set.

where $h \otimes \overline{k}(l) := (l \mid k)h$ (Proposition 1.1). Now suppose that $\pi : \mathcal{K}(\mathcal{H}) \to B(\mathcal{H}_\pi)$ is an irreducible representation, and fix a unit vector $e \in \mathcal{H}$. Then $e \otimes \overline{e}$ is the projection of $\mathcal{H}$ onto the span of e, and hence $P = \pi(e \otimes \overline{e})$ satisfies $P^2 = P = P^*$, and is the projection onto the closed subspace $P\mathcal{H}_\pi$. Fix a unit vector $\xi \in P(\mathcal{H}_\pi)$, and consider the subspace $\overline{\pi(\mathcal{K})\xi} = \overline{\operatorname{span}}\{\, \pi(T)\xi : T \in \mathcal{K}(\mathcal{H}) \,\}$. This is invariant, is non-zero because $\xi = \pi(e \otimes \overline{e})\xi$ is in it, and hence by the irreducibility of π is all of $\mathcal{H}_\pi$. Define $U : \mathcal{H} \to \mathcal{H}_\pi$ by $Uh := \pi(h \otimes \overline{e})\xi$. Then U is isometric:

$$\begin{aligned}\big(Ug \mid Uh\big) &= \big(\pi(g \otimes \overline{e})\xi \mid \pi(h \otimes \overline{e})\xi\big) = \big(\xi \mid \pi((g \otimes \overline{e})^*(h \otimes \overline{e}))\xi\big) \\ &= \big(\xi \mid \pi((e \otimes \overline{g})(h \otimes \overline{e}))\xi\big) = \big(\xi \mid \pi((h \mid g)(e \otimes \overline{e}))\xi\big) \\ &= (g \mid h)\big(\xi \mid \pi(e \otimes \overline{e})\xi\big) = (g \mid h)\|\xi\|^2 = (g \mid h).\end{aligned}$$

It is surjective because every element in the spanning set $\{\, \pi(h \otimes \overline{k})\xi \,\}$ is in the range of U:

$$\pi(h \otimes \overline{k})\xi = \pi(h \otimes \overline{k})\pi(e \otimes \overline{e})\xi = (e \mid k)\pi(h \otimes \overline{e})\xi = (e \mid k)Uh.$$

Thus U is a unitary transformation of $\mathcal{H}$ onto $\mathcal{H}_\pi$, and since

$$\begin{aligned}\pi(h \otimes \overline{k})(Ug) &= \pi\big((h \otimes \overline{k})(g \otimes \overline{e})\big)\xi = (g \mid k)\pi(h \otimes \overline{e})\xi \\ &= (g \mid k)Uh = U\big((g \mid k)h\big) = U\big((h \otimes \overline{k})g\big),\end{aligned}$$

we have $\pi(T)U = U\,\mathrm{id}(T)$ for all $T \in \mathcal{K}(\mathcal{H})$. Thus π is indeed unitarily equivalent to id.

Example A.16. We claim that if T is a compact Hausdorff space, then every irreducible representation π of $C(T)$ is 1-dimensional, and given by an evaluation map $\epsilon_t : f \mapsto f(t)$. Since the equivalence relation on 1-dimensional representations is trivial[†], we can therefore write

$$C(T)^\wedge = \{\, \epsilon_t : t \in T \,\}.$$

To see the claim, note that $\pi(C(T))$ is commutative, and hence every operator $\pi(f)$ is in the commutant $\pi(C(T))'$. Thus if π is irreducible, then $\pi : C(T) \to \mathbb{C}1_{\mathcal{H}_\pi}$, and every subspace of $\mathcal{H}_\pi$ is invariant; this is only possible if $\dim \mathcal{H}_\pi = 1$, and then π is essentially a nonzero homomorphism of $C(T)$ into $\mathbb{C}$. But every such homomorphism is an evaluation map ϵ_t (see, for example, [125, 4.2.5]).

A powerful tool in the analysis of commutative C^*-algebras is the identification of the sets of nonzero homomorphisms and maximal ideals, via the map $\phi \mapsto \ker\phi$ [125, 4.2.2]. In exactly the same way, we are now interested in the ideals which are the kernels of irreducible representations, which we call *primitive ideals*. The set of primitive ideals in a C^*-algebra A is called the *primitive ideal space* of A,

[†]If $\mathcal{H}$ is 1-dimensional, it is isomorphic to the Hilbert space $\mathbb{C}$. Since $B(\mathbb{C}) \cong \mathbb{C}$ is commutative, conjugating by a unitary has no effect. Thus the number in $\mathbb{C} \cong B(\mathbb{C})$ we get from an operator in $B(\mathbb{C})$ by identifying $\mathcal{H}$ with $\mathbb{C}$ is independent of the identification. In particular, this means that each evaluation map gives rise to a representation in a canonical way.

and denoted Prim A. (But we caution that this will soon stand for the topological space obtained by endowing the set of primitive ideals with a natural topology: see Definition A.21 on page 214.)

Before discussing the topology, we list some basic properties of primitive ideals. Recall that for us, all ideals in C^*-algebras are closed and two-sided unless otherwise stated; primitive ideals are closed because representations of C^*-algebras are automatically continuous.

Proposition A.17. *Let A be a C^*-algebra A. Then*

(a) *every closed ideal I in A is the intersection of the primitive ideals containing it;*

(b) *if I is a primitive ideal in A, and J, K are two ideals such that $J \cap K \subset I$, then either $J \subset I$ or $K \subset I$.*

Proof. (a) We have to show that if $a \in A$ and $a \notin I$, then there is a primitive ideal P of A with $I \subset P$ and $a \notin P$. Consider the element $a + I$ of the quotient C^*-algebra A/I, which is nonzero because $a \notin I$. There is an irreducible representation π of A/I such that $\|\pi(a + I)\| = \|a + I\| \neq 0$. But if $q : A \to A/I$ is the quotient map, then the composition $\pi \circ q$ is an irreducible representation of A with $a \notin \ker \pi \circ q$.

(b) Let $\pi : A \to B(\mathcal{H})$ be an irreducible representation with $\ker \pi = I$. If $J \not\subset I$, then $\pi(J) \neq 0$, and hence $\mathcal{V} := \overline{\pi(J)\mathcal{H}}$ is nonzero. Because J is an ideal, $\mathcal{V}$ is invariant: for any $j \in J$, $h \in \mathcal{H}$,

$$\pi(a)\big(\pi(j)h\big) = \pi(aj)h \in \pi(J)\mathcal{H},$$

and the linearity and continuity of $\pi(a)$ force $\pi(a)\mathcal{V} \subset \mathcal{V}$. Thus the irreducibility of π implies $\overline{\pi(J)\mathcal{H}} = \mathcal{H}$. But now

$$\pi(K)\big(\pi(J)\mathcal{H}\big) \subset \pi(K \cap J)\mathcal{H} \subset \pi(I)\mathcal{H} = 0$$

implies $K \subset \ker \pi = I$. □

Remark A.18. (a) Part (b) of Proposition A.17 says that every primitive ideal is *prime*. The concept of prime ideals comes from ring theory, and especially the ring $\mathbb{Z}$. The ideals in $\mathbb{Z}$ have the form $n\mathbb{Z}$ for some $n \in \mathbb{N}$, and $m\mathbb{Z} \subset n\mathbb{Z} \iff n \mid m$. The ideal $n\mathbb{Z}$ is prime exactly when n is a prime number.

(b) In any separable C^*-algebra, the converse of (b) is also true: any closed prime ideal is also primitive (Theorem A.49). Whether the same is true in an arbitrary C^*-algebra is not known.

Example A.16 says that the spectrum of $C(T)$ is naturally in one-to-one correspondence with T via the map $t \mapsto \epsilon_t$. Since the compact space T is normal, two representations ϵ_t and ϵ_s have distinct kernels, and the map $t \mapsto \ker \epsilon_t$ is a bijection of T onto Prim $C(T)$. It is important to remember that the topology we are about to define on the primitive ideal space Prim A is intended to be the usual topology on T when $A = C(T)$, in the sense that $t \mapsto \ker \epsilon_t$ is a homeomorphism.

Definition A.19. If A is a C^*-algebra, and F is a subset of $\operatorname{Prim} A$, we define the *closure* $\overline{F}$ of F to be

$$\overline{F} := \{ P \in \operatorname{Prim} A : \bigcap_{I \in F} I \subset P \}.$$

For motivation, consider the algebra $C(T)$, where $F = \{ \ker \epsilon_t : t \in S \}$ for some uniquely defined subset S of T. Then

$$\begin{aligned} \bigcap_{I \in F} I = \bigcap_{t \in S} \ker \epsilon_t &= \{ f \in C(T) : f(t) = 0 \text{ for } t \in S \} \\ &= \{ f \in C(T) : f|_S = 0 \} = \{ f \in C(T) : f|_{\overline{S}} = 0 \}. \end{aligned}$$

So we trivially have $\bigcap_{I \in F} I \subset \ker \epsilon_t$ for all $t \in \overline{S}$. Conversely, if $s \notin \overline{S}$, Urysohn's Lemma gives $f \in C(T)$ such that $f|_{\overline{S}} = 0$ and $f(s) = 1$, and hence an element of $\bigcap_{I \in F} I$ which is not in $\ker \epsilon_s$. Thus $\overline{F} = \{ \ker \epsilon_t : t \in \overline{S} \}$ corresponds to the usual closure of S in T.

Lemma A.20. *Let A be a C^*-algebra. The subsets F of* $\operatorname{Prim} A$ *such that $F = \overline{F}$ are the closed sets in a topology for* $\operatorname{Prim} A$.

Proof. We verify that $F \mapsto \overline{F}$ satisfies the Kuratowski closure axioms:

(a) $\overline{\emptyset} = \emptyset$;

(b) $F \subset \overline{F}$ for all $F \subset \operatorname{Prim} A$;

(c) $\overline{\overline{F}} = \overline{F}$ for all $F \subset \operatorname{Prim} A$;

(d) $\overline{F \cup G} = \overline{F} \cup \overline{G}$ for all $F, G \subset \operatorname{Prim} A$.

It is then quite easy to check from first principles that $\{ F : F = \overline{F} \}$ are the closed sets in a topology. Parts (a) and (b) are trivial, provided one remembers that $\bigcap_{I \in \emptyset} I$ is all of A. For (c), just observe that $\bigcap_{I \in F} I \subset P$ for all $P \in \overline{F}$ implies $\bigcap_{I \in F} I \subset \bigcap_{P \in \overline{F}} P$, which is only compatible with $F \subset \overline{F}$ if $\bigcap_{I \in F} I = \bigcap_{P \in \overline{F}} P$. If $F \subset H$ then $\overline{F} \subset \overline{H}$, and it follows that $\overline{F} \cup \overline{G} \subset \overline{F \cup G}$. So it remains to prove that $\overline{F \cup G} \subset \overline{F} \cup \overline{G}$. But

$$\bigcap_{I \in F \cup G} I \subset P \Longleftrightarrow (\bigcap_{I \in F} I) \cap (\bigcap_{J \in G} J) \subset P,$$

which because P is prime (Proposition A.17(b)) implies that $P \in \overline{F}$ or $P \in \overline{G}$. □

This topology on $\operatorname{Prim} A$ is known variously as the *Jacobson topology* (after its inventor, in the context of ring theory), or the *hull-kernel topology* (after the constructions in the first part of Proposition A.27 below). In Corollary A.28, we will describe the open sets explicitly.

Since the kernels of equivalent representations are the same, there is a canonical surjection $\pi \mapsto \ker \pi$ of $\hat{A}$ onto $\operatorname{Prim} A$. This map is *not* always an injection (see Hooptedoodle A.33) but is for many C^*-algebras of interest to us, such as $C(T)$. Using the map $\pi \mapsto \ker \pi$, we can pull the hull-kernel topology back to a topology on $\hat{A}$. When one talks about the spectrum of a C^*-algebra, one usually means the set $\hat{A}$ endowed with this topology. Thus we formally define:

Definition A.21. The *primitive ideal space* $\operatorname{Prim} A$ of a C^*-algebra A is the set of primitive ideals of A endowed with the hull-kernel topology. The *spectrum* of A is the set of unitary equivalence classes of irreducible representations of A with the topology inherited from $\operatorname{Prim} A$: thus $S \subset \hat{A}$ is open if and only if $\{ \ker \pi : \pi \in S \}$ is open in $\operatorname{Prim} A$.

Remark A.22. Notice that we have already slipped into a standard sloppiness: we confuse an irreducible representation π with its equivalence class $[\pi]$ in $\hat{A}$. This will only cause problems if we forget that we are really making a choice of representative for the class $[\pi]$, or forget to check that some mapping is well-defined on $\hat{A}$. Thus, for example, we were careful above to note that equivalent representations have the same kernel, so that $[\pi] \mapsto \ker \pi$ is well-defined.

We are now ready to discuss the examples most relevant to this book. We have already seen that $\operatorname{Prim} C(T)$ is just T, in the sense that $t \mapsto \ker \epsilon_t$ is a homeomorphism. (Before Lemma A.20, we saw that this map converts the usual closure operation to the one we have defined on $\operatorname{Prim} C(T)$; it follows that it swaps closed sets with closed sets and open sets with open sets, and is therefore a homeomorphism.) Since $\epsilon_t \mapsto \ker \epsilon_t$ is a also a bijection, we also have $C(T)^\wedge = T$. Example A.15 says that $\mathcal{K}(\mathcal{H})^\wedge = \{ \mathrm{id} \}$, and hence $\operatorname{Prim} \mathcal{K}(\mathcal{H}) = \{ 0 \}$. The topology here is not interesting, but Proposition A.17(a) is: it implies that $\{ 0 \}$ is the only proper ideal in $\mathcal{K}(\mathcal{H})$, or in other words that $\mathcal{K}(\mathcal{H})$ is *simple*.

Our next examples are combinations of these two, and the techniques we use to determine their spectra are used throughout the book. The simplest is the algebra $C(T, M_n(\mathbb{C}))$ of continuous functions from a compact Hausdorff space T into the C^*-algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices, which is a C^*-algebra under the pointwise operations inherited from $M_n(\mathbb{C})$ (so that, for example, $(fg)(t)$ is the usual product of the matrices $f(t)$ and $g(t)$), and the sup norm

$$\|f\|_\infty := \sup\{ \|f(t)\|_{M_n(\mathbb{C})} : t \in \mathbb{T} \}.$$

A function $f \in C(T, M_n(\mathbb{C}))$ can be naturally viewed as a matrix of continuous functions, and hence as an element of the C^*-algebra $M_n(C(T))$ discussed on page 34; indeed, this gives a isomorphism of $C(T, M_n(\mathbb{C}))$ onto $M_n(C(T))$, which has to be isometric because both algebras carry a unique C^*-norm. We shall use this isomorphism freely in the examples to follow.

Example A.23. Suppose A is the C^*-algebra $C(T, M_n(\mathbb{C}))$ described above. Then each irreducible representation is equivalent to an evaluation map $\epsilon_t : a \mapsto a(t)$ of A into $M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$, and there is exactly one such t. Thus we can write $\hat{A} = \{ \epsilon_t : t \in T \}$, provided as usual that we remember that we have only written down a representative for each class: we get different representations if we change the identification of $M_n(\mathbb{C})$ with $B(\mathbb{C}^n)$, but they are unitarily equivalent[‡]. The description of $\operatorname{Prim} A$ as $\{ \ker \epsilon_t : t \in T \}$, on the other hand, is canonical. The maps $t \mapsto [\epsilon_t]$ and $t \mapsto \ker \epsilon_t$ are both homeomorphisms.

[‡]If S, T are two isomorphisms of $M_n(\mathbb{C})$ onto $B(\mathbb{C}^n)$, then $S \circ T^{-1}$ is an automorphism of $B(\mathbb{C}^n)$, and hence has the form $\operatorname{Ad} U$ for some unitary operator U on $\mathbb{C}^n$ (Proposition 1.6). The unitary operator U implements an equivalence between the representations $T \circ \epsilon_t : a \mapsto T(a(t))$ and $S \circ \epsilon_t : a \mapsto S(a(t))$.

Proof. The arguments we use to see all this will come up again and again. First suppose that $\pi : A \to B(\mathcal{H})$ is an irreducible representation. There is a copy of $C(T)$ sitting inside A, as multiples $\{ f1_n \}$ of the identity matrix $1_n \in M_n(\mathbb{C})$: we can view $f1_n$ either as the function $t \mapsto f(t)1_n$ in $C(T, M_n(\mathbb{C}))$, or as the diagonal matrix

$$\begin{pmatrix} f & 0 & \dots & 0 \\ 0 & f & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f \end{pmatrix}$$

of continuous functions in $M_n(C(T))$. The map $f \mapsto f1_n$ is a unital homomorphism, and hence the representation $f \mapsto \pi(f1_n)$ is nondegenerate. Further, since $(f1_n)a = a(f1_n)$ for all $a \in A$, $\pi(f1_n)$ commutes with the range of π; since π is irreducible, this implies that $\pi(f1_n) \in \mathbb{C}1_{\mathcal{H}_\pi}$ for all $f \in C(T)$. Since π is certainly nonzero, Example A.16 implies that there exists $t \in T$ such that $\pi(f1_n) = f(t)1_{\mathcal{H}_\pi}$ for all $f \in C(T)$.

We next claim that $I_t := \{ a \in C(T, M_n(\mathbb{C})) : a(t) = 0 \} \subset \ker \pi$. For suppose $a(t) = 0$, and $\varepsilon > 0$. We can choose a neighbourhood N of t such that $\|a(s)\| < \varepsilon$ for $s \in N$, and $f \in C(T)$ such that $0 \leq f \leq 1$, $f(t) = 0$ and $f(s) = 1$ for $s \notin N$. Then $\|a(s) - f(s)a(s)\| < \varepsilon$ for all $s \in T$, and hence $\|\pi(a) - \pi(f1_n)\pi(a)\| < \varepsilon$. Since we know that $\pi(f1_n) = f(t)1_{\mathcal{H}_\pi} = 0$, we deduce that $\|\pi(a)\| < \varepsilon$; because ε was arbitrary, we must have $\|\pi(a)\| = 0$.

Since $I_t \subset \ker \pi$, there is a representation $\widetilde{\pi}$ of A/I_t on the same space $\mathcal{H}_\pi$ such that $\widetilde{\pi}(a + I_t) = \pi(a)$; $\widetilde{\pi}$ is irreducible because it has the same range as π. Since $a \mapsto a(t)$ induces an isomorphism ϕ_t of A/I_t onto $M_n(\mathbb{C})$, the composition $\widetilde{\pi} \circ \phi_t^{-1}$ is an irreducible representation of $M_n(\mathbb{C})$, and hence equivalent to the identity by Example A.15 above. We deduce that the representation $\pi = \widetilde{\pi} \circ \phi_t^{-1} \circ \epsilon_t$ is equivalent to ϵ_t. Since for $s \neq t$ we can easily find a such that $a(s) = 0$ but $a(t) \neq 0$, ϵ_s and ϵ_t have different kernels and must be inequivalent. This justifies the descriptions of $\hat{A}$ and $\operatorname{Prim} A$, at least as sets.

It remains to establish the last statement. Because $[\epsilon_t] \mapsto \ker \epsilon_t$ is a bijection of $\hat{A}$ onto $\operatorname{Prim} A$, it is enough to show that $t \mapsto \ker \epsilon_t$ is a homeomorphism, and hence enough to show that

$$\overline{\{ \ker \epsilon_t : t \in N \}} = \{ \ker \epsilon_t : t \in \overline{N} \}$$

for every subset N of T. We know that the left-hand side has the form $\{ \ker \epsilon_t : t \in M \}$ for $M := \{ s : \bigcap_{t \in N} \ker \epsilon_t \subset \ker \epsilon_s \}$, and we have to prove $M = \overline{N}$. Since the elements of A are continuous functions on T, $a = 0$ on N implies $a = 0$ on $\overline{N}$, so certainly $\overline{N} \subset M$. On the other hand, if $s \notin \overline{N}$, there exists $f \in C(T)$ such that $f|_{\overline{N}} = 0$ and $f(s) = 1$, and then $f1_n$ is in $\ker \epsilon_t$ for all $t \in \overline{N}$ but not in $\ker \epsilon_s$, which implies $s \notin M$. Thus we must have $M = \overline{N}$, and the claim is proved. □

Example A.24. Now suppose that $A = C_0\big(T, \mathcal{K}(\mathcal{H})\big)$ for some locally compact space T and infinite-dimensional Hilbert space $\mathcal{H}$. As for the previous example, $t \mapsto [\epsilon_t]$ is a homeomorphism of T onto $\hat{A}$, but there are technical complications in

following the same line of argument. First, $C_0(T)$ does not sit inside A as multiples of the identity: rather we have a homomorphism ι of $C_0(T)$ into the centre $ZM(A)$ of the multiplier algebra of A, given by $(\iota(f)a)(t) := f(t)a(t)$. Then, since T is locally compact rather than compact, ι is nondegenerate in the sense of Section 2.2 rather than unital: we can approximate any $a \in A$ by $\iota(f)a$ for suitable f in $C_0(T)$. (Choose f to be 1 on $\{t : \|a(t)\| \geq \varepsilon\}$.) By Proposition 2.50, we can extend π to a unital representation $\bar{\pi}$ of the multiplier algebra $M(A)$, and consider the representation $f \mapsto \bar{\pi}(\iota(f))$, which is nondegenerate because ι is, and in particular is nonzero. Because $\iota(f)$ is in $ZM(A)$, $\bar{\pi}(\iota(f))$ is in $\pi(A)' = \mathbb{C}1_{\mathcal{H}_\pi}$, and as before there exists $t \in T$ such that $\bar{\pi}(\iota(f)) = f(t)1_{\mathcal{H}_\pi}$. As in the previous example, we can see that $I_t := \{a \in A : a(t) = 0\}$ is contained in $\ker\pi$, and the result of Example A.15 implies that π is equivalent to the evaluation map $\epsilon_t : a \mapsto a(t)$. One can also modify the argument of the previous example to see that $t \mapsto [\epsilon_t]$ is a homeomorphism, using elements of the form $f(e \otimes \bar{e})$ rather than $f1_n$.

Example A.25. Consider three C^*-subalgebras of $C([0,1], M_2(\mathbb{C}))$:

$$A_1 = \{f \in C([0,1], M_2(\mathbb{C})) : f(0) \in \mathbb{C}1_2\}$$
$$A_2 = \{f \in C([0,1], M_2(\mathbb{C})) : f(0) = \left(\begin{smallmatrix}\lambda & 0\\ 0 & 0\end{smallmatrix}\right) \text{ for some } \lambda \in \mathbb{C}\}$$
$$A_3 = \{f \in C([0,1], M_2(\mathbb{C})) : f(0) = \left(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\right) \text{ for some } \lambda, \mu \in \mathbb{C}\}$$

Using the techniques of the previous examples, one can see that $t \mapsto [\epsilon_t]$ is a homeomorphism of $[0,1]$ onto both $\hat{A}_1$ and $\hat{A}_2$. On the other hand, while every irreducible representation of A_3 factors through an evaluation map ϵ_t, there are two which factor through ϵ_0: the representations $\pi_1 : f \mapsto f(0)_{11}$ and $\pi_2 : f \mapsto f(0)_{22}$ are inequivalent. As a topological space, $\hat{A}_3 = (0,1] \cup \{\pi_1, \pi_2\}$, where $(0,1]$ has its usual topology, and sets of the form $\{\pi_1\} \cup (0,\delta)$ and $\{\pi_2\} \cup (0,\delta)$ are also open. Thus sequences converging to 0 in $(0,1]$ converge to both π_1 and π_2 in $\hat{A}_3$, and $\hat{A}_3$ is definitely not Hausdorff. However, it is T_1, because both the unusual points are closed. (Curiously, one is more likely to come across algebras like A_2 and A_3 than the unital algebra A_1.)

We shall now discuss the main properties of these topologies on $\operatorname{Prim} A$ and $\hat{A}$. First, we consider their behavior with respect to ideals and quotients; this is of fundamental importance for our localization arguments in Chapter 5. We begin with set-theoretic descriptions in the next Proposition, and then prove that these descriptions respect the topologies in Proposition A.27.

Proposition A.26. *Let I be an ideal in a C^*-algebra A, and let $q : A \to A/I$ be the quotient map. Then*

(a) *the map $\pi \mapsto \pi|_I$ is a bijection of $\{\pi \in \hat{A} : \pi|_I \neq 0\}$ onto $\hat{I}$, and $P \mapsto P \cap I$ is a bijection of $\{P \in \operatorname{Prim} A : I \not\subset P\}$ onto $\operatorname{Prim} I$;*

(b) *the map $\pi \mapsto \pi \circ q$ is a bijection of $(A/I)^\wedge$ onto $\{\rho \in \hat{A} : \rho|_I = 0\}$, and $Q \mapsto q^{-1}(Q)$ is a bijection of $\operatorname{Prim} A/I$ onto $\{P \in \operatorname{Prim} A : I \subset P\}$ with inverse $P \mapsto P/I$.*

Proof. (a) Suppose π is irreducible and $\pi|_I \neq 0$. Since

$$\overline{\pi(I)\mathcal{H}_\pi} := \overline{\text{span}}\{\, \pi(a)h : a \in I,\ h \in \mathcal{H}_\pi \,\}$$

is a closed invariant subspace of $\mathcal{H}_\pi$, we must have $\overline{\pi(I)\mathcal{H}_\pi} = \mathcal{H}_\pi$. Thus $\pi|_I$ is nondegenerate. If $\mathcal{V}$ is invariant for $\pi|_I$, then the equation

$$\pi(a)\big(\pi(i)k\big) = \pi(ai)k \tag{A.5}$$

implies that $\mathcal{V}$ is invariant for π on all of A. Hence $\mathcal{V}$ must be $\{\,0\,\}$ or $\mathcal{H}_\pi$, and $\pi|_I$ is irreducible. If $\pi \sim \rho$, then we trivially have $\pi|_I \sim \rho|_I$, so the map $\pi \mapsto \pi|_I$ is a well-defined map of $\{\, \pi \in \hat{A} : \pi|_I \neq 0 \,\}$ into $\hat{I}$. The equation (A.5) also implies that if $\pi|_I \sim \rho|_I$, then $\pi \sim \rho$ on all of A, so the map $\pi \mapsto \pi|_I$ is injective.

If we start with a representation $\rho \in \hat{I}$, then because I is an ideal, ρ extends to a representation $\bar{\rho}$ of A (apply Proposition 2.50 with $\mathsf{X} = \mathcal{H}$). The irreducibility of ρ trivially implies that of π, so this proves that the map $\pi \mapsto \pi|_I$ is onto.

Since $(\ker\pi) \cap I = \ker(\pi|_I)$, it follows immediately that $P \mapsto P \cap I$ is a well-defined map of $\operatorname{Prim} A$ onto $\operatorname{Prim} I$. To see that this map is injective, suppose $P_1, P_2 \in \operatorname{Prim} A$ have $I \not\subset P_k$ and $P_1 \cap I = P_2 \cap I$. Then the primeness of P_1 implies that $P_2 \subset P_1$, and by symmetry we must have $P_1 = P_2$.

(b) For any representation π of A/I, the representation $\pi \circ q$ of A has exactly the same range as π; hence $\pi \circ q$ is irreducible if and only if π is, and $\pi \circ q \sim \rho \circ q$ if and only if $\pi \sim \rho$. Thus $\pi \mapsto \pi \circ q$ is a well-defined injective map. If $\rho \in \hat{A}$ has $I \subset \ker\rho$, then there is a representation $\tilde{\rho}$ of A/I on $\mathcal{H}_\rho$ such that $\tilde{\rho}(a + I) = \rho(a)$, and hence such that $\tilde{\rho} \circ q = \rho$; then $\tilde{\rho}$ is irreducible because ρ is. Thus $\pi \mapsto \pi \circ q$ is surjective from $(A/I)^\wedge$ onto $\{\, \rho \in \hat{A} : \rho|_I = 0 \,\}$. The corresponding facts for $\operatorname{Prim} A/I$ follow easily. □

Proposition A.27. *Let A be a C^*-algebra.*

(a) *$F \mapsto k(F) := \bigcap\{\, J : J \in F \,\}$ is a bijection between the closed subsets F of $\operatorname{Prim} A$ and the ideals in A, with inverse given by*

$$h(I) := \{\, P \in \operatorname{Prim} A : I \subset P \,\}.$$

One calls $k(F)$ the kernel *of F and $h(I)$ the* hull *of I.*

(b) *If I is an ideal in A, then $P \mapsto P \cap I$ is a homeomorphism of the open set $\{\, P \in \operatorname{Prim} A : I \not\subset P \,\}$ onto $\operatorname{Prim} I$, and $\pi \mapsto \pi|_I$ is a homeomorphism of the open set $\{\, \pi \in \hat{A} : \pi|_I \neq 0 \,\}$ onto $\hat{I}$.*

(c) *If I is an ideal in A, and $q : A \to A/I$ is the quotient map, then $Q \mapsto q^{-1}(Q)$ is a homeomorphism of $\operatorname{Prim} A/I$ onto the closed subset $\{\, P \in \operatorname{Prim} A : I \subset P \,\}$ of $\operatorname{Prim} A$ with inverse $P \mapsto P/I$, and $\pi \mapsto \pi \circ q$ is a homeomorphism of $(A/I)^\wedge$ onto the closed subset $\{\, \pi \in \hat{A} : \pi|_I = 0 \,\}$ of $\hat{A}$.*

Proof of Proposition A.27. (a) Recall from Proposition A.17(a) that for any ideal I,

$$I = \bigcap\{\, P \in \operatorname{Prim} A : I \subset P \,\}.$$

This implies, first, that the set $F := \{ P \in \operatorname{Prim} A : I \subset P \}$ is closed:

$$P \in \overline{F} \iff \bigcap_{J \in F} J \subset P \iff I \subset P \iff P \in F;$$

and, second, that $I = k(F)$, so that $k \circ h$ is the identity on the set of ideals. On the other hand, $k(F) \subset P \iff P \in \overline{F}$, so that we can recover a closed set F as $\{ P : k(F) \subset P \}$, and $h \circ k$ is the identity on the set of closed subsets of $\operatorname{Prim} A$.

(b) The canonical maps θ from $\hat{A}$ to $\operatorname{Prim} A$ are open and continuous, and we have a commutative diagram

$$\begin{array}{ccc} \{ \pi \in \hat{A} : \pi|_I \neq 0 \} & \xrightarrow{\pi \mapsto \pi|_I} & \hat{I} \\ \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle \theta} \\ \{ P \in \operatorname{Prim} A : I \not\subset P \} & \xrightarrow[P \mapsto P \cap I]{} & \operatorname{Prim} I \end{array}$$

in which the horizontal arrows are bijections by Proposition A.26 and

$$\{ \pi \in \hat{A} : \pi|_I \neq 0 \} = \theta^{-1} \{ P \in \operatorname{Prim} A : I \not\subset P \}.$$

Since $S := \{ P \in \operatorname{Prim} A : I \not\subset P \}$ is open by (a), it is enough to show that $P \mapsto P \cap I$ is bicontinuous. But a subset N is open in $\operatorname{Prim} A$ if and only if there is an ideal J in A such that $N = \{ P \in \operatorname{Prim} A : J \not\subset P \}$, and hence M is open in S if and only if there is an ideal J in A such that

$$M = \{ P \in \operatorname{Prim} A : I \not\subset P \text{ and } J \not\subset P \}.$$

Since primitive ideals are prime,

$$I \not\subset P \text{ and } J \not\subset P \iff I \cap J \not\subset P,$$

so we can deduce that the open sets in S have the form

$$M = \{ P \in \operatorname{Prim} A : I \cap J \not\subset P \}$$

for some ideal J of A. But

$$\begin{aligned} P \in M &\iff I \cap J \not\subset P \iff I \cap J \not\subset I \cap P \\ &\iff I \cap P \in \{ Q \in \operatorname{Prim} I : I \cap J \not\subset Q \}, \end{aligned}$$

and hence $P \mapsto P \cap I$ is bicontinuous.

(c) The closed sets of $T = \{ P \in \operatorname{Prim} A : I \subset P \}$ have the form

$$T \cap \bigcap \{ P \in \operatorname{Prim} A : J \subset P \} = \{ P \in \operatorname{Prim} A : I \cup J \subset P \}$$

for some ideal J of A. By writing K for the ideal generated by I and J, we see that the closed sets of T have the form

$$T_K = \{ P \in \operatorname{Prim} A : K \subset P \}$$

for some ideal K of A containing I. Since the closed subsets $\operatorname{Prim} A/I$ are precisely those of the form $S_L = \{ Q \in \operatorname{Prim} A/I : L \subset Q \}$ for some ideal L of A/I, we see that q^{-1} maps the closed set S_L to $T_{q^{-1}(L)}$, and its inverse maps T_K to $S_{K/I}$. □

Corollary A.28. *The open sets in* $\operatorname{Prim} A$ *are precisely those of the form*

$$\mathcal{O}_J = \{ P : J \not\subset P \}$$

for some ideal J *of* A.

Remark A.29. A considerable effort has gone into describing the topology on the primitive ideal spaces of interesting C^*-algebras (for example, see [48, 50, 51, 10, 170, 135, 114, 74, 33, 35, 38, 151]). The first part of the previous proposition shows why: when we know which subsets of $\operatorname{Prim} A$ are closed, we know all the ideals in A. Indeed, since the correspondence $F \mapsto k(F)$ is order reversing, we even know the lattice of ideals in A.

Theorem A.14 implies that the functions $\pi \mapsto \|\pi(a)\| : \hat{A} \to \mathbb{R}$ determine the norm of a via $\|a\| = \sup_\pi \|\pi(a)\|$. These functions also have some interesting continuity properties which we shall need; the following general results can be substantially sharpened when the spectrum is Hausdorff (see Lemma 5.2).

Lemma A.30. *Suppose that* A *is a* C^**-algebra and that* $a \in A$.

(a) *The function* $\pi \mapsto \|\pi(a)\|$ *is lower semicontinuous on* $\hat{A}$*; that is,* $\{ \pi \in \hat{A} : \|\pi(a)\| \leq k \}$ *is closed for all* $k \geq 0$.

(b) *For each* $k > 0$, $\{ \pi \in \hat{A} : \|\pi(a)\| \geq k \}$ *is compact.*

Proof. (a) Let $C = \{ \pi \in \hat{A} : \|\pi(a)\| \leq k \}$. Since $\|\pi(a)\|^2 = \|\pi(a^*a)\|$, it suffices to prove the assertion for $a \geq 0$. Let $\sigma(a)$ be the spectrum of a. Because $a \geq 0$, the norm of a is its spectral radius, so

$$C = \{ \pi \in \hat{A} : \sigma(\pi(a)) \subset [0, k] \}.$$

If C is not closed, then there are $\rho \in \overline{C}$ and $\alpha \notin [0, k]$ such that $\alpha \in \sigma(\rho(a))$. However, there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f = 0$ on $[0, k]$ and $f(\alpha) = 1$, and then $f(\rho(a)) = \rho(f(a)) \neq 0$. On the other hand, $\pi(f(a)) = 0$ for all $\pi \in C$; since $\rho \in \overline{C}$ means $\ker \rho \supset \bigcap_{\pi \in C} \ker \pi$, this implies $\rho\big(f(a)\big) = 0$, which is a contradiction. This proves (a).

(b) To prove that

$$C = \{ \pi \in \hat{A} : \|\pi(a)\| \geq k \} \tag{A.6}$$

is compact, let $\{ C_i \}_{i \in I}$ be a family of closed sets in C with the finite intersection property. Let $J_i = \bigcap_{\pi \in C_i} \ker \pi$, and let J be the ideal generated by $\bigcup_{i \in I} J_i$, so that $J = \bigcap_{\pi \in \cap C_i} \ker \pi$. Fix $n \in \mathbb{N}$. By definition of the quotient norm there exists b in the linear span of $\bigcup_{i \in I} J_i$ such that

$$\|a + J\| \geq \|a + b\| - \frac{k}{n}. \tag{A.7}$$

Note that $b \in \operatorname{span}\left(\bigcup_{i\in F} J_i\right)$ for some finite subset F of I. If $\pi \in \bigcap_{i\in F} C_i$, which is nonempty by the finite intersection property of $\{ C_i \}$, then $\pi(b) = 0$; since $\pi \in C$, this implies that

$$\|a + b\| \geq \|\pi(a + b)\| = \|\pi(a)\| \geq k. \tag{A.8}$$

Since n was arbitrary, (A.7) and (A.8) imply $\|a + J\| \geq k$. It now follows from Theorem A.14 that there is an irreducible representation ρ of A/J such that $\|\rho(a + J)\| \geq k$; thus there exists $\eta \in \hat{A}$ such that $\ker\eta \supset J$ and $\|\eta(a)\| \geq k$ (see Proposition A.26(b)). In other words, $\eta \in \overline{C_i}$ for all i (because $J \supset J_i$ for all i) and $\eta \in C$. Since each C_i is closed in C, we deduce that $\eta \in C_i$ for all i. Thus C is compact, as claimed. □

At this point, we have the basic facts about the spectrum which we need for our localization arguments. However, we would be leaving a false impression of the possible complications if we only mention algebras of continuous functions with values in algebras of compact operators. So we shall briefly survey other important results in the area, without any claims to completeness. Among other things, we shall explain the acronyms CCR and GCR which occasionally crop up in the text.

First we want to discuss the effect of the previous Proposition on the extension problem for C^*-algebras: to what extent do an ideal I and the quotient A/I determine a C^*-algebra A? Even for commutative algebras, this is a complicated question. An ideal in $C_0(T)$ has the form $I_F := \{ f : f|_F = 0 \} = C_0(T\setminus F)$ for some closed subset F of T, and the map $f \mapsto f|_F$ induces an isomorphism of $C_0(T)/I_F$ onto $C_0(F)$. Thus the extension problem asks: given two locally compact Hausdorff spaces S and F, how can we embed S as an open subset of a locally compact space T so that the complement $T \setminus S$ is homeomorphic to F? There is a trivial solution (take T to be the disjoint union of S and F), but there could be many more.

Given two C^*-algebras B, C, the extension problem seeks algebras A containing a copy of B as an ideal with the quotient isomorphic to C. Again there is a trivial solution: the C^*-algebra direct sum $B \oplus C$, with $\|(b, c)\| := \max\{ \|b\|, \|c\| \}$. Proposition A.27 says that this problem is at least as complicated as the topological problem for Prim B and Prim C, and in fact it is even more so: primitive ideal spaces need not be Hausdorff, so there are potentially even more ways to fit the two bits together. The fundamental example here is the *Toeplitz algebra* $\mathcal{T}$; we shall have no need of its realization as the C^*-algebra generated by a family of Toeplitz operators, so we merely define $\mathcal{T}$ to be the C^*-algebra generated by the unilateral shift, and mention that a famous theorem of Coburn justifies the name (see, for example, [111, 3.5.18]).

Example A.31. Let $\mathcal{H} = \ell^2$ have the usual basis $\{ e_n \}_{n=1}^{\infty}$, so that the unilateral shift S is the isometry characterized by $Se_n = e_{n+1}$. We claim that the C^*-algebra $\mathcal{T}$ generated by S is an irreducible subalgebra of $B(\mathcal{H})$ containing the algebra $\mathcal{K}(\mathcal{H})$ of compact operators, with corresponding quotient $\mathcal{T}/\mathcal{K}(\mathcal{H})$ isomorphic to $C(\mathbb{T})$. Since $\mathcal{T}$ is irreducible, $\{ 0 \}$ is a dense subset of Prim $\mathcal{T}$, so $\mathcal{T}$ cannot be isomorphic to $\mathcal{K}(\mathcal{H}) \oplus C(\mathbb{T})$.

Proof of the claim. We first prove that $\mathcal{T}$ acts irreducibly on $\mathcal{H}$. So suppose $\mathcal{V}$ is a non-zero closed subspace of $\mathcal{H}$ which is invariant under $\mathcal{T}$. Since $S^*e_1 = 0$ and $S^*e_n = e_{n-1}$ for $n > 1$, the operator $S^*S - SS^*$ is the rank-one projection $e_1 \otimes \bar{e}_1$ onto the span of e_1. Since $S^n e_1 = e_n$, the space $\mathcal{T}e_1 := \{ Te_1 : T \in \mathcal{T} \}$ is dense in $\mathcal{H}$. Write $e_1 = h + k$ with $h \in \mathcal{V}$ and $k \in \mathcal{V}^\perp$. If $(k \mid e_1) \neq 0$, then $(e_1 \otimes \bar{e}_1)(k) = (k \mid e_1)e_1$ and hence e_1 also belongs to $\mathcal{V}^\perp$; this implies that $\mathcal{T}e_1 \subset \mathcal{V}^\perp$ and $\mathcal{V}^\perp = \mathcal{H}$, which contradicts $\mathcal{V} \neq \{0\}$. Thus $(k \mid e_1) = 0$, $e_1 = h$ belongs to $\mathcal{V}$, and $\mathcal{T}e_1 \subset \mathcal{V}$, so $\mathcal{V}$ is all of $\mathcal{H}$.

To see that $\mathcal{T}$ contains the compacts, it is enough to show that each rank-one operator $h \otimes \bar{k}$ is in $\mathcal{T}$. Because $\mathcal{T}e_1$ is dense in $\mathcal{H}$, we can find $R, T \in \mathcal{T}$ such that $Re_1 \sim h$ and $Te_1 \sim k$. But now $h \otimes \bar{k} \sim R(e_1 \otimes \bar{e}_1)T^* \in \mathcal{T}$, as required.

It remains to identify the quotient $\mathcal{T}/\mathcal{K}(\mathcal{H})$ with $C(\mathbb{T})$. Since $\mathcal{T}$ is generated by S, the quotient is generated by the image $q(S)$ under the quotient map q. Because $S^*S = 1$ and $SS^* - S^*S$ is compact, $u = q(S)$ is unitary, and hence has spectrum $\sigma(u)$ contained in $\mathbb{T}$; since $C^*(u) \cong C(\sigma(u))$, it is enough to prove that $\sigma(u)$ is all of $\mathbb{T}$. Since it is certainly nonempty, it is enough to prove that $\lambda \in \sigma(u)$ implies $z\lambda \in \sigma(u)$ for all $z \in \mathbb{T}$.

So suppose $\|z\| = 1$. There is a unitary operator U_z on $\mathcal{H}$ such that $U_z(e_n) = z^n e_n$. Then $U_z^* e_n = \bar{z}^n e_n$, so

$$USU^*(e_n) = \bar{z}^n U(Se_n) = \bar{z}^n U e_{n+1} = \bar{z}^n z^{n+1} e_{n+1} = zSe_n;$$

in other words, $U_z S U_z^* = zS$. Since zS also generates $\mathcal{T} = C^*(S)$, $\operatorname{Ad} U_z$ restricts to an automorphism α_z of $\mathcal{T}$ such that $\alpha_z(S) = zS$; further, since $\operatorname{Ad} U_z$ maps $\mathcal{K}(\mathcal{H})$ onto $\mathcal{K}(\mathcal{H})$, α_z induces an automorphism β_z of the quotient $\mathcal{T}/\mathcal{K}(\mathcal{H})$ such that $\beta_z(u) = zu$. We then have $\sigma(u) = \sigma(\beta_z(u)) = \sigma(zu) = z\sigma(u)$, which gives what we needed. □

A C^*-algebra A is called CCR or *liminal* if the image of each irreducible representation π is precisely $\mathcal{K}(\mathcal{H}_\pi)$, and GCR or *postliminal* if the image of each irreducible representation π contains $\mathcal{K}(\mathcal{H}_\pi)$. Thus every CCR algebra is GCR. Examples A.23–A.25 are all CCR algebras. The Toeplitz algebra $\mathcal{T}$ of Example A.31 is not CCR — the identity representation is irreducible but properly contains the compacts — but is GCR. If A is a GCR algebra, then the canonical map $[\pi] \mapsto \ker \pi$ is a bijection of $\hat{A}$ onto $\operatorname{Prim} A$, and hence a homeomorphism. This is easy to see if A is CCR, since each primitive quotient is isomorphic to $\mathcal{K}(\mathcal{H})$ and hence has an essentially unique irreducible representation by Example A.15, and relatively straightforward for GCR algebras ([111, 5.6.3] or [2, 1.5.4]). A much harder theorem says that injectivity of the canonical map of $\hat{A}$ onto $\operatorname{Prim} A$ characterizes the GCR algebras among separable C^*-algebras; this and other characterizations of GCR algebras are summarized in [124, Theorem 6.8.7] and the following "Notes and Remarks."

To finish off this section, we give an example of a C^*-algebra which is not GCR, and then discuss a class of examples where one can see directly that the canonical map of $\hat{A}$ onto $\operatorname{Prim} A$ is many-to-one.

Example A.32. The quotient $\mathcal{C}(\mathcal{H}) := B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is called the *Calkin algebra.* If $\mathcal{H}$ is a separable infinite-dimensional Hilbert space, then $\mathcal{C}(\mathcal{H})$ is simple and

every representation of $\mathcal{C}(\mathcal{H})$ contains no nonzero compact operators. It follows that $\operatorname{Prim} B(\mathcal{H}) = \{\, 0, \mathcal{K}(\mathcal{H}) \,\}$, and that $B(\mathcal{H})$ is not GCR.

Proof. To see that $\mathcal{C}(\mathcal{H})$ is simple, it is enough to show that every ideal I of $B(\mathcal{H})$ which properly contains $\mathcal{K}(\mathcal{H})$ is all of $B(\mathcal{H})$. If $I \supset \mathcal{K}(\mathcal{H})$ but $I \neq \mathcal{K}(\mathcal{H})$, then I contains a non-compact operator T; since I is the span of its self-adjoint elements, we may assume that T is self-adjoint. We aim to prove that I contains an invertible operator. Since I is an ideal, it is enough to find an infinite-dimensional subspace M of $\mathcal{H}$ on which T is invertible: for then if $V : \mathcal{H} \to M$ is an isometry, V^*TV will be an invertible operator in I.

For $\epsilon > 0$, let

$$M_\epsilon := \overline{\operatorname{span}}\{\, f(T)h : h \in \mathcal{H},\ f \in C\big(\sigma(T)\big), \text{ and } f(\lambda) = 0 \text{ for } |\lambda| \leq \epsilon \,\}.$$

We first claim that $\|Th\| \geq \epsilon\|h\|$ for all $h \in M_\epsilon$. To see this, fix $h \in M_\epsilon$ and let $J_\epsilon = \{\, f \in C\big(\sigma(T)\big) : f(\lambda) = 0 \text{ for } |\lambda| \leq \epsilon \,\}$. If $f \in J_\epsilon$ and $g(\lambda) = \lambda$ for all $\lambda \in \sigma(T)$, then $g^2|f|^2 \geq \epsilon^2|f|^2$; thus $T^2 f(T)^* f(T) \geq \epsilon^2 f(T)^* f(T)$, and

$$\|Tf(T)h\|^2 = \big(T^2 f(T)^* f(T)h \mid h\big) \geq \epsilon^2 \big(f(T)^* f(T)h \mid h\big) = \epsilon^2 \|f(T)h\|^2.$$

Let $\{\, f_i \,\}$ be an approximate identity in J_ϵ. Then $f_i(T)h \to h$, so

$$\|Th\|^2 = \lim_i \|Tf_i(T)h\| \geq \epsilon^2 \lim_i \|f_i(T)h\|^2 = \epsilon^2 \|h\|^2,$$

as claimed. Thus T is bounded below on M_ϵ, and since T is self-adjoint, it follows that T is invertible on M_ϵ.

Now let P_n be the projection onto $M_{1/n}$, and define

$$f_n(\lambda) = \begin{cases} 0 & \text{if } 0 \leq \lambda < \frac{1}{n}, \\ 2(\lambda - \frac{1}{n}) & \text{if } \frac{1}{n} \leq \lambda < \frac{2}{n}, \\ \lambda & \text{if } \lambda \geq \frac{2}{n}. \end{cases}$$

Then $f_n \in J_{1/n}$ and $f_n \to g$ uniformly on $\sigma(T)$, where $g(\lambda) = \lambda$ for all $\lambda \in \sigma(T)$. Thus $f_n(T) \to T$ and $P_n f_n(T) = f_n(T)$. If each $M_{1/n}$ were finite-dimensional, then P_n and hence $f_n(T)$ would be finite-rank operators, and T would be compact. So there exists n such that $M_{1/n}$ is infinite-dimensional, and this is the required subspace M. This proves that I is all of $B(\mathcal{H})$, as required.

Suppose that π is an irreducible representation of $\mathcal{C}(\mathcal{H})$ whose range contains a compact operator. The range must then be contained in $\mathcal{K}(\mathcal{H}_\pi)$: otherwise the inverse image of $\mathcal{K}(\mathcal{H}_\pi)$ would be a proper nonzero ideal in $\mathcal{C}(\mathcal{H})$. For the same reason, π must be faithful. On the other hand, $\mathcal{C}(\mathcal{H})$ has an identity, so the space $\mathcal{H}_\pi$ must be finite-dimensional, which is impossible because one can find infinitely many mutually orthogonal projections in $\mathcal{C}(\mathcal{H})$ (just take the images of infinitely many mutually orthogonal projections in $B(\mathcal{H})$ of infinite rank). Thus no irreducible representation of $\mathcal{C}(\mathcal{H})$ can contain a compact operator, as claimed. □

Hooptedoodle A.33. We describe a family of C^*-algebras A for which $\operatorname{Prim} A = \{\, 0 \,\}$ but $\hat{A}$ is uncountable. Let κ be a sequence of positive integers $\{\, \kappa_n \,\}_{n=1}^{\infty}$ with each $\kappa_n \geq 2$,

and set $\kappa!(n) := \kappa_1\kappa_2\cdots\kappa_n$. There is a natural unital monomorphism ϕ_n of $M_{\kappa!(n)}$ into $M_{\kappa!(n+1)}$ obtained by sending $a \in M_{\kappa!(n)}$ to the block diagonal matrix in $M_{\kappa!(n+1)}$ with a repeated κ_{n+1} times down the main diagonal; alternatively, identify $M_{\kappa!(n+1)}$ with $M_{\kappa!(n)} \otimes M_{\kappa_{n+1}}$ (see Example B.19), and define $\phi_n(a) := a \otimes 1$. Now fix a separable infinite-dimensional Hilbert space $\mathcal{H}$ and view each $M_{\kappa!(n)}$ as a unital subalgebra of $B(\mathcal{H})$ by viewing $a \in M_{\kappa!(n)}$ as an infinite block diagonal matrix with a's on the diagonal; each ϕ_n is now given by inclusion. The *Glimm algebra* or *UHF-algebra* associated to κ is the norm closure $A(\kappa)$ of $\bigcup_n M_{\kappa!(n)}$ in $B(\mathcal{H})$. If we define $\epsilon_\kappa : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by

$$\epsilon_\kappa(r) = \sup\{\, m \in \mathbb{N} : r^m \text{ divides some } \kappa!(n) \,\},$$

then some hard work shows that $A(\kappa)$ is isomorphic to $A(\kappa')$ if and only if $\epsilon_\kappa = \epsilon_{\kappa'}$ [111, Theorem 6.2.3 and Example 7.3.3]. It follows that there are uncountably many nonisomorphic Glimm algebras.

It is easy to see that each Glimm algebra is simple [111, Theorem 6.1.3], so that $\operatorname{Prim} A(\kappa) = \{0\}$. We shall now see that the spectrum $A(\kappa)^\wedge$ is uncountable. For each n, fix a set $\Lambda(n) = \{\lambda_1^n, \ldots, \lambda_{\kappa_n}^n\} \subset [0,1]$ with $\sum_{r=1}^{\kappa_n} \lambda_r^n = 1$. Then there is a state ρ^n on M_{κ_n} such that

$$\rho^n(a) = \sum\nolimits_{i=1}^{\kappa_n} \lambda_i^n a_{ii};$$

if each $\lambda_i^n = 1/\kappa_n$, then ρ^n is the usual normalized trace on M_{κ_n}. Let τ^n be the product state $\rho^1 \otimes \cdots \otimes \rho^n$ on $M_{\kappa!(n)} = M_{\kappa_1} \otimes \cdots \otimes M_{\kappa_n}$ (see Proposition B.7). For example, if $\kappa_1 = \kappa_2 = 2$, $\Lambda(1) = \{\alpha, \beta\}$ and $\Lambda(2) = \{\gamma, \delta\}$, then

$$\tau^2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma\tau^1(a) + \delta\tau^1(d) = \alpha\gamma a_{11} + \beta\gamma a_{22} + \alpha\delta d_{11} + \beta\delta d_{22}.$$

In general, we have $\tau^n = \tau^{n+1}|_{M_{\kappa!(n)}}$, so the τ_n combine to give a functional on $\bigcup_n M_{\kappa!(n)}$; it turns out that there is a unique state τ^Λ of the closure $A(\kappa)$ such that $\tau^\Lambda|_{M_{\kappa!(n)}} = \tau^n$ for all n. If each λ_n^n is either 0 or 1, then τ^Λ is a pure state [124, Lemma 6.5.5], and the associated irreducible representations are equivalent if and only if the sets $\Lambda(n)$ are the same for all large n [124, Proposition 6.5.6].

A.3 The Dauns-Hofmann Theorem

The Gelfand-Naimark Theorem identifies a commutative C^*-algebra A with an algebra of continuous functions on its maximal ideal space. One might hope for a similar theorem identifying a noncommutative C^*-algebra as an algebra of operator-valued functions on its spectrum or primitive ideal space, and indeed this was the original motivation for developing the theory of C^*-bundles. This program was not completely successful — the resulting generalizations of the Gelfand-Naimark Theorem had nontrivial hypotheses and required the development of an entirely new kind of bundle — and this is one reason we have preferred a more algebraic approach in this book. However, the theory certainly had successes, and the Dauns-Hofmann theorem is a particularly striking one. It says that every C^*-algebra is a module over the algebra of continuous functions on its primitive ideal space. While this formulation is purely algebraic, the result was motivated by the correspondence between bundles over a space T and $C(T)$-modules, and was originally proved by

bundle-theoretic methods [23]. The present direct proof was developed by Dixmier [27] and Elliott-Olesen [45].

Theorem A.34 (Dauns-Hofmann). *Let A be a C^*-algebra. For each $P \in \operatorname{Prim} A$, let $\pi_P : A \to A/P$ be the quotient map. Then there is an isomorphism ϕ of $C_b\big(\operatorname{Prim} A\big)$ onto the center $ZM(A)$ of $M(A)$ such that for all $f \in C_b(\operatorname{Prim} A)$ and $a \in A$*

$$\pi_P\big(\phi(f)a\big) = f(P)\pi_P(a) \quad \text{for every } P \in \operatorname{Prim} A. \tag{A.9}$$

We usually write $f \cdot a$ for $\phi(f)a$.

For the proof we need some lemmas. For the first, recall that if I and J are ideals in a C^*-algebra, then $I + J$ is also a closed ideal: if $\pi_J : A \to A/J$ is the quotient map, then $I + J = \pi_J^{-1}(\pi_J(I))$.

Lemma A.35. *Suppose I and J are ideals in a C^*-algebra A, and a is a positive element of $I + J$. Then there are positive elements $b \in I$ and $c \in J$ such that $a = b + c$.*

Proof. Since I and J are ideals in A^1, we can assume that A has an identity. By definition, there are not-necessarily-positive elements $b \in I$ and $c \in J$ such that $a = b + c$. Since $a = (a + a^*)/2$, we can assume $b = b^*$ and $c = c^*$. Fix $\epsilon > 0$. Then $h := |b| + |c| + \epsilon 1$ is positive and invertible. Let $d = a^{1/2}h^{-1/2}$. Then $|b| + |c| \le h$, $h^{-1/2}\big(|b|+|c|\big)h^{-1/2} \le 1$, and $d\big(|b|+|c|\big)d^* \le a$. Now let $b_1 = d|b|d^*$ and $c_1 = d|c|d^*$; then $b_1 \in I$, $c_1 \in J$, both are positive, and $b_1 + c_1 \le a$. Since

$$1 - d^*d = h^{-1/2}(h - a)h^{-1/2} = h^{-1/2}(|b| - b + |c| - c + \epsilon 1)h^{-1/2} \ge 0,$$

we have $dd^* \le 1$ and

$$b_1 + c_1 = d\big(|b| + |c|\big)d^* = d(h - \epsilon 1)d^* = a - \epsilon dd^* \ge a - \epsilon 1.$$

In other words, $0 \le a - b_1 - c_1 \le \epsilon 1$. This implies that $\|b_1\| \le \|a\|$ and $\|c_1\| \le \|a\|$.

We can repeat the above construction with a replaced by $a_1 = a - b_1 - c_1$ and ϵ by $\epsilon/2$, and then continue inductively to obtain positive elements $b_k \in I$ and $c_k \in J$ such that

$$0 \le a - (b_1 + b_2 + \cdots + b_n) - (c_1 + c_2 + \cdots + c_n) \le (\epsilon/2^n)1,$$

$\|b_k\| \le 2^{-(k-1)}\|a\|$, and $\|c_k\| \le 2^{-(k-1)}\|a\|$. The required elements are $b' = \sum_{k=1}^{\infty} b_k$ and $c' = \sum_{k=1}^{\infty} c_k$. □

Lemma A.36. *Let $f \in C_b(\operatorname{Prim} A)$, $a \in A$ and $\epsilon > 0$. Then there exists $b_\epsilon \in A$ such that*

$$\|\pi_P(b_\epsilon) - f(P)\pi_P(a)\| < \epsilon \quad \text{for all } P \in \operatorname{Prim} A. \tag{A.10}$$

Any other such b'_ϵ satisfies $\|b_\epsilon - b'_\epsilon\| < 2\epsilon$.

Proof. It is enough to do this for $f : \operatorname{Prim} A \to [0,1]$. Fix n, and define open sets

$$O_k = \{\, P \in \operatorname{Prim} A : \frac{k-1}{n} < f(P) < \frac{k+1}{n} \,\} \quad \text{for } k = 0, 1, \ldots, n.$$

Note that $\operatorname{Prim} A = \bigcup_{k=0}^{n} O_k$ and that each $P \in \operatorname{Prim} A$ belongs to at most two O_k's. Let

$$J_k = \bigcap \{\, P \in \operatorname{Prim} A : P \notin O_k \,\}$$

be the ideal in A with primitive ideal space O_k, so that $P \in O_k$ if and only if $J_k \not\subset P$. Since every P belongs to some O_k, we have $J_k \not\subset P$ for some k, and

$$J_0 + J_1 + \cdots + J_n = A.$$

By Lemma A.35, there are positive elements $a_k \in J_k$ such that $a = a_0 + a_1 + \cdots + a_n$; notice that $\|a_k\| \le \|a\|$ for all k. Let $b = \sum_{k=0}^{n} \frac{k}{n} a_k$. Then for every $P \in \operatorname{Prim} A$, we have

$$\begin{aligned} \|\pi_P(b) - f(P)\pi_P(a)\| &= \Big\| \sum\nolimits_{k=0}^{n} \frac{k}{n} \pi_P(a_k) - f(P)\pi_P(a_k) \Big\| \\ &= \Big\| \sum\nolimits_{k=0}^{n} \Big(\frac{k}{n} - f(P) \Big) \pi_P(a_k) \Big\|. \end{aligned} \tag{A.11}$$

But if $P \in O_k$, then $|k/n - f(P)| < 1/n$. Since each P belongs to at most two O_k's, (A.11) is bounded by $(2/n)\|a\|$. Now the existence of b_ϵ follows by taking $n > 2\|a\|/\epsilon$.

If b'_ϵ also satisfies (A.10), then

$$\begin{aligned} \|b_\epsilon - b'_\epsilon\| &= \sup_P \|\pi_P(b_\epsilon) - \pi_P(b'_\epsilon)\| \\ &\le \sup_P \big(\|\pi_P(b_\epsilon) - f(P)\pi_P(a)\| + \|f(P)\pi_P(a) - \pi_P(b'_\epsilon)\| \big) \\ &< 2\epsilon, \end{aligned}$$

as required. □

Proof of Theorem A.34. Given f and a, we consider the sequence $\{b_{2^{-n}}\}$ generated by Lemma A.36. The uniqueness clause implies that this sequence is Cauchy, and hence converges to an element $T(f,a)$. Equation (A.10) implies that

$$\pi_P(T(f,a)) = f(P)\pi_P(a) \quad \text{for all } P \in \operatorname{Prim} A,$$

and this equation characterizes $T(f,a)$ because $\bigcap \ker \pi_P = \{0\}$. The map $b \mapsto T(\bar{f}, b)$ is an adjoint for $T : a \mapsto T(f,a)$: to see this, we just need to verify that

$$\pi_P\big(\langle T(f\ ,\ a), b\rangle_A\big) = \overline{f(P)}\pi_P(a^*b) = \pi_P\big(\langle a\ ,\ T(\bar{f}, b)\rangle_A\big)$$

for $P \in \operatorname{Prim} A$. Thus we have a multiplier $\phi(f) := T(f, \cdot)$ of A. A similar calculation shows that $\phi(f) \in ZM(A)$. It is immediate from (A.9) and $\bigcap \ker \pi_P = \{0\}$ that ϕ is injective, so it only remains to check that ϕ is surjective.

If $\pi \in \hat{A}$ and $z \in ZM(A)$, then $\bar{\pi}(z)$ commutes with $\pi(A)$ and must be a scalar multiple of $1_{\mathcal{H}_\pi}$: say $\bar{\pi}(z) := w_\pi(z)1_{\mathcal{H}_\pi}$. The map $w_\pi : ZM(A) \to \mathbb{C}$ is a homomorphism, and is nonzero because $w_\pi(1) = 1$; hence w_π belongs to the maximal ideal space Δ of $ZM(A)$. Since

$$\ker w_\pi = \{\, z \in ZM(A) : z \cdot A \subset \ker \pi \,\},$$

w_π depends only on $\ker \pi$, and we have a well-defined map $\alpha : \operatorname{Prim} A \to \Delta$ such that $\alpha\big(\ker \pi\big) = w_\pi$. Recall that if J is an ideal in $ZM(A)$, then

$$\mathcal{O}_J := \{\, w \in \Delta : \ker w \not\supset J \,\}$$

is a typical open set in $\Delta = \operatorname{Prim} ZM(A)$ (Proposition A.27). On the other hand, $\overline{J \cdot A} := \overline{\operatorname{span}}\{\, za : z \in J, a \in A \,\}$ is an ideal in A, and the corresponding open set

$$\mathcal{V}_{\overline{J \cdot A}} := \{\, P \in \operatorname{Prim} A : P \not\supset \overline{J \cdot A} \,\}$$

in $\operatorname{Prim} A$ satisfies

$$\begin{aligned} \alpha^{-1}(\mathcal{O}_J) &= \{\, \ker \pi : \pi \in \hat{A} \text{ and } \ker w_\pi \not\supset J \,\} \\ &= \{\, \ker \pi : \pi \in \hat{A} \text{ and } \overline{J \cdot A} \not\subset \ker \pi \,\} \\ &= \mathcal{V}_{\overline{J \cdot A}}. \end{aligned}$$

Thus α is continuous, and we can define $\psi : ZM(A) \to C_b(\operatorname{Prim} A)$ by $\psi(z) = \hat{z} \circ \alpha$, or equivalently

$$\psi(z)\big(\ker \pi\big)1_{\mathcal{H}_\pi} = \hat{z}(w_\pi)1_{\mathcal{H}_\pi} = \bar{\pi}(z) \quad \text{for } \pi \in \hat{A}.$$

Then for $\pi \in \hat{A}$ and $a \in A$ we have

$$\pi(za) = \bar{\pi}(z)\pi(a) = \psi(z)\big(\ker \pi\big)\pi(a) = \pi\big(\phi(\psi(z))a\big),$$

so $z = \phi(\psi(z))$. Thus ϕ is surjective, as required. □

Hooptedoodle A.37. If X is any topological space, then a compact Hausdorff space βX together with a continuous map $\beta : X \to \beta X$ is called a *Stone-Čech compactification* of X if, for every continuous map k from X into a compact Hausdorff space Y, there is a unique continuous map $k' : \beta X \to Y$ such that $k = k' \circ \beta$. Stone-Čech compactifications are essentially unique, and always exist — even for potentially pathological spaces such as $\operatorname{Prim} A$ — and $C_b(X) \cong C(\beta X)$. When A has an identity, $\operatorname{Prim} A$ is compact and $\beta(\operatorname{Prim} A) = \Delta_{ZA}$ is also known as a complete regularization of $\operatorname{Prim} A$. Further references and details can be found in [112].

A.4 The State Space of a C^*-Algebra

For our motivating example $A = C(T)$, the topology on $\operatorname{Prim} A = \hat{A}$ can be alternatively described by viewing $\hat{A} = \{\, \epsilon_t : t \in T \,\}$ as a subset of the Banach space dual A^* and restricting the weak-$*$ topology — indeed, both topologies reduce to

the usual one on T (see, respectively, the motivating discussion before Lemma A.20, and [125, 4.2.5]). In this section we shall give a similar description of the topology on the spectrum $\hat{A}$ of a noncommutative C^*-algebra A. For this, we realize $\hat{A}$ as the quotient of the space $P(A)$ of pure states under the map $\rho \mapsto \pi_\rho$ which sends a pure state ρ to its GNS-representation; the weak-$*$ topology on $P(A)$ then descends to a quotient topology on $\hat{A}$. Of course, there is a good deal of redundancy in this reaization of $\hat{A}$: any vector state $\omega : a \mapsto (\pi(a)h \mid h)$ associated to an irreducible representation π gives a GNS-representation π_ω equivalent to π (by Corollary A.8).

Theorem A.38. *Let A be a C^*-algebra, and give the set $P(A)$ the weak-$*$ topology. Then the GNS-map $\Lambda : \rho \mapsto \pi_\rho$ is a continuous open surjection of $P(A)$ onto $\hat{A}$.*

After proving Theorem A.38, we discuss some results which we want to use in two of our peripheral sections. The first says, conversely to Proposition A.17, that in a separable C^*-algebra, prime ideals are primitive. This is used in the proof of Theorem B.45. The second is a technical result about states and ideals which is needed for the proof of the Brown-Green-Rieffel Theorem.

For the proof of Theorem A.38 we need some more convex analysis. We have already met the Krein-Milman Theorem, which says that every weak-$*$ compact set C in A^* is the weak-$*$ closed convex hull of its extreme points; now we need to know that if C is the weak-$*$ closed convex hull of some other set D, then every extreme point of C is in the weak-$*$ closure of D. Detailed proofs of this result are given in [111, Theorem A.14] and [81, Theorem 1.4.5]. We also need a version of the Hahn-Banach Separation Theorem obtained by applying the usual one to the dual of a Banach space X.

Lemma A.39. *Suppose that X is a Banach space, and ϕ is a weak-$*$ continuous linear functional on X^*. Then there exists $x \in X$ such that $\phi(g) = g(x)$ for all $g \in X^*$.*

Proof. The set $S = \{\, g \in X^* : |\phi(g)| < 1 \,\}$ is a weak-$*$ neighbourhood of 0 in X^*, so there are $x_1, \ldots x_n \in X$ and $\epsilon > 0$ such that

$$\bigcap_i \{\, g \in X^* : |g(x_i)| < \epsilon \,\} \subset S.$$

Then $g(x_i) = 0$ for all i implies $\phi(g) = 0$. (If $\phi(g) \neq 0$, then $\phi(g)^{-1}g$ belongs to the intersection but not to S.) Thus $\Phi(g) = \big(g(x_i), \ldots, g(x_n)\big)$ defines a linear map $\Phi : X^* \to \mathbb{C}^n$ such that $\ker \Phi \subset \ker \phi$, and hence there is a linear functional $T : \mathbb{C}^n \to \mathbb{C}$ such that $\phi = T \circ \Phi$. Thus there is a vector $(z_1, \ldots, z_n) \in \mathbb{C}^n$ such that $\phi(g) = \sum_i z_i g(x_i)$ for all $g \in X^*$. Then $x := \sum z_i x_i$ will do. □

Corollary A.40. *Suppose that X is a Banach space and that C and D are disjoint nonempty convex subsets of the dual space X^*. If D is open in the weak-$*$ topology, then there exist $x \in X$ and $\alpha \in \mathbb{R}$ such that*

$$\operatorname{Re} f(x) < \alpha \leq \operatorname{Re} g(x) \quad \textit{for all } f \in D \textit{ and } g \in C.$$

Proof. This follows immediately from the Lemma and the usual Hahn-Banach Separation Theorem (see, for example, [125, Theorem 2.4.7]). □

Lemma A.41. *Let A be a C^*-algebra with identity. Suppose D is a subset of $S(A)$ such that*

$$a = a^* \text{ and } \rho(a) \geq 0 \text{ for all } \rho \in D \Longrightarrow a \geq 0. \tag{A.12}$$

Then $P(A)$ is contained in the weak-$*$ closure of D.

Proof. In view of the comments preceding Lemma A.39, it is enough to prove that the weak-$*$ closed convex hull C of D is $S(A)$. Since A has an identity, $C \subset S(A)$. If $C \neq S(A)$, then by Corollary A.40 there exist $\tau \in S(A)$, $\alpha \in \mathbb{R}$, and $a \in A$ such that

$$\operatorname{Re}\tau(a) < \alpha \leq \operatorname{Re}\rho(a)$$

for all $\rho \in C$. Replacing a by $a_0 = (a+a^*)/2$ and using that all positive functionals satisfy $f(a^*) = \overline{f(a)}$, we have $\tau(a_0) < \alpha \leq \rho(a_0)$ for all $\rho \in C$. But if $\rho(a_0) \geq \alpha$ for all $\rho \in D \subset C$, then the assumption (A.12) implies $a_0 - \alpha 1_A \geq 0$, which in turn forces $\tau(a_0) \geq \alpha$, a contradiction. Therefore $C = S(A)$, as required. □

Let $\pi : A \to B(\mathcal{H}_\pi)$ be a representation of a C^*-algebra A. We say that a state $\tau \in S(A)$ is *associated to* π if $\tau(a) = (\pi(a)h \mid h)$ for some $h \in \mathcal{H}_\pi$. Notice that if π and ρ are equivalent, then a state is associated to π if and only if it is associated to ρ. If π is irreducible, then a state τ is associated to π iff π is equivalent to π_τ (this follows from Proposition A.6 and Corollary A.8). More generally, if S is a subset of $\hat{A}$, then the set of states associated to elements of S is precisely the set $\Lambda^{-1}(S)$ of states τ such that $\pi_\tau \in S$.

The following result relating the hull-kernel closure of a subset F of $\hat{A}$ to the weak-$*$ closure of $\Lambda^{-1}(F)$ is the key to the proof of Theorem A.38.

Proposition A.42. *Let π be an irreducible representation of a C^*-algebra A, and let F be a subset of $\hat{A}$. Then the following statements are equivalent:*

(a) *π belongs to the closure $\overline{F}$ of F in $\hat{A}$;*

(b) *at least one state associated to π is in the weak-$*$ closure of $\Lambda^{-1}(F)$;*

(c) *every state associated to π is in the weak-$*$ closure of $\Lambda^{-1}(F)$.*

Proof. Let $\mu = \oplus_{\rho \in F}\rho$ be the direct sum of a family of representatives for the classes in F. Let $\mu(A)^\sim = \mu(A) + \mathbb{C}I_\mathcal{H}$ be the C^*-algebra generated by the image of μ and the identity operator 1 (so that $\mu(A)^\sim = \mu(A)$ if A has an identity). Then we can view $\Lambda^{-1}(F)$ as the subset of $S(\mu(A)^\sim)$ consisting of the vector states associated to vectors in one of the direct summands $\mathcal{H}_\rho$ of $\mathcal{H}_\mu$. Thus if $\mu(a) + \lambda 1_\mathcal{H}$ is a self-adjoint element of $\mu(A)^\sim$ and if $\tau(\mu(a) + \lambda 1_\mathcal{H}) \geq 0$ for all $\tau \in \Lambda^{-1}(F)$, then $\rho(a) + \lambda 1_{\mathcal{H}_\rho} \geq 0$ for all $\rho \in F$, and $\mu(a) + \lambda 1_{\mathcal{H}_\mu} = \bigoplus(\rho(a) + \lambda 1_{\mathcal{H}_\rho}) \geq 0$. Therefore we can apply Lemma A.41 with $D = \Lambda^{-1}(F)$, and deduce that every pure state on $\mu(A)^\sim$ is in the weak-$*$ closure of $\Lambda^{-1}(F)$.

Now suppose that $\pi \in \overline{F}$. Then

$$\ker \pi \supset \bigcap_{\rho \in F} \ker \rho = \ker \mu, \tag{A.13}$$

and any state τ associated to π may be viewed as a pure state on $\mu(A)$. (It is pure on $\mu(A)$ because it is a vector state associated to an irreducible representation of $A/\ker \mu \cong \mu(A)$.) Since τ extends to a state of $\mu(A)^\sim$ by Lemma A.7, and since the extension of a pure state is easily seen to be pure, it follows from the previous paragraph that (a) implies (c).

Since (c) trivially implies (b), it will suffice to show that (b) implies (a). This amounts to proving that (A.13) holds. By assumption there is a pure state τ on A of the form $\tau(a) = (\pi(a)h \mid h)$ which belongs to the weak-$*$ closure of $\Lambda^{-1}(F)$. Now suppose that $a \in A$ belongs to the right-hand side of (A.13), and $b \in A$ is arbitrary. Then $b^*a^*ab \in \bigcap \ker \rho$, so every ϕ in $\Lambda^{-1}(F)$ vanishes on b^*a^*ab, and τ must too. Since $\tau(b^*a^*ab) = \|\pi(a)\pi(b)h\|^2$ and $\overline{\pi(A)h} = \mathcal{H}_\pi$, we conclude that $\pi(a) = 0$. Thus (A.13) is valid, and $\pi \in \overline{F}$. □

Proof of Theorem A.38. It follows from Lemma A.12 that Λ is surjective. To see that Λ is continuous, suppose F is closed in $\hat{A}$ and $\tau \in \overline{\Lambda^{-1}(F)}$. Then by the (b) $\Longrightarrow$ (a) implication in the Proposition, we must have $\pi_\tau \in \overline{F} = F$, so $\tau \in \Lambda^{-1}(F)$. Thus $\Lambda^{-1}(F)$ is closed. To see that Λ is open, suppose that U is open in $P(A)$ and $\pi \in \Lambda(U)$; say $\pi = \pi_\tau$ for $\tau \in U$. Since τ is not in the closure of $P(A) \setminus U$, it is not in the closure of the smaller set $\Lambda^{-1}(\hat{A} \setminus \Lambda(U))$. But now the (a) $\Longrightarrow$ (c) implication in Proposition A.42 says that π_τ is not in the closure of $\hat{A} \setminus \Lambda(U)$, so $\Lambda(U)$ is open. □

Hooptedoodle A.43. With Theorem A.38 in hand, we have two rather different definitions of the topology on the spectrum of a C^*-algebra. There is yet another. In essence, the idea is to view $\hat{A}$ as a quotient of the set $\operatorname{Irr} A$ of irreducible representations of A in a fixed Hilbert space of sufficiently large dimension; $\operatorname{Irr} A$ has a natural topology, in which it is a complete separable metric space when A is separable. Unfortunately, complications arise because C^*-algebras can have irreducible representations of different dimensions. This "third definition of the topology on the spectrum" is discussed in [46].

A topological space P is *separable** if it has a countable dense subset. A metric d on P is *compatible* with the topology if d induces that topology on P, and a topological space which admits a compatible metric is called *metrizable*. The Urysohn Metrization Theorem says that every normal second countable space, and hence every second countable locally compact Hausdorff space, is metrizable [125, 1.6.14 and 1.7.9]. A separable topological space is called a *Polish space* if there exists a compatible complete metric. For example, $(0, 1)$ is a Polish space in its usual topology; the usual metric $d(x, y) = |x - y|$ is not complete, but there is a

*Some authors call a locally compact group separable when its topology is second countable, i.e., has a countable base for the topology. This is potentially confusing: every second countable space is certainly separable, but there are locally compact groups which are separable as topological spaces but are not second countable (for example, the dual of the group $\mathbb{R}_d$ of additive reals with the discrete topology). We only use separable to mean the existence of a countable dense set.

homeomorphism of $(0,1)$ onto $\mathbb{R}$, so we can pull back the complete metric from $\mathbb{R}$. Since closed subsets of complete spaces are complete, every closed subset of a Polish space is Polish. We can do rather better than this; recall that the countable intersection of open sets is called a G_δ-set.

Lemma A.44. *Every G_δ-subset of a Polish space P is Polish.*

Outline of the proof. (For the details, see [2, §3].) We first claim that every open subset U of P is Polish. To see this, let d be a complete metric on P and define $d(x, P \setminus U) := \inf\{\, d(x,y) : y \in P \setminus U \,\}$. Then $f(x) = 1/d(x, P \setminus U)$ is a continuous function on U and U is homeomorphic to the graph $\Gamma(f)$ of f in $P \times \mathbb{R}$. Since $\Gamma(f)$ is closed in the Polish space $P \times \mathbb{R}$, it follows that U is Polish.

Now suppose A is a G_δ-set in P. There are open sets $U_n \subset P$ such that $A = \bigcap_n U_n$, and by the previous paragraph there are complete metric spaces P_n and homeomorphisms $\phi_n : P_n \to U_n$. Let

$$B = \{\, (p_n) \in \prod_n P_n : \phi_1(p_1) = \phi_k(p_k) \text{ for } k = 1, 2, \dots \,\}.$$

Again, it is not hard to see that B is a closed subset of the complete metric space $\prod_n P_n$, and is therefore itself a complete metric space. It is not difficult to check that the map $\phi : B \to A$ defined by $\phi\big((p_n)\big) := \phi_1(p_1)$ is a homeomorphism. □

Hooptedoodle A.45. The converse is also true [124, Lemma 4.2.3]; thus a subset E of a Polish space P is Polish in the relative topology if and only if E is a G_δ-subset of P.

The Baire Category Theorem implies that every Polish space has the *Baire property*: the countable intersection of open dense sets is again dense [125, 2.2.2]. Notice that a continuous, open image of a space with the Baire property also has the Baire property.

Now let A_1^* be the unit ball in the dual A^* of a C^*-algebra A, which is weak-$*$ compact by the Banach-Alaoglu Theorem [125, 2.5.2]. If A is separable, then A_1^* is second countable and therefore separable. In fact, A_1^* is a separable metric space. (We can either invoke the Urysohn Metrization Theorem, or write down a compatible metric: fix a countable dense set $\{a_n\}$ in the unit ball of A, and define $d(f,g) := \sum_n |f(a_n) - g(a_n)|/2^{-n}$.) Define

$$Q(A) := \{\, \rho \in A^* : \rho \text{ is positive and } \|\rho\| \leq 1 \,\};$$

Pedersen [124] calls $Q(A)$ the quasi-state space of A. Notice that $Q(A)$ is a weak-$*$ closed subset of A_1^*, and is therefore Polish.

Proposition A.46. *If A is a separable C^*-algebra, then $P(A)$ is a Polish space.*

Proof. Let d be a complete metric on $Q(A)$, and note that $Q(A)$ is convex. Thus

$$F_n := \{\, 0 \,\} \cup \{(\phi + \psi)/2 : \phi, \psi \in Q(A) \text{ and } d(\phi, \psi) \geq 1/n\}$$

are subsets of $Q(A)$ for $n \in \mathbb{N}$. It will suffice to see that each F_n is closed, and that $P(A) = Q(A) \setminus \bigcup_n F_n$; for then $P(A)$ is a G_δ-subset of the Polish space $Q(A)$, and the result will follow from Lemma A.44.

Suppose that $\{\rho_k\} \subset F_n \setminus \{0\}$ and $\rho_k \to \rho$. For each k there are $\phi_k, \psi_k \in Q(A)$ such that $\rho_k = \frac{1}{2}(\phi_k + \psi_k)$ and $d(\phi_k, \psi_k) \geq 1/n$. Since $Q(A)$ is a closed subset of A_1^*, it is compact, so by passing to a subsequence we can assume that $\phi_k \to \phi$ and $\psi_k \to \psi$. Then $d(\phi, \psi) \geq 1/n$ and $\rho = \frac{1}{2}(\phi + \psi) \in F_n$. Thus F_n is closed.

Now suppose that $\rho \in Q(A) \setminus \{0\}$ and that $\|\rho\| < 1$. Choose δ such that $0 < \delta < \min\{\|\rho\|, 1 - \|\rho\|\}$, and $n \in \mathbb{N}$ such that $d\big((1-\delta)\rho, (1+\delta)\rho\big) \geq 1/n$. Since $\rho = \frac{1}{2}\big((1-\delta)\rho + (1+\delta)\rho\big)$, $\rho \in F_n$. On the other hand, if $\|\rho\| = 1$, then $\rho \in S(A)$. If $\rho \in S(A) \setminus P(A)$, then there are states ϕ, τ and $\lambda \in [\frac{1}{2}, 1)$ such that $\rho = \lambda\phi + (1-\lambda)\tau$. Then $\zeta = (2\lambda - 1)\phi + (1 - (2\lambda - 1))\tau \in Q(A)$ and $\zeta \neq \phi$. Thus there is an $n \in \mathbb{N}$ such that $d(\phi, \zeta) \geq 1/n$. Since $\rho = \frac{1}{2}(\phi + \zeta)$ we have $\rho \in F_n$. Since no pure state can possibly lie in any F_n, we deduce that $Q(A) \setminus \bigcup_n F_n$ is precisely the set $P(A)$ of pure states. □

Since Theorem A.38 implies that $\hat{A}$ is the continuous open image of the space $P(A)$, we can immediately deduce:

Corollary A.47. *If A is a separable C^*-algebra, then every countable intersection of open dense subsets of $\hat{A}$ is again dense.*

Hooptedoodle A.48. The separability hypothesis is known to be unnecessary in Corollary A.47: Choquet has proved that the set of extreme points of a compact convex subset of a locally convex Hausdorff space always has the Baire property [28, Appendix B14].

Theorem A.49. *Every prime ideal in a separable C^*-algebra is primitive.*

Since Proposition A.46 and Theorem A.38 imply that the primitive ideal space of a separable C^*-algebra is second countable, this Theorem is a corollary of the next result. A topological space is called *almost Hausdorff* if every closed subset has a dense open subset which is Hausdorff in the relative topology.

Theorem A.50. *Suppose that A is a C^*-algebra such that* Prim A *is either second countable or almost Hausdorff. Then every prime ideal is primitive.*

Proof. Every closed set in Prim A has the form $C_I := \{P \in \operatorname{Prim} A : P \supset I\}$ for some ideal in $\mathcal{I}(A)$ (Proposition A.27), and $k(C_I) := \bigcap\{P : P \in C_I\} = I$ by Proposition A.17. Since primitive ideals are always prime (Proposition A.17), $C_{IJ} = C_I \cup C_J$ for any ideals $I, J \in \mathcal{I}(A)$. Notice that if Q is a prime ideal, then C_Q is *irreducible* in the sense that, if F_1 and F_2 are closed sets with $C = F_1 \cup F_2$, then $C = F_1$ or $C = F_2$; to see this, note that if $C_Q = C_I \cup C_J = C_{IJ}$, then $Q = IJ$, and primeness forces $Q = I$ or $Q = J$. We will show that every irreducible closed set is the closure of a point $P \in \operatorname{Prim} A$. Then $C_Q = \overline{\{P\}} = C_P$, and $Q = k(C_Q) = k(C_P) = P$ is primitive.

If Prim A is second countable, so is C_Q. Since C_Q is the primitive ideal space of a C^*-algebra (Proposition A.26), it is the continuous open image of a space with the Baire property, and therefore itself has the property. Since C_Q is irreducible, every nonempty open subset is dense, so the Baire property implies that the intersection of all the nonempty open sets is dense. Any point contained in this intersection is dense. If, on the other hand, Prim A is almost Hausdorff, the same is true of C_Q. If a dense open Hausdorff subset of C_Q contained more than one point, then removing

a point would give a nonempty proper open subset of C_Q, which is impossible since C_Q is irreducible. □

Hooptedoodle A.51. If A is a GCR C^*-algebra, then $\hat{A}$ always has an open dense relatively Hausdorff subset [28, Theorem 4.4.5]. Since quotients of GCR algebras are again GCR ([111, Theorem 5.6.2] or [28, Proposition 4.3.5]), it follows that GCR algebras have almost Hausdorff spectra, and therefore almost Hausdorff primitive ideal spaces. Thus a prime ideal in a GCR algebra is always primitive. It is not known whether Theorem A.49 is true for arbitrary (nonseparable) C^*-algebras.

Suppose that J and K are ideals in a C^*-algebra A such that $J \subset K$. If every state $\rho \in S(A)$ which vanishes on J also vanishes on K, then $J = K$ (apply Lemma A.13 to A/J). In Section 5.5 we need to know that this is still true when J and K are right ideals (Proposition A.54). The proof requires some nontrivial facts about right ideals. The first of these says, among other things, that closed right ideals are hereditary: if $b \in J$ and $a \in A$ satisfy $0 \leq a \leq b$, then $a \in J$. This is not hard to prove when $A = C(X)$ for some compact space X, because the ideals of A are determined by their zero sets. But we need some tricky manipulations with the functional calculus to establish the general result.

Lemma A.52. *Suppose that A is a C^*-algebra with identity, that J is a closed right ideal in A, and that a is a positive element of A. If for every $\epsilon > 0$, there is a positive element a_ϵ of J such that $a \leq a_\epsilon + \epsilon 1$, then $a \in J$.*

Proof. Fix $\epsilon > 0$ and a_ϵ as above. Let $b = a_\epsilon^{1/2}$, which belongs to J because it is the norm limit of polynomials in a_ϵ with no constant terms. The functional calculus implies that $b + \sqrt{\epsilon}1$ is invertible. Since $\big(b(b+\sqrt{\epsilon}1)^{-1} - 1\big)(b+\sqrt{\epsilon}1) = -\sqrt{\epsilon}1$,

$$\begin{aligned} \|b(b+\sqrt{\epsilon}1)^{-1}a^{\frac{1}{2}} - a^{\frac{1}{2}}\|^2 &= \|\big(b(b+\sqrt{\epsilon}1)^{-1} - 1\big)a^{\frac{1}{2}}\|^2 \\ &= \|\sqrt{\epsilon}(b+\sqrt{\epsilon}1)^{-1}a^{\frac{1}{2}}\|^2 \\ &= \epsilon\|(b+\sqrt{\epsilon}1)^{-1}a(b+\sqrt{\epsilon}1)^{-1}\|. \end{aligned} \tag{A.14}$$

On the other hand, $0 \leq a \leq a_\epsilon + \epsilon 1 = b^2 + \epsilon 1$, and since $d \leq f$ implies that $cdc^* \leq cfc^*$, we have

$$\begin{aligned} 0 &\leq (b+\sqrt{\epsilon}1)^{-1}a(b+\sqrt{\epsilon}1)^{-1} \leq (b+\sqrt{\epsilon}1)^{-1}(b^2+\epsilon 1)(b+\sqrt{\epsilon}1)^{-1} \\ &= (b^2+\epsilon 1)(b^2+2\sqrt{\epsilon}b+\epsilon 1)^{-1} \leq 1. \end{aligned} \tag{A.15}$$

Now (A.14) and (A.15) imply that $\|b(b+\sqrt{\epsilon}1)^{-1}a^{\frac{1}{2}} - a^{\frac{1}{2}}\| < \epsilon$. Since ϵ is arbitrary and $b \in J$, this implies that $a^{1/2} \in J$, which in turn implies that $a \in J$. □

Lemma A.53. *Suppose that J is a closed right ideal in a C^*-algebra A. Then there is a net $\{e_\lambda\}$ of positive elements in J such that $a = \lim_\lambda e_\lambda a$ for all $a \in J$. In particular, J is the closed right ideal generated by the positive elements in J.*

Proof. Since $B = J \cap J^*$ is a C^*-subalgebra, there is an approximate identity $\{e_\lambda\}$ for B. If $a \in J$, then $aa^* \in B$ and $0 = \lim_\lambda (1 - e_\lambda)aa^*$ (in A^1 if necessary). Since each e_λ is a positive element of norm one, $\|1 - e_\lambda\| \leq 1$, and

$$\lim_\lambda \|a - e_\lambda a\|^2 = \lim_\lambda \|(1-e_\lambda)aa^*(1-e_\lambda)\| \leq \lim_\lambda \|aa^*(1-e_\lambda)\| = 0.$$

The last statement follows from the existence of the e_λ. □

Proposition A.54. *Suppose that A is a C^*-algebra and that J and K are closed right ideals in A with $J \subset K$. If every state on A which vanishes on J also vanishes on K, then $J = K$.*

Proof. Since a right ideal in A is still a right ideal in A^1, we may assume that A has an identity. We will show that every positive element of K is in J; this will suffice in view of Lemma A.53.

Fix $\epsilon > 0$ and $a \in K$ such that $a \geq 0$. The set S_ϵ of states ρ such that $\rho(a) \geq \epsilon$ is a (possibly empty) weak-$*$ compact set. If $\rho \in S_\epsilon$, then ρ is not identically zero on K, and by assumption cannot vanish identically on J. Therefore there exist $a_\rho \in J$ and a weak-$*$ neighbourhood U_ρ of ρ in S_ϵ such that $\tau(a_\rho) \neq 0$ for all $\tau \in U_\rho$. By compactness, there are elements $a_1, \ldots, a_n \in J$ such that the open sets $U_{a_1}, \ldots, U_{a_n}$ cover S_ϵ. Since by Cauchy-Schwarz we have

$$0 < |\rho(a_i)|^2 \leq \rho(a_i^* a_i) \quad \text{for all } \rho \in U_{a_i},$$

it follows that $\rho(a_1^* a_1 + \cdots + a_n^* a_n) > 0$ for all $\rho \in S_\epsilon$. Since S_ϵ is compact, we can multiply the a_i by a sufficiently large scalar and assume that $\rho(a_1^* a_1 + \cdots + a_n^* a_n) \geq \rho(a)$ for all $\rho \in S_\epsilon$. If $\rho \notin S_\epsilon$, then $\rho(\epsilon 1 - a) \geq 0$, so we can deduce that

$$\rho(a_1^* a_1 + \cdots + a_n^* a_n + \epsilon 1 - a) \geq 0$$

for all states ρ in $S(A)$. It follows (see Remark 2.6) that

$$a \leq a_1^* a_1 + \cdots + a_n^* a_n + \epsilon 1,$$

and Lemma A.52 implies that $a \in J$. □

Hooptedoodle A.55. In general a C^*-subalgebra $B \subset A$ is called *hereditary* if $0 \leq a \leq b$ in A and $b \in B$ imply that $a \in B$. Hereditary subalgebras of A are in one-to-one correspondence with closed right ideals: the subalgebra corresponding to a right ideal J is $J \cap J^*$ [111, Theorem 3.2.1].

Appendix B

Tensor Products of C^*-Algebras

In Chapters 5 and 6 we use tensor products of C^*-algebras, and in this Appendix we provide a self-contained treatment of the facts we have used. Since this is a technically complicated aspect of the general theory of C^*-algebras, we begin by reviewing the basic ideas.

We discussed tensor products of vector spaces at the start of Section 2.4. One can view the tensor product $V \odot W$ of two vector spaces either as the vector space spanned by certain elementary tensors $v \otimes w$ subject to rules which make the map $(v, w) \mapsto v \otimes w$ bilinear, or as an abstract vector space on which linear maps are given by bilinear maps on $V \times W$. When $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, there is a natural inner product on the vector-space tensor product $\mathcal{H} \odot \mathcal{K}$, and hence a natural way to complete $\mathcal{H} \odot \mathcal{K}$ to get a Hilbert-space tensor product $\mathcal{H} \otimes \mathcal{K}$ (see Lemma 2.59). In this case, though, it was definitely easier to use the construction of $\mathcal{H} \otimes \mathcal{K}$ rather than a universal characterization of $\mathcal{H} \otimes \mathcal{K}$ in terms of a class of bounded bilinear maps on $\mathcal{H} \times \mathcal{K}$ (see Remark 2.63).

If A and B are C^*-algebras, there is an obvious way to make the vector-space tensor product $A \odot B$ into a $*$-algebra (see Lemma B.1), but there are several different ways of norming $A \odot B$ and completing it to give a C^*-algebra. In practical terms, we have to choose whether we want a tensor product which is well-suited to concrete C^*-algebras acting on Hilbert spaces, or one which has good universal properties and is better-suited to abstract C^*-algebras. This is unfortunate: C^*-algebras are a powerful tool largely because both concrete and abstract methods are available. On the other hand, there is a large class of *nuclear* C^*-algebras for which all C^*-algebra tensor products coincide, and one can often work entirely within this class. Thus, for example, the continuous-trace algebras studied in this book are nuclear, and almost anything we can do to them will give us other nuclear algebras (see Section B.5).

To define the *spatial tensor product*, we represent A and B faithfully on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and show how to represent $A \odot B$ faithfully on $\mathcal{H} \otimes \mathcal{K}$; we can

then pull the norm from $B(\mathcal{H} \otimes \mathcal{K})$ back to $A \odot B$, and complete to get a C^*-algebra $A \otimes_\sigma B$. The hard bit is to prove that the norm does not depend on the choice of representations (Theorem B.9). We discuss the spatial tensor product in Section B.1, and give some important examples in Section B.2.

In many applications, such as those in Chapters 5 and 6, only the spatial tensor product appears. However, many of the arguments used to establish properties of $A \otimes_\sigma B$ work in greater generality, so we have digressed a little in Section B.3 to consider other C^*-norms on $A \odot B$. (Those looking for the most efficient route to the main text can read $\|\cdot\|_\sigma$ for $\|\cdot\|_\gamma$ in the first part of Section B.3, and skip to Section B.4 after Corollary B.22.) In particular, we discuss the *maximal tensor product* $A \otimes_{\max} B$, which is universal for commuting pairs of representations of A and B (Theorem B.27). As the name suggests, this is the largest tensor product in a sense to be made precise, but this point of view is not necessarily helpful: to establish properties of $\otimes_{\max}$, one manipulates commuting pairs of representations and invokes the universal property.

In Section B.4, we prove the main theorem about $A \otimes_\sigma B$ needed for our work on the Brauer group, which says that for C^*-algebras A and B with Hausdorff spectrum, there is a natural homeomorphism of $\hat{A} \times \hat{B}$ onto the spectrum of $A \otimes_\sigma B$. In the last section, we discuss extensions of this result to more general classes of C^*-algebras. We have included this material because these theorems are hard to track through the literature, but we have not used them elsewhere in the book. The most satisfactory generalizations are for the classes of GCR (= Type I) and nuclear C^*-algebras.

B.1 The Spatial Tensor Product

We begin by showing that $A \odot B$ is a $*$-algebra.

Lemma B.1. *If A and B are $*$-algebras, then there is a unique $*$-algebra structure on the vector-space tensor product $A \odot B$ such that*

$$(a \otimes b)(c \otimes b) = ac \otimes bd \quad \text{and} \quad (a \otimes b)^* = a^* \otimes b^*$$

for all $a, c \in A$ and $b, d \in B$.

Proof. Fix $a \in A$ and $b \in B$. Then $(c, d) \mapsto ac \otimes bd$ is bilinear from $A \times B$ into $A \odot B$, and determines a linear operator $L_a \otimes L_b$ on $A \odot B$. The map $(a, b) \mapsto L_a \otimes L_b$ is bilinear from $A \odot B$ into the vector space $L(A \odot B)$ of linear operators on $A \odot B$, and hence induces a linear map $M : A \odot B \to L(A \odot B)$. The required product of $s, t \in A \odot B$ is given by $st := M(s)(t)$.

Let $(A \odot B)^\sim$ be the conjugate of the vector space $A \odot B$. Then $(a, b) \mapsto a^* \otimes b^*$ is a bilinear map from $A \times B$ to $(A \odot B)^\sim$. Thus there is a linear map from $A \odot B$ to $(A \odot B)^\sim$ with the right property. The product and adjoint are uniquely determined by their values on elementary tensors because the elementary tensors span $A \odot B$. □

A natural strategy for endowing $A \odot B$ with a C^*-norm is to embed it as a $*$-subalgebra of some $B(\mathcal{H})$: the norm of an element in $A \odot B$ will then be the operator norm of the associated bounded operator. The key lemma shows how to construct an operator $S \hat{\otimes} T$ on a tensor product $\mathcal{H} \otimes \mathcal{K}$ from an operator S on $\mathcal{H}$ and an operator T on $\mathcal{K}$; we use the notation $S \hat{\otimes} T$ to distinguish this operator from the element $S \otimes T$ of the tensor product $B(\mathcal{H}) \odot B(\mathcal{K})$, though we will shortly show that this distinction is unnecessary.

Lemma B.2. *Suppose that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, $S \in B(\mathcal{H})$, and $T \in B(\mathcal{K})$. Then there is a unique bounded operator $S \hat{\otimes} T$ on $\mathcal{H} \otimes \mathcal{K}$ such that*

$$S \hat{\otimes} T(h \otimes k) = Sh \otimes Tk \quad \text{for } h \in \mathcal{H} \text{ and } k \in \mathcal{K}, \tag{B.1}$$

and $\|S \hat{\otimes} T\| = \|S\| \, \|T\|$.

Proof. The map $(h, k) \mapsto Sh \otimes Tk$ is bilinear, so there is a unique linear operator $S \hat{\otimes} T$ on $\mathcal{H} \odot \mathcal{K}$ satisfying (B.1); the issue is to show that $S \hat{\otimes} T$ is bounded with norm $\|S\| \, \|T\|$, and hence extends to an operator on $\mathcal{H} \otimes \mathcal{K}$ with the same properties.

We first assume that S and T are unitary, and let $\zeta \in \mathcal{H} \odot \mathcal{K}$. We can write $\zeta = \sum_{i=1}^n h_i \otimes k_i$ with the vectors k_i mutually orthogonal. Since T is unitary, the vectors Tk_i are also mutually orthogonal, and hence so are $Sh_i \otimes Tk_i$. Thus

$$\begin{aligned} \|S \hat{\otimes} T(\zeta)\|^2 &= \Big\| \sum\nolimits_{i=1}^n Sh_i \otimes Tk_i \Big\|^2 = \sum\nolimits_{i=1}^n \|Sh_i \otimes Tk_i\|^2 \\ &= \sum\nolimits_{i=1}^n \|Sh_i\|^2 \|Tk_i\|^2 = \sum\nolimits_{i=1}^n \|h_i\|^2 \|k_i\|^2 \\ &= \|\zeta\|^2, \end{aligned}$$

so $\|S \hat{\otimes} T\| = 1$. Since $B(\mathcal{H})$ and $B(\mathcal{K})$ are spanned by unitaries*, it follows that $S \hat{\otimes} T$ is a bounded operator for all S and T.

The maps $S \mapsto S \hat{\otimes} 1_{\mathcal{K}}$ and $T \mapsto 1_{\mathcal{H}} \hat{\otimes} T$ are injective homomorphisms of $B(\mathcal{H})$ and $B(\mathcal{K})$ into $B(\mathcal{H} \otimes \mathcal{K})$. Since injective homomorphisms between C^*-algebras are isometric, we have $\|S \hat{\otimes} 1_{\mathcal{K}}\| = \|S\|$ and $\|1_{\mathcal{H}} \hat{\otimes} T\| = \|T\|$. Furthermore it is easily checked that $(S \hat{\otimes} 1_{\mathcal{K}})(1_{\mathcal{H}} \hat{\otimes} T) = S \hat{\otimes} T$, and therefore

$$\|S \hat{\otimes} T\| = \|(S \hat{\otimes} 1_{\mathcal{K}})(1_{\mathcal{H}} \hat{\otimes} T)\| \leq \|S\| \, \|T\|.$$

On the other hand, we can approximate $\|S\|$ by $\|Sh\|$ and $\|T\|$ by $\|Tk\|$ for unit vectors h and k; then $h \otimes k$ is also a unit vector, and it follows from (B.1) that $\|S \hat{\otimes} T\| \geq \|S\| \, \|T\|$. □

Lemma B.3. *Suppose $\mathfrak{A} \subset B(\mathcal{H})$ and $\mathfrak{B} \subset B(\mathcal{K})$ are concrete C^*-algebras (i.e., nondegenerate C^*-subalgebras). Then there is an injective $*$-homomorphism ι of $\mathfrak{A} \odot \mathfrak{B}$ into $B(\mathcal{H} \otimes \mathcal{K})$ such that $\iota(S \otimes T) = S \hat{\otimes} T$.*

*It is enough to show that if $T \in B(\mathcal{H})$ and if $0 \leq T \leq 1$, then T is a linear combination of unitaries. But $U = T + i\sqrt{1 - T^2}$ is a unitary and $2T = U + U^*$.

Proof. Since $(S,T) \mapsto S \hat{\otimes} T$ is a bilinear map into $B(\mathcal{H}\otimes\mathcal{K})$, there is a well-defined linear map ι of $\mathfrak{A} \odot \mathfrak{B}$ into $B(\mathcal{H} \otimes \mathcal{K})$, which takes $S \otimes T$ to $S \hat{\otimes} T$ and is easily checked to be a $*$-homomorphism for the $*$-algebra structure of Lemma B.1. So it remains to prove that ι is injective. To see this, suppose $t \in \mathfrak{A}\odot\mathfrak{B}$ satisfies $\iota(t) = 0$, and write $t = \sum_{j=1}^n S_j \otimes T_j$ with $\{T_1, \ldots, T_n\}$ linearly independent. Fix $h \in \mathcal{H}$ and choose an orthonormal basis $\{e_1, \ldots, e_m\}$ for the span of $\{S_1h, \ldots, S_nh\}$ in $\mathcal{H}$. Then there are complex scalars λ_{ij} such that $S_jh = \sum_i \lambda_{ij}e_i$ for each $1 \le j \le n$, and for any $k \in \mathcal{K}$ we have

$$\begin{aligned} 0 &= \sum\nolimits_{j=1}^n S_jh \otimes T_jk = \sum\nolimits_{j=1}^n \sum\nolimits_{i=1}^m \lambda_{ij}e_i \otimes T_jk \\ &= \sum\nolimits_{i=1}^m e_i \otimes \Big(\sum\nolimits_{j=1}^n \lambda_{ij}T_jk\Big). \end{aligned}$$

Since the vectors e_i are independent, this implies $\sum_j \lambda_{ij}T_j = 0$ for each $1 \le i \le m$. Now the independence of the T_j implies that $\lambda_{ij} = 0$ for all i, j, and we have $S_jh = 0$ for all j. Since h was arbitrary, this implies $S_j = 0$ for all j, and $t = \sum_j S_j \otimes T_j = 0$. Thus ι is injective. □

This Lemma allows us to identify $\mathfrak{A} \odot \mathfrak{B}$ with a subalgebra of $B(\mathcal{H} \otimes \mathcal{K})$, and we will no longer distinguish between $S \otimes T$ in $\mathfrak{A} \odot \mathfrak{B}$ and $S \hat{\otimes} T$ in $B(\mathcal{H} \otimes \mathcal{K})$. We write $\mathfrak{A} \otimes_c \mathfrak{B}$ for the closure of $\mathfrak{A} \odot \mathfrak{B}$ in $B(\mathcal{H} \otimes \mathcal{K})$.

Remark B.4. At first glance, it would seem that we have solved the problem of finding a tensor product norm for abstract C^*-algebras A and B. There are nondegenerate faithful representations $\pi : A \to B(\mathcal{H}_\pi)$ and $\eta : B \to B(\mathcal{H}_\eta)$, and then $\pi(A)$ and $\eta(B)$ are concrete C^*-algebras isomorphic to A and B, respectively. Since $(a, b) \mapsto \pi(a) \otimes \eta(b)$ is bilinear, there is a unique linear map $\pi \otimes \eta : A \odot B \to B(\mathcal{H}_\pi \otimes \mathcal{H}_\eta)$ characterized by $\pi \otimes \eta(a \otimes b) = \pi(a) \otimes \eta(b)$, which turns out to be a $*$-homomorphism. Since π and η are faithful, $\pi \otimes \eta$ is injective — it is the composition of the vector space isomorphism of $A \odot B$ with $\pi(A) \odot \eta(B)$ and the identification of the latter with operators on $\mathcal{H}_\pi \otimes \mathcal{H}_\eta$. We can now define a C^*-norm on $A \odot B$ by $\|t\|_{\pi,\eta} = \|\pi \otimes \eta(t)\|_{B(\mathcal{H}_\pi \otimes \mathcal{H}_\eta)}$. The completion of $A \odot B$ in this norm is (isomorphic to) $\pi(A) \otimes_c \eta(B)$. However, we need to prove that $\|t\|_{\pi,\eta}$ is independent of our choice of representations π and η. We do this by computing $\|\pi \otimes \eta(t)\|_{B(\mathcal{H}_\pi \otimes \mathcal{H}_\eta)}$ in terms of positive linear functionals on A and B (Theorem B.9). For this computation, we need to look at positive linear functionals on concrete C^*-algebras.

If $\mathfrak{A} \subset B(\mathcal{H})$ is a concrete C^*-algebra, a *vector state* on $\mathfrak{A}$ is a state $\rho \in S(\mathfrak{A})$ of the form $\rho : S \mapsto (Sh \mid h)$ for some unit vector $h \in \mathcal{H}$. Although $\mathfrak{A}$ will have lots of states which aren't vector states, for many purposes it suffices to consider vector states:

Proposition B.5. *If $\mathfrak{A} \subset B(\mathcal{H})$ is a nondegenerate C^*-subalgebra, then every state ρ on $\mathfrak{A}$ is in the weak-$*$ closed convex hull of the vector states of $\mathfrak{A}$.*

The key lemma says that the norm of a self-adjoint element of $\mathfrak{A}$ is determined by its values on vector states. This lemma will also be used in Section B.5.

Lemma B.6. *Suppose that a is a self-adjoint element of a C^*-algebra A. Then*

$$\|a\| = \sup\{\, |\rho(a)| : \rho \in S(A)\,\}.$$

Moreover, if $C \subset S(A)$ and D is the weak-$*$ closed convex hull of C, then $S(A) \subset D$ if and only if

$$\|a\| = \sup\{\, |\rho(a)| : \rho \in C\,\}$$

for every self-adjoint element a of A.

Proof. The first assertion is part of Lemma A.10. For the second, let $s(a) = \sup\{\, |\rho(a)| : \rho \in C\,\}$; we want to show that $s(a) = \|a\|$ if and only if $S(A) \subset D$. However,

$$\begin{aligned} s(a) &\le \sup\{\, |\rho(a)| : \rho \in D\,\} = \sup\{\, |\rho(a)| : \rho \in \operatorname{conv}(C)\,\} \\ &= \sup\left\{ \left| \textstyle\sum_i \lambda_i \rho_i(a) \right| : \rho_i \in C,\ \lambda_i > 0, \text{ and } \textstyle\sum \lambda_i = 1 \right\} \\ &\le \sup\left\{ \textstyle\sum \lambda_i s(a) : \lambda_i > 0 \text{ and } \sum_i \lambda_i = 1 \right\} = s(a). \end{aligned}$$

Therefore if $S(A) \subset D$, then $s(a) \ge \sup\{\, |\rho(a)| : \rho \in S(A)\,\} = \|a\|$.

To prove the other implication, suppose that $\rho_0 \in S(A) \setminus D$. Then there is a convex weak-$*$ open neighbourhood U of ρ_0 disjoint from D. By Corollary A.40, there are $\alpha \in \mathbb{R}$ and $a \in A$ such that $\operatorname{Re}\rho(a) \le \alpha < \operatorname{Re}\rho_0(a)$ for all $\rho \in D$. Now replace a by $a_0 = (a + a^*)/2$. Since every positive linear functional satisfies $\rho(a^*) = \overline{\rho(a)}$, we have $\rho(a_0) \le \alpha < \rho_0(a_0)$ for all $\rho \in D$. Thus $s(a_0) \le \alpha < \rho_0(a_0) \le \|a_0\|$. □

Proof of Proposition B.5. Let C be the collection of vector states of $\mathfrak{A}$. If T is in $B(\mathcal{H})$, define the numerical range of T to be

$$\operatorname{nr}(T) = \sup\{\, |(Th \mid h)| : h \in \mathcal{H}, \|h\| = 1\,\}.$$

If T is positive, $\operatorname{nr}(T) = \|T\|$ (since $T = S^*S$ for some $S \in B(\mathcal{H})$). If $T = T^*$, then it follows from the functional calculus that either $T + \|T\|1$ or $-T + \|T\|1$ is positive with norm $2\|T\|$; suppose $T + \|T\|1 \ge 0$. Then

$$\begin{aligned} 2\|T\| &= \sup\{\, \big((T + \|T\|1)h \mid h\big) : \|h\| = 1\,\} \\ &= \sup\{\, \big(Th \mid h\big) : \|h\| = 1\,\} + \|T\|. \end{aligned}$$

It follows that for all self-adjoint operators T,

$$\|T\| = \operatorname{nr}(T) = \sup\{\, \rho(T) : T \in C\,\}.$$

The result now follows from Lemma B.6. □

Proposition B.7. *Suppose that $\mathfrak{A} \subset B(\mathcal{H})$ and $\mathfrak{B} \subset B(\mathcal{K})$ are concrete C^*-algebras, that $\rho \in S(\mathfrak{A})$, and that $\tau \in S(\mathfrak{B})$. Then there is a unique state $\rho \otimes \tau$ on $\mathfrak{A} \otimes_c \mathfrak{B}$ such that*

$$\rho \otimes \tau(S \otimes T) = \rho(S)\tau(T) \quad \text{for all } S \in \mathfrak{A} \text{ and } T \in \mathfrak{B}. \tag{B.2}$$

A state of this form is called a product state *on $\mathfrak{A} \otimes_c \mathfrak{B}$.*

Proof. Since (B.2) determines $\rho \otimes \tau$ on $\mathfrak{A} \odot \mathfrak{B}$, uniqueness is clear. We claim that it will suffice to produce a positive functional f such that $f(S \otimes T) = \rho(S)\tau(T)$ and $\|f\| \leq 1$. To see this, just choose S and T such that $\|S\| = 1 = \|T\|$, $\rho(S) \sim 1$ and $\tau(T) \sim 1$. Then $\|S \otimes T\| = 1$ and $f(S \otimes T) = \rho(S)\tau(T) \sim 1$, so f has norm 1 and is a state.

Suppose first that both ρ and τ are vector states, so there are unit vectors $h \in \mathcal{H}$ and $k \in \mathcal{K}$ such that $\rho(S) = (Sh \mid h)$ and $\tau(T) = (Tk \mid k)$. The unit vector $h \otimes k$ determines a vector state $\omega : T \mapsto \big(T(h \otimes k) \mid h \otimes k\big)$ on $B(\mathcal{H} \otimes \mathcal{K})$, and we can take for $\rho \otimes \tau$ the restriction of ω to $\mathfrak{A} \otimes_c \mathfrak{B}$. If $\rho = \sum_i \lambda_i \rho_i$ and $\tau = \sum_i \mu_i \tau_i$ are convex combinations of vector states, then $f = \sum_{ij} \lambda_i \mu_j \rho_i \otimes \tau_j$ is a state on $\mathfrak{A} \otimes_c \mathfrak{B}$ with the required property.

For the general case, let C and D be the collections of convex combinations of vector states on $\mathfrak{A}$ and $\mathfrak{B}$, respectively. By Proposition B.5, C and D are weak-$*$ dense in $S(\mathfrak{A})$ and $S(\mathfrak{B})$, respectively, and there is a net $\{ (\rho_\lambda, \tau_\lambda) \}_{\lambda \in \Lambda}$ in $C \times D$ converging to (ρ, τ) in the product topology. But $\{ \rho_\lambda \otimes \tau_\lambda \}_{\lambda \in \Lambda}$ is a net in the in the unit ball of $(\mathfrak{A} \otimes_c \mathfrak{B})^*$, which is compact in the weak-$*$ topology. Thus by passing to a subnet and relabelling, we can assume that $\{ \rho_\lambda \otimes \tau_\lambda \}_{\lambda \in \Lambda}$ converges to a functional f of norm at most one, which is positive because each $\rho_\lambda \otimes \tau_\lambda$ is. Since

$$f(S \otimes T) = \lim_\lambda \rho_\lambda \otimes \tau_\lambda (S \otimes T) = \lim_\lambda \rho_\lambda(S)\tau_\lambda(T) = \rho(S)\tau(T),$$

this f will suffice. □

Suppose $\mathfrak{A} \subset B(\mathcal{H})$ is a concrete C^*-algebra and that h is a unit vector in $\mathcal{H}$. We will write $[\mathfrak{A}h]$ for the projection onto the subspace $\overline{\mathfrak{A}h}$; we call $[\mathfrak{A}h]$ a *cyclic projection* and h a *cyclic vector for* $[\mathfrak{A}h]$. Note that since $\overline{\mathfrak{A}h}$ is invariant for $\mathfrak{A}$, any cyclic projection P is in the commutant $\mathfrak{A}' := \{ T \in B(\mathcal{H}) : TS = ST \text{ for all } S \in \mathfrak{A} \}$.

By a Zorn's lemma argument using the nondegeneracy of $\mathfrak{A}$, there is a family$\{ P_i \}_{i \in I}$ of orthogonal cyclic projections such that $\sum_i P_i = 1_\mathcal{H}$ (with convergence in the strong operator topology); we call it a *maximal family of cyclic projections for* $\mathfrak{A}$. Since $\|h\|^2 = \sum_i \|P_i h\|^2$ for all $h \in \mathcal{H}$, and since each P_i belongs to the commutant $\mathfrak{A}'$, we have

$$\|T\| = \sup_i \|TP_i\| \qquad \text{for all } T \in \mathfrak{A}. \tag{B.3}$$

Proposition B.8. *Suppose that $\mathfrak{A} \subset B(\mathcal{H})$ and $\mathfrak{B} \subset B(\mathcal{K})$ are concrete C^*-algebras and that $T \in \mathfrak{A} \odot \mathfrak{B}$. Then as an operator on $\mathcal{H} \otimes \mathcal{K}$, we have*

$$\|T\|^2 = \sup\{ \rho \otimes \tau(S^*T^*TS)/\rho \otimes \tau(S^*S) : \rho \in S(\mathfrak{A}),\ \tau \in S(\mathfrak{B}) \\ \text{and } S \in \mathfrak{A} \odot \mathfrak{B} \text{ satisfy } \rho \otimes \tau(S^*S) \neq 0 \}. \tag{B.4}$$

Proof. If $\phi \in S(\mathfrak{A} \otimes_c \mathfrak{B})$ and $S \in \mathfrak{A} \otimes_c \mathfrak{B}$ satisfies $\phi(S^*S) \neq 0$, then the equation $\mu_\phi(T) := \phi(S^*TS)/\phi(S^*S)$ defines a positive linear functional. It is a state since $\lim_\lambda \mu_\phi(U_\lambda) = 1$ for any approximate identity $\{ U_\lambda \}$ for $\mathfrak{A} \otimes_c \mathfrak{B}$ [111, Theorems

3.3.1 and 3.3.3][†]. Therefore the right-hand side of (B.8) is dominated by $\|T^*T\| = \|T\|^2$.

Now let $\{ P_i \}$ be any maximal family of cyclic projections for $\mathfrak{A} \otimes_c \mathfrak{B}$ with (unit) cyclic vectors z_i for each P_i. If $T \in \mathfrak{A} \odot \mathfrak{B}$ (or $\mathfrak{A} \otimes_c \mathfrak{B}$), then

$$\begin{aligned}
\|T\|^2 &= \sup\{ \|TP_i\|^2 : i \in I \} \\
&= \sup\{ \|TSz_i\|^2 : i \in I,\ S \in \mathfrak{A} \odot \mathfrak{B},\ \|Sz_i\| \leq 1 \} \\
&= \sup\{ \|TSz_i\|^2 / \|Sz_i\|^2 : i \in I,\ S \in \mathfrak{A} \odot \mathfrak{B},\ Sz_i \neq 0 \} \\
&= \sup\{ \omega_i(S^*T^*TS)/\omega_i(S^*S) : i \in I,\ S \in \mathfrak{A} \odot \mathfrak{B},\ \omega_i(S^*S) \neq 0 \},
\end{aligned}$$

where ω_i is the vector state $A \mapsto (Az_i \mid z_i)$. Thus it is enough to show that we can choose the P_i and z_i so that each ω_i is a product state. It is certainly enough to arrange that each z_i is an elementary tensor.

Choose unit vectors $\{ h_\lambda \}_{\lambda \in \Lambda}$ in $\mathcal{H}$ and $\{ k_\mu \}_{\mu \in M}$ in $\mathcal{K}$ such that $1_\mathcal{H} = \sum_\lambda [\mathfrak{A}h_\lambda]$ and $1_\mathcal{K} = \sum_\mu [\mathfrak{B}k_\mu]$. Put $N = \Lambda \times M$, $z_{(\lambda,\mu)} = h_\lambda \otimes k_\mu$, and for each $\nu \in N$, let $P_\nu = [(\mathfrak{A} \otimes_c \mathfrak{B})z_\nu]$. Then $\{ P_\nu \}_{\nu \in N}$ is an orthogonal family of cyclic projections, and the range of the projections $\sum_\nu P_\nu$ contains all elementary tensors $h \otimes k$ with $h \in \bigcup_\lambda \mathfrak{A}h_\lambda$ and $k \in \bigcup_\mu \mathfrak{B}k_\mu$. Since these tensors span a dense subspace, $\sum_\nu P_\nu = 1_{\mathcal{H} \otimes \mathcal{K}}$ and we are done. □

Now suppose that A and B are (abstract) C^*-algebras. If $\rho \in S(A)$ and $\tau \in S(B)$, then $(a, b) \mapsto \rho(a)\tau(b)$ is bilinear, and the universal property of tensor products gives a well-defined linear functional $\rho \odot \tau$ on $A \odot B$ satisfying $\rho \odot \tau(a \otimes b) = \rho(a)\tau(b)$.

Theorem B.9. *Suppose that A and B are C^*-algebras.*

(a) *If $t \in A \odot B$, $\rho \in S(A)$, and $\tau \in S(B)$, then $\rho \odot \tau(t^*t) \geq 0$.*

(b) *For each $t \in A \odot B$, define*

$$\|t\|_\sigma^2 := \sup\{ \rho \odot \tau(s^*t^*ts)/\rho \odot \tau(s^*s) : \rho \in S(A),\ \tau \in S(B) \text{ and } s \in A \odot B \text{ satisfy } \rho \odot \tau(s^*s) \neq 0 \}.$$

Then $\| \cdot \|_\sigma$ is a C^*-norm on $A \odot B$ satisfying $\|a \otimes b\|_\sigma = \|a\| \, \|b\|$.

(c) *If $\pi : A \to B(\mathcal{H}_\pi)$ and $\eta : B \to B(\mathcal{H}_\eta)$ are nondegenerate representations, then there is a unique $*$-homomorphism $\pi \otimes \eta : A \odot B \to B(\mathcal{H}_\pi \otimes \mathcal{H}_\eta)$ such that $\pi \otimes \eta(a \otimes b) = \pi(a) \otimes \eta(b)$. Moreover, for every $t \in A \odot B$ we have*

$$\|\pi \otimes \eta(t)\| \leq \|t\|_\sigma.$$

[†]Alternatively, the Cauchy-Schwarz inequality for positive linear functionals says that if $\omega \in S(A)$ and if $a, b \in A$, then $|\omega(a^*b)| \leq \omega(a^*a)^{1/2}\omega(b^*b)^{1/2}$. Thus

$$\phi(S^*TS) \leq \phi(S^*S)^{1/2}\phi(S^*T^*TS)^{1/2} \leq \|T\|\phi(S^*S),$$

where the last inequality follows from $S^*T^*TS \leq \|T\|^2 S^*S$. Therefore $\|\phi\| \leq 1$. The approximate identity argument shows that $\|\phi\| = 1$.

(d) *If π and η are faithful representations, then equality holds in* (c). *That is,*

$$\left\|\sum_{i=1}^{k} a_i \otimes b_i\right\|_\sigma = \left\|\sum_{i=1}^{k} \pi(a_i) \otimes \eta(b_i)\right\|_{B(H_\pi \otimes H_\eta)}$$

for all $a_i \in A$ and $b_i \in B$.

(e) *If $\rho \in S(A)$ and $\tau \in S(B)$, then $\rho \odot \tau$ is σ-continuous; that is, $|\rho \odot \tau(t)| \leq \|t\|_\sigma$ for all $t \in A \odot B$.*

Proof. Suppose that π and η are faithful nondegenerate representations. Since $(a, b) \mapsto \pi(a) \otimes \eta(b)$ is bilinear, there is a linear map $\pi \otimes \eta : A \odot B \to B(\mathcal{H}_\pi \otimes \mathcal{H}_\eta)$ characterized by $\pi \otimes \eta(a \otimes b) = \pi(a) \otimes \eta(b)$, which is a $*$-homomorphism and is faithful (Remark B.4). The states of A are exactly those of the form $\rho \circ \pi$ for states ρ of $\pi(A)$, and those of B have the form $\tau \circ \eta$ for states τ of $\eta(B)$. Notice that $(\rho \circ \pi) \odot (\tau \circ \eta)(t) = \rho \otimes \tau(\pi \otimes \eta(t))$. Since $\pi \otimes \eta$ is a $*$-homomorphism, $\pi \otimes \eta(t^*t)$ is a positive element of $\pi(A) \otimes_c \eta(B)$, and part (a) follows from Proposition B.7. Proposition B.8 implies that $\|t\|_\sigma$ is the operator norm of $\pi \otimes \eta(t)$; (d) follows immediately, and the equality at the end of (b) follows from Lemma B.2. Part (e) follows because Proposition B.7 implies that product states are bounded with respect to the operator norm.

On the other hand, even if π and η are not faithful, $\rho \circ \pi$ and $\tau \circ \eta$ are still states whenever $\rho \in S(\pi(A))$ and $\tau \in S(\eta(B))$, though not every state of A and B need arise this way. Thus (c) follows from (b). □

Definition B.10. If A and B are C^*-algebras, then the norm $\|\cdot\|_\sigma$ defined in Theorem B.9 is called the *spatial norm on $A \odot B$*. The completion of $A \odot B$ in $\|\cdot\|_\sigma$ is denoted by $A \otimes_\sigma B$, and is called the *spatial tensor product.*

Corollary B.11. *Suppose A and B are C^*-algebras and $\pi : A \to B(\mathcal{H}_\pi)$ and $\eta : B \to B(\mathcal{H}_\eta)$ are representations. Then there is a unique representation $\pi \otimes \eta : A \otimes_\sigma B \to B(\mathcal{H}_\pi \otimes \mathcal{H}_\eta)$ satisfying $\pi \otimes \eta(a \otimes b) = \pi(a) \otimes \eta(b)$. If π and η are faithful, then so is $\pi \otimes \eta$.*

Proof. When π and η are nondegenerate, this is part (d) of the Theorem. Any representation is nondegenerate on its essential subspace, so we can decompose $\pi = \pi_1 \oplus 0$ and $\eta = \eta_1 \oplus 0$ where π_1 and η_1 are nondegenerate. The tensor product $\mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\eta_1}$ of the essential subspaces naturally embeds as a closed subspace of $\mathcal{H}_\pi \otimes \mathcal{H}_\eta$, and then $\pi(a) \otimes \eta(b) = \big(\pi_1(a) \otimes \eta_1(b)\big) \oplus 0$. So we can define $\pi \otimes \eta := (\pi_1 \otimes \eta_1) \oplus 0$, and the result follows. □

Corollary B.12. *If A and B are C^*-algebras, $\tau \in S(A)$, and $\rho \in S(B)$, then $\rho \odot \tau$ extends to a state $\rho \otimes_\sigma \tau$ on $A \otimes_\sigma B$.*

Proof. This follows immediately from part (a) of the Theorem. □

Proposition B.13. *Suppose that $\phi : A \to C$ and $\psi : B \to D$ are homomorphisms between C^*-algebras. Then there is a unique homomorphism $\phi \otimes \psi : A \otimes_\sigma B \to C \otimes_\sigma D$ such that $\phi \otimes \psi(a \otimes b) = \phi(a) \otimes \psi(b)$ for all $a \in A$ and $b \in B$. If ϕ and ψ are injective, then so is $\phi \otimes \psi$.*

Proof. As usual, the universal properties of the algebraic tensor product imply that there is a unique $*$-homomorphism $\phi \otimes \psi$ on $A \odot B$ satisfying $\phi \otimes \psi(a \otimes b) = \phi(a) \otimes \psi(b)$. Suppose that $\pi : C \to B(\mathcal{H}_\pi)$ and $\eta : D \to B(\mathcal{H}_\eta)$ are faithful nondegenerate representations. Then $\pi \otimes \eta$ is isometric with respect to the spatial norm on $C \odot D$, and $(\pi \otimes \eta) \circ (\phi \otimes \psi) = (\pi \circ \phi) \otimes (\eta \circ \psi)$ is a representation of $C \odot D$ which is bounded for the spatial norm (Corollary B.11). It follows that $\phi \otimes \psi$ is bounded with respect to the spatial norm on $A \odot B$ and extends to a homomorphism of $A \otimes_\sigma B$, as asserted. If ϕ and ψ are faithful, then $\pi \circ \phi$ and $\eta \circ \psi$ are faithful representations, so the final assertion follows also from Corollary B.11. □

Corollary B.14. *Suppose B and C are C^*-algebras, and A is a C^*-subalgebra of B. Then the inclusion i of A in B extends to an embedding $i \otimes \mathrm{id}$ of $A \otimes_\sigma C$ into $B \otimes_\sigma C$.*

Proof. We just apply the Proposition to $i : A \to B$ and the identity homomorphism $\mathrm{id} : C \to C$; since both are injective, so is $i \otimes \mathrm{id}$. □

In Section 6.1, we use the spatial tensor product to define an abelian group operation on a set of C^*-algebras, and we will have to prove that this is commutative and associative.

Proposition B.15. *Suppose that A, B and C are C^*-algebras.*

(a) *There is an isomorphism γ of $A \otimes_\sigma B$ onto $B \otimes_\sigma A$ such that $\gamma(a \otimes b) = b \otimes a$; γ is sometimes called the* flip isomorphism.

(b) *There is an isomorphism α of $(A \otimes_\sigma B) \otimes_\sigma C$ onto $A \otimes_\sigma (B \otimes_\sigma C)$ such that $\alpha((a \otimes b) \otimes c) = a \otimes (b \otimes c)$.*

The key is to realize that there are similar isomorphisms at the vector space and Hilbert space levels. So first let V, W, and Z be vector spaces. The map $(v, w) \mapsto w \otimes v$ is bilinear, and hence there is a linear map $\gamma : V \odot W \to W \odot V$ such that $\gamma(v \otimes w) = w \otimes v$; doing the same thing to $(w, v) \mapsto v \otimes w$ gives an inverse for γ. In the same way, for fixed z, the map $(v, w) \mapsto v \otimes (w \otimes z)$ is bilinear, and thus there is a linear map $L_z : V \odot W \to V \odot (W \odot Z)$ such that $L_z(v \otimes w) = v \otimes (w \otimes z)$. Now $(t, z) \mapsto L_z(t)$ is bilinear on $(V \odot W) \times Z$, and there is a linear map $\alpha : (V \odot W) \odot Z \to V \odot (W \odot Z)$ such that $\alpha\big((v \otimes w) \otimes z)\big) = v \otimes (w \otimes z)$. The same process gives a linear map $\beta : V \odot (W \odot Z) \to (V \odot W) \odot Z$ which is an inverse for α, so α is an isomorphism. If now $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{W}$ are Hilbert spaces, it is easy to check that the isomorphisms of $\mathcal{H} \odot \mathcal{K}$ onto $\mathcal{K} \odot \mathcal{H}$ and $(\mathcal{H} \odot \mathcal{K}) \odot \mathcal{W}$ onto $\mathcal{H} \odot (\mathcal{K} \odot \mathcal{W})$ preserve the inner products, and hence extend to unitary operators $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}$ and $V : (\mathcal{H} \otimes \mathcal{K}) \otimes \mathcal{W} \to \mathcal{H} \otimes (\mathcal{K} \otimes \mathcal{W})$.

To see that the vector-space isomorphisms $\gamma : A \odot B \to B \odot A$ and $\alpha : (A \odot B) \odot C \to A \odot (B \odot C)$ extend to C^*-algebra isomorphisms, choose faithful representations $\pi : A \to B(\mathcal{H}_\pi)$, $\eta : B \to B(\mathcal{H}_\eta)$, and $\zeta : C \to B(\mathcal{H}_\zeta)$. Then $\pi \otimes \eta$ and $\eta \otimes \pi$ are faithful representations of $A \otimes_\sigma B$ and $B \otimes_\sigma A$, by Corollary B.11. If $U : \mathcal{H}_\pi \otimes \mathcal{H}_\eta \to \mathcal{H}_\eta \otimes \mathcal{H}_\pi$ is the flip isomorphism on Hilbert spaces, then

$$\eta \otimes \pi(\gamma(t)) = U\pi \otimes \eta(t)U^* \tag{B.5}$$

for all $t \in A \odot B$; to check this, write $t = \sum a_i \otimes b_i$ and apply both sides to an elementary tensor in $\mathcal{H}_\eta \odot \mathcal{H}_\pi$. Since the right-hand side of (B.5) is an isometric $*$-homomorphism in t and $\eta \otimes \pi$ is isometric, (B.5) implies both that γ is a $*$-homomorphism and that γ is isometric. Thus γ extends to a C^*-algebra isomorphism of $A \otimes_\sigma B$ onto $B \otimes_\sigma A$, proving part (a) of Proposition B.15. We can similarly verify that

$$\pi \otimes (\eta \otimes \zeta)(\alpha(t)) = V\big((\pi \otimes \eta) \otimes \zeta\big)(t)V^*$$

for all $t \in (A \odot B) \odot C$, and deduce that α extends to a C^*-algebra isomorphism of the spatial tensor products. This completes the proof of Proposition B.15.

B.2 Fundamental Examples

There are some basic examples where the tensor product of two C^*-algebras takes a familiar form, and these examples provide motivation and insight for the general theory. When one algebra is commutative, for example, we can identify the tensor product with an algebra of vector-valued functions. If T is a locally compact Hausdorff space and A is a C^*-algebra, then the set $C_0(T, A)$ of continuous functions $f : T \to A$ such that $t \mapsto \|f(t)\|$ vanishes at infinity is a C^*-algebra with pointwise operations and the sup-norm:

$$(fg)(t) := f(t)g(t), \quad f^*(t) := f(t)^*, \quad \text{and} \quad \|f\| := \sup \|f(t)\|$$

(the completeness is established as in Example 1.7).

Proposition B.16. *Let T be a locally compact Hausdorff space and A a C^*-algebra. Then there is an isomorphism ϕ of $C_0(T) \otimes_\sigma A$ onto $C_0(T, A)$ such that $\phi(f \otimes a)(t) = f(t)a$ for $a \in A$ and $f \in C_0(T)$.*

Proof. The map sending (f, a) to the function $t \mapsto f(t)a$ is bilinear, and so there is a linear map $\phi : C_0(T) \odot A \to C_0(T, A)$ with the required property, which turns out to be a $*$-homomorphism. A partition of unity argument like that of Lemma 5.60 shows that ϕ has dense range. Therefore it will suffice to show that ϕ is isometric.

Fix a faithful representation $\pi : A \to B(\mathcal{H}_\pi)$, and let $M : C_0(T) \to B\big(\ell^2(T)\big)$ be the usual (faithful) representation of $C_0(T)$ by multiplication operators: $M_f(g)(t) = f(t)g(t)$ for $f \in C_0(T)$ and $g \in \ell^2(T)$. Corollary B.11 implies that $M \otimes \pi$ is a faithful representation of $C_0(T) \otimes_\sigma A$, and hence is isometric. There is also a faithful representation R of $C_0(T, A)$ on $\ell^2(T, \mathcal{H}_\pi)$ by multiplication: $R(F)(\xi)(t) = \pi\big(F(t)\big)\xi(t)$ for $F \in C_0(T, A)$ and $\xi \in \ell^2(T, \mathcal{H}_\pi)$. Let U be the isomomorphism of $\ell^2(T) \otimes \mathcal{H}_\pi$ onto $\ell^2(T, \mathcal{H}_\pi)$ satisfying $U(g \otimes h)(t) = g(t)h$ (see Example 2.61). Then a straightforward calculation shows that

$$R(\phi(f \otimes a))U(g \otimes h) = U\big(M \otimes \pi(f \otimes a)\big)(g \otimes h),$$

and since $M \otimes \pi$, R and $\operatorname{Ad} U$ are isometric, so is ϕ. □

Corollary B.17. *If S and T are locally compact Hausdorff spaces, then there is an isomorphism ψ of $C_0(S) \otimes_\sigma C_0(T)$ onto $C_0(S \times T)$ such that $\phi(f \otimes g)(s,t) = f(s)g(t)$ for $f \in C_0(S)$, $g \in C_0(T)$.*

Proof. In view of Proposition B.16, it suffices to prove that the map sending $F \in C_0(S \times T)$ to the function $s \mapsto F(s, \cdot)$ is an isomorphism of $C_0(S \times T)$ onto $C_0\big(S, C_0(T)\big)$. We like to think this is a routine exercise in general topology. □

Our next examples involve the algebra $M_n := M_n(\mathbb{C})$ of $n \times n$-matrices with entries in $\mathbb{C}$. Let $e_{ij} \in M_n$ be the *matrix unit* with 1 in the $(i,j)^{\text{th}}$ slot and zeros elsewhere, so that $\{ e_{ij} \}_{i,j=1}^n$ is a vector-space basis for M_n: for any $(\lambda_{ij}) \in M_n$, we have $(\lambda_{ij}) = \sum_{ij} \lambda_{ij} e_{ij}$. The multiplication on M_n is then determined by

$$e_{ij} e_{k\ell} = \begin{cases} e_{i\ell} & \text{if } j = k \\ 0 & \text{if } j \neq k, \end{cases}$$

and the adjoint by $e_{ij}^* = e_{ji}$.

In the discussion preceding Lemma 2.65, we showed how the $*$-algebra $M_n(A)$ of $n \times n$-matrices with entries from a C^*-algebra A can be made into a C^*-algebra (in a necessarily unique way: there is only one complete C^*-norm on any $*$-algebra). Every elementary tensor in $M_n \odot A$ can be written as $(\lambda_{ij}) \otimes a = \sum \lambda_{ij} e_{ij} \otimes a = \sum e_{ij} \otimes \lambda_{ij} a$, so the map $(a_{ij}) \mapsto \sum e_{ij} \otimes a_{ij}$ is a vector-space isomorphism of $M_n(A)$ onto $M_n \odot A$; it is easy to check using the formula for multiplying matrix units that it is actually an isomorphism of $*$-algebras. Pulling the norm over from $M_n(A)$ gives a C^*-norm on $M_n \odot A$ in which it is already complete. In fact this is the only C^*-norm on $M_n \odot A$:

Proposition B.18. *If A is a C^*-algebra, then $M_n \odot A$ has a unique C^*-norm. It is complete in this norm, so we can safely write $M_n \otimes A$ for $M_n \odot A$, and the map $(a_{ij}) \mapsto \sum e_{ij} \otimes a_{ij}$ is an isomorphism of the C^*-algebra $M_n(A)$ onto $M_n \otimes A$.*

Proof. Let $\| \cdot \|$ denote the norm on $M_n \odot A$ pulled over from $M_n(A)$, and suppose $\| \cdot \|_\gamma$ is another C^*-norm on $A \odot M_n$ with corresponding completion $M_n \otimes_\gamma A$. Then the inclusion of $(M_n \odot A, \| \cdot \|)$ in $M_n \otimes_\gamma A$ is an injection of C^*-algebras, and is therefore isometric — in other words, $\| \cdot \|_\gamma = \| \cdot \|$ on $M_n \odot A$. The rest we proved above. □

Example B.19. If m and n are positive integers, we obtain a $*$-isomorphism of $M_m \otimes M_n$ with M_{mn} by mapping an element of $M_m \otimes M_n$ to an $m \times m$ block matrix in $M_m(M_n)$ and viewing this as an $mn \times mn$-matrix. More formally, let $\{ e_{ij} \}$ be a set of matrix units for M_m, $\{ f_{kl} \}$ for M_n, and $\{ g_{rs} \}$ for M_{mn}. Then we define a linear isomorphism by sending the basis element $e_{ij} \otimes f_{kl}$ for $M_m \otimes M_n$ to $g_{(i-1)m+k,(j-1)m+l}$, and verify by computation that this is actually a $*$-isomorphism.

Since $M_n \cong B(\mathbb{C}^n)$, the previous example says that $B(\mathbb{C}^m) \otimes B(\mathbb{C}^n) \cong B(\mathbb{C}^{mn})$. For a finite-dimensional Hilbert space we have $B(\mathcal{H}) = \mathcal{K}(\mathcal{H})$, so we can also view this as the finite-dimensional case of the next example.

Example B.20. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, then applying Corollary B.11 to the identity representations of $\mathcal{K}(\mathcal{H}_i)$ on $\mathcal{H}_i$ gives a faithful representation π of $\mathcal{K}(\mathcal{H}_1) \otimes_\sigma \mathcal{K}(\mathcal{H}_2)$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfying

$$\pi\big((h_1 \otimes \overline{k_1}) \otimes (h_2 \otimes \overline{k_2})\big) = (h_1 \otimes h_2) \otimes \overline{(k_1 \otimes k_2)}.$$

Since the various rank-one operators span the various algebras of compacts, it follows that π is an isomorphism of $\mathcal{K}(\mathcal{H}_1) \otimes_\sigma \mathcal{K}(\mathcal{H}_2)$ onto $\mathcal{K}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

B.3 Other C^*-Norms

Since the spatial norm on $A \odot B$ is defined using the tensor products of operators on Hilbert space, it is convenient when A and B have natural representations on Hilbert space. Unfortunately, the spatial tensor product $A \otimes_\sigma B$ does not have a universal property like the one which characterizes the algebraic tensor product of vector spaces. In this section we introduce another norm, called the maximal norm (Proposition B.25), for which the associated C^*-tensor product is universal for commuting representations of A and B (Theorem B.27).

One should immediately wonder whether the spatial and maximal norms are really the same — and if not, how many other norms there are. We insist that all norms $\|\cdot\|_\gamma$ on $A \odot B$ are *submultiplicative*, in the sense that $\|ts\|_\gamma \le \|t\|_\gamma \|s\|_\gamma$ for $s, t \in A \odot B$; this will ensure that the completion $A \otimes_\gamma B$ is a Banach algebra. To ensure that $A \otimes_\gamma B$ is a C^*-algebra, we also require that $\|\cdot\|_\gamma$ satisfies the C^*-norm identity $\|t^*t\|_\gamma = \|t\|_\gamma^2$, and then we say $\|\cdot\|_\gamma$ is a *C^*-norm*.

These C^*-algebra tensor products $A \otimes_\gamma B$ have some common features. The next proposition says that they are always generated by commuting copies of A and B, and this implies that their representations are always given by pairs of commuting representations of A and B (Corollary B.22). From this point of view, the maximal norm of Proposition B.25 is the most natural: the representations of the associated *maximal tensor product* are in one-to-one correspondence with pairs of commuting representations (Corollary B.28).

Proposition B.21. *Let A and B be C^*-algebras and suppose that $\|\cdot\|_\gamma$ is a C^*-norm on $A \odot B$. Then there are injective homomorphisms $i_A : M(A) \to M(A \otimes_\gamma B)$ and $i_B : M(B) \to M(A \otimes_\gamma B)$ such that*

$$i_A(c)(a \otimes b) = ca \otimes b, \quad i_B(d)(a \otimes b) = a \otimes db, \text{ and}$$
$$i_A(a) i_B(b) = i_B(b) i_A(a) = a \otimes b$$

for $a \in A$, $b \in B$, $c \in M(A)$, and $d \in M(B)$.

Proof. Fix $c \in M(A)$. Since $(a, b) \mapsto ca \otimes b$ is bilinear, there is a unique linear map $L_c : A \odot B \to A \otimes_\gamma B$ such that $L_c(a \otimes b) = ca \otimes b$. We first want to show that for all $t \in A \odot B$,

$$\|L_c(t)\|_\gamma \le \|c\| \, \|t\|_\gamma. \tag{B.6}$$

Let $t = \sum_{i=1}^n a_i \otimes b_i$, and define $X = (x_{ij}) \in M_n(A)$ by $x_{1j} = a_j$ and $x_{ij} = 0$ for $i > 1$. Then since $M_n(A)$ is a C^*-subalgebra of $M_n(M(A))$, we have

$$(a_i^* c^* c a_j) = X^*(c^*c \cdot 1_n)X \le \|c\|^2 X^* X = \|c\|^2 (a_i^* a_j)$$

in $M_n(A)$. Thus there exists $Y = (y_{ij}) \in M_n(A)$ such that the $(i,j)^{\text{th}}$ entry of Y^*Y is

$$\sum_{k=1}^n y_{ki}^* y_{kj} = a_i^* \big(\|c\|^2 - c^*c\big) a_j.$$

Let $Z = (z_{ij}) := (y_{ij} \otimes b_j) \in M_n(A \otimes_\gamma B)$. Then the $(i,j)^{\text{th}}$ entry of Z^*Z is

$$\sum_{k=1}^n z_{ki}^* z_{kj} = \sum_{k=1}^n y_{ki}^* y_{kj} \otimes b_i^* b_j = a_i^* \big(\|c\|^2 - c^*c\big) a_j \otimes b_i^* b_j.$$

Since $Z^*Z \ge 0$ in $M_n(A \otimes_\gamma B)$, the sum of its entries is positive in $A \otimes_\gamma B$. (Suppose $(c_{ij}) \ge 0$ in $M_n(C)$, and represent C faithfully on $\mathcal{H}$, so that $M_n(C)$ acts faithfully on $\mathcal{H}^n$. For each $h \in \mathcal{H}$, let $n \cdot h = (h, \dots, h) \in \mathcal{H}^n$. Then $0 \le \big((c_{ij})(n \cdot h) \mid n \cdot h\big) = \big((\sum_{ij} c_{ij})(h) \mid h\big)$, and $\sum_{ij} c_{ij} \ge 0$ in C.) Therefore

$$\sum_{i,j=1}^n a_i^* \big(\|c\|^2 - c^*c\big) a_j \otimes b_i^* b_j \ge 0$$

in $A \otimes_\gamma B$, and

$$\begin{aligned} \|L_c(t)\|_\gamma^2 = \|L_c(t)^* L_c(t)\|_\gamma &= \Big\| \sum\nolimits_{ij} a_i^* c^* c a_j \otimes b_i^* b_j \Big\|_\gamma \\ &\le \|c\|^2 \Big\| \sum\nolimits_{ij} a_i^* a_j \otimes b_i^* b_j \Big\|_\gamma = \|c\|^2 \|t\|_\gamma^2. \end{aligned}$$

This proves (B.6), and hence that L_c extends to all of $A \otimes_\gamma B$. The similarly defined map L_{c^*} is an adjoint for L_c as a linear map on the right Hilbert $A \otimes_\gamma B$-module $(A \otimes_\gamma B)_{A \otimes_\gamma B}$, so $L_c \in M(A \otimes_\gamma B) := \mathcal{L}\big((A \otimes_\gamma B)_{A \otimes_\gamma B}\big)$, and we can define $i_A(c) = L_c$.

It is easy to see that i_A is a homomorphism. If $i_A(c) = 0$, then $ca \otimes b = 0$ for all $a \in A$ and $b \in B$. By choosing $\tau \in B^*$ and $b \in B$ such that $\tau(b) = 1$, we see that $\rho(ca) = \rho \odot \tau(ca \otimes b) = 0$ for all $\rho \in A^*$; thus $ca = 0$ for all $a \in A$, and $c = 0$. Therefore i_A is injective.

A similar argument shows the existence and injectivity of i_B, and the remaining assertions follow quickly. For example if $a \in A$ and $b \in B$, then $i_A(a) i_B(b)(a' \otimes b') = aa' \otimes bb' = (a \otimes b)(a' \otimes b')$ for all $a' \in A$ and $b' \in B$, so $i_A(a) i_B(b)$ and $a \otimes b$ agree as transformations on $A \otimes_\gamma B$, and hence in $M(A \otimes_\gamma B)$. □

Corollary B.22. *Suppose A and B are C^*-algebras, $\|\cdot\|_\gamma$ is a C^*-norm on $A \odot B$, and $\pi : A \otimes_\gamma B \to B(\mathcal{H})$ is a nondegenerate representation.*

(a) *There are unique representations $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ such that*

$$\pi(a \otimes b) = \pi_A(a)\pi_B(b) = \pi_B(b)\pi_A(a)$$

for all $a \in A$ and $b \in B$.

(b) *Both* π_A *and* π_B *are nondegenerate, and their canonical extensions* $\bar{\pi}_A : M(A) \to B(\mathcal{H})$ *and* $\bar{\pi}_B : M(B) \to B(\mathcal{H})$ *also have commuting ranges.*

(c) *If* $\bar{\pi}$ *is the canonical extension of* π *to* $M(A \otimes_\gamma B)$*, and if* i_A *and* i_B *are as in Proposition B.21, then* $\pi_A = \bar{\pi} \circ i_A$ *and* $\pi_B = \bar{\pi} \circ i_B$*; if* π *is faithful, so are* π_A *and* π_B*.*

(d) *If* $\{ e_i \}$ *and* $\{ u_j \}$ *are approximate identities for* A *and* B*, respectively, then in the strong operator topology*

$$\pi_A(a) = \lim_j \pi(a \otimes u_j), \qquad \text{and} \qquad \pi_B(b) = \lim_i \pi(e_i \otimes b).$$

Proof. If we define $\pi_A = \bar{\pi} \circ i_A$ and $\pi_B = \bar{\pi} \circ i_B$, then π_A and π_B satisfy the first part of (c) by definition, are faithful if π is because i_A and i_B are injective, and satisfy the identity in (a) because i_A and i_B do. We claim that any representations π'_A and π'_B satisfying the identity in (a) must be nondegenerate. Suppose $h \in \mathcal{H}$ satisfies $\pi'_A(a)h = 0$ for all $a \in A$; then

$$\pi'_B(b)\pi'_A(a)h = \pi(a \otimes b)h = 0$$

for all $a \in A$ and $b \in B$, and $h = 0$ because π is nondegenerate. A similar statement holds for π'_B, so the claim is proved. Since π'_A and π'_B are nondegenerate, $\pi'_A(e_i)$ and $\pi'_B(u_j)$ both converge to $1_{\mathcal{H}}$ in the strong operator topology, and part (d) and the uniqueness in (a) follow easily.

It remains to prove (b), and we have already seen that π_A and π_B are nondegenerate. Suppose that $m \in M(A)$ and $n \in M(B)$. By nondegeneracy, it is enough to compute on vectors of the form $\pi_A(a)\pi_B(b)h$ for $a \in A$, $b \in B$, and then

$$\begin{aligned}
\bar{\pi}_A(m)\bar{\pi}_B(n)\big(\pi_A(a)\pi_B(b)h\big) &= \bar{\pi}_A(m)\bar{\pi}_B(n)\big(\pi_B(b)\pi_A(a)h\big) \\
&= \bar{\pi}_A(m)\big(\pi_B(nb)\pi_A(a)h\big) \\
&= \bar{\pi}_A(m)\big(\pi_A(a)\pi_B(nb)h\big) \\
&= \pi_A(ma)\pi_B(nb)h \\
&= \pi_B(nb)\pi_A(ma)h \\
&= \bar{\pi}_B(n)\big(\pi_B(b)\pi_A(ma)h\big) \\
&= \bar{\pi}_B(n)\big(\pi_A(ma)\pi_B(b)h\big) \\
&= \bar{\pi}_B(n)\bar{\pi}_A(m)\big(\pi_A(a)\pi_B(b)h\big),
\end{aligned}$$

so $\bar{\pi}_A(m)$ commutes with $\bar{\pi}_B(n)$. □

Example B.23. If π is a tensor product representation $\rho \otimes \eta$ of the spatial tensor product $A \otimes_\sigma B$, as in Corollary B.11, then π_A is the representation $\rho \otimes 1_{\mathcal{H}_\eta} : a \mapsto \rho(a) \otimes 1_{\mathcal{H}_\eta}$, and $\pi_B = 1_{\mathcal{H}_\rho} \otimes \eta$.

One attractive property of the spatial norm $\| \cdot \|_\sigma$ is the equality

$$\|a \otimes b\|_\sigma = \|a\| \, \|b\| \quad \text{for all } a \in A \text{ and } b \in B; \tag{B.7}$$

we sum this up by saying that $\|\cdot\|_\sigma$ is a *cross-norm*. We will later prove that every C^*-norm on $A \odot B$ is a cross-norm (Theorem B.38), but this is nontrivial, and for the main goals of this book it is enough to see that every C^*-norm is *subcross*: $\|a \otimes b\|_\gamma \leq \|a\| \, \|b\|$ for all $a \in A$ and $b \in B$ (Corollary B.24). This condition is important because it implies that all such norms are jointly continuous in the A- and B-variables. In fact, we will prove that every *C^*-seminorm* is subcross: a C^*-seminorm is a seminorm which is submultiplicative and satisfies the C^*-norm identity.

Corollary B.24. *Let A and B be C^*-algebras. Every C^*-seminorm $\|\cdot\|_\alpha$ on $A \odot B$ is subcross: $\|a \otimes b\|_\alpha \leq \|a\| \, \|b\|$ for all $a \in A$ and $b \in B$.*

Proof. Let* $\|\cdot\|_\sigma$ be the spatial norm on $A \odot B$. Then $\|t\|_\gamma := \max\{\|t\|_\sigma, \|t\|_\alpha\}$ defines a C^*-norm on $A \odot B$. If $\|\cdot\|_\gamma$ is subcross, so is $\|\cdot\|_\alpha$. Let $\|\cdot\|_\gamma$ also denote the corresponding norm on $M(A \otimes_\gamma B)$. Then, since the norm on any multiplier algebra is submultiplicative,

$$\|a \otimes b\|_\gamma = \|i_A(a) i_B(b)\|_\gamma \leq \|i_A(a)\|_\gamma \|i_B(b)\|_\gamma = \|a\| \, \|b\|,$$

where the last equality holds because i_A and i_B are injective and therefore isometric. □

Now that we know all C^*-norms are subcross, we can define what is obviously the largest possible C^*-norm on $A \odot B$.

Proposition B.25. *Let A and B be C^*-algebras. Then*

$$\begin{aligned} \|t\|_{\max} :&= \sup\{\|t\|_\gamma : \|\cdot\|_\gamma \text{ is a } C^*\text{-seminorm on } A \odot B\} \\ &= \sup\{\|t\|_\gamma : \|\cdot\|_\gamma \text{ is a } C^*\text{-norm on } A \odot B\} \end{aligned}$$

is a C^*-cross-norm majorizing all C^*-seminorms on $A \odot B$; it is called the maximal norm *on $A \odot B$. The completion of $A \odot B$ in $\|\cdot\|_{\max}$ is a C^*-algebra called the* maximal tensor product *of A and B, and denoted by $A \otimes_{\max} B$.*

Proof. Suppose $t = \sum_i a_i \otimes b_i$ and $\|\cdot\|_\gamma$ is a C^*-seminorm on $A \odot B$. Because $\|\cdot\|_\gamma$ is subcross, we have $\|t\|_\gamma \leq \sum_i \|a_i\| \, \|b_i\|$ by Corollary B.24; thus the sup is well-defined and we have

$$\Big\| \sum\nolimits_i a_i \otimes b_i \Big\|_{\max} \leq \sum\nolimits_i \|a_i\| \, \|b_i\|.$$

The resulting function $\|\cdot\|_{\max}$ is a seminorm, is submultiplicative, and satisfies the C^*-identity because each $\|\cdot\|_\gamma$ has these properties. Since the spatial norm $\|\cdot\|_\sigma$ is a C^*-norm, $\|\cdot\|_{\max}$ is a C^*-norm, and it is enough to sup over C^*-norms; in particular, $\|\cdot\|_{\max}$ majorizes all C^*-norms. Finally,

$$\|a\| \, \|b\| = \|a \otimes b\|_\sigma \leq \|a \otimes b\|_{\max} \leq \|a\| \, \|b\|,$$

so $\|\cdot\|_{\max}$ is a cross-norm. □

*There is a proof in [92] which does not use multipliers; see also [167, Corollary T.6.2 and Exercise T.J].

Remark B.26. The notation $\|\cdot\|_{\max}$ is standard, but it should be used with caution. For example, suppose that A_0 and B_0 are C^*-subalgebras of A and B, respectively, and that $\|\cdot\|_\gamma$ is a C^*-norm on $A \odot B$. Then the restriction of $\|\cdot\|_\gamma$ to the subalgebra $A_0 \odot B_0$ is a C^*-norm, which one would instinctively continue to denote by $\|\cdot\|_\gamma$, and the completion $A_0 \otimes_\gamma B_0$ in this norm is naturally a C^*-subalgebra of $A \otimes_\gamma B$. But if $\|\cdot\|_\gamma$ is the maximal norm on $A \odot B$, its restriction to $A_0 \odot B_0$ need not be the maximal norm for $A_0 \odot B_0$; in particular, the inclusion map $A_0 \odot B_0 \hookrightarrow A \odot B$ is not generally bounded with respect to the respective maximal norms, and $A_0 \otimes_{\max} B_0$ is not naturally a subalgebra of $A \otimes_{\max} B$. This is not a problem for the spatial tensor product, and we can safely identify $A_0 \otimes_\sigma B_0$ with a C^*-subalgebra of $A \otimes_\sigma B$ (see Corollary B.14); nor is it a problem for the maximal tensor product and ideals rather than subalgebras (see Proposition B.30).

Theorem B.27. *Suppose A and B are C^*-algebras. Then there are nondegenerate homomorphisms $i_A : A \to M(A \otimes_{\max} B)$, $i_B : B \to M(A \otimes_{\max} B)$ such that*

(a) *$i_A(a)i_B(b) = i_B(b)i_A(a) = a \otimes b$ for $a \in A$, $b \in B$;*

(b) *if ϕ and ψ are representations of A and B with commuting ranges, then there is a representation $\phi \otimes_{\max} \psi$ of $A \otimes_{\max} B$ such that*

$$\phi \otimes_{\max} \psi(i_A(a)i_B(b)) = \phi(a)\psi(b) \quad \text{for } a \in A, b \in B;$$

(c) *$A \otimes_{\max} B = \overline{\operatorname{span}}\{\, i_A(a)i_B(b) : a \in A \text{ and } b \in B \,\}$.*

If D is a C^*-algebra and $j_A : A \to M(D)$, $j_B : B \to M(D)$ are homomorphisms satisfying the analogues of (a), (b) *and* (c), *then there is an isomorphism θ of $A \otimes_{\max} B$ onto D such that $\theta(a \otimes b) = j_A(a)j_B(b)$.*

Proof. The existence of the maps i_A, i_B was established in Proposition B.21. To see that they are nondegenerate, note that if $\{e_i\}$ is an approximate identity for A, then

$$i_A(e_i)(a \otimes b) = (e_i a) \otimes b \to a \otimes b$$

in $A \otimes_{\max} B$ because $\|\cdot\|_{\max}$ is subcross. If ϕ, ψ are commuting representations, the universal property of the algebraic tensor product implies that there is a unique linear map $\pi : A \odot B \to B(\mathcal{H})$ such that $\pi(a \otimes b) = \phi(a)\psi(b)$, and π is a representation because ϕ and ψ have commuting ranges. Pulling back the norm from $B(\mathcal{H})$ gives a C^*-seminorm $\|t\| := \|\pi(t)\|$ on $A \odot B$, so $\|\pi(t)\| \le \|t\|_{\max}$, and we deduce that π extends to a representation $\phi \otimes_{\max} \psi$ of $A \otimes_{\max} B$ with the required property. Part (c) follows from the equation $i_A(a)i_B(b) = a \otimes b$ because $A \otimes_{\max} B$ is by definition a completion of $A \odot B = \operatorname{span}\{a \otimes b\}$.

For the last part, choose a faithful nondegenerate representation π of D on $\mathcal{H}$. Then the analogue of (a) implies that $\bar{\pi} \circ j_A$ and $\bar{\pi} \circ j_B$ are representations of A and B with commuting ranges, and hence by (b) there is a representation $(\bar{\pi} \circ j_A) \otimes_{\max} (\bar{\pi} \circ j_B)$ of $A \otimes_{\max} B$ on $\mathcal{H}$ such that

$$\big((\bar{\pi} \circ j_A) \otimes_{\max} (\bar{\pi} \circ j_B)\big)(a \otimes b) = \bar{\pi}(j_A(a))\bar{\pi}(j_B(b)) = \pi(j_A(a)j_B(b))$$

(note that $j_A(a)j_B(b)$ belongs to D by the analogue of (c)). So we can define $\theta := \pi^{-1} \circ \big((\bar{\pi} \circ j_A) \otimes_{\max} (\bar{\pi} \circ j_B)\big)$, and θ is a homomorphism of $A \otimes_{\max} B$ into D. On the other hand, by putting $A \otimes_{\max} B$ faithfully on Hilbert space and applying the analogue of (b) to the commuting pair (i_A, i_B), we obtain a homomorphism ρ of D into $A \otimes_{\max} B$ such that $\rho(j_A(a)j_B(b)) = i_A(a)i_B(b) = a \otimes b$. Now (c) and its analogue imply that θ is an isomorphism with inverse ρ. □

Corollary B.28. *Let A and B be C^*-algebras. Then $\pi \mapsto (\bar{\pi} \circ i_A, \bar{\pi} \circ i_B)$ is a bijection between the nondegenerate representations of $A \otimes_{\max} B$ and commuting pairs of nondegenerate representations of A and B. In particular, we have*

$$\Big\| \sum\nolimits_i a_i \otimes b_i \Big\|_{\max} = \sup \Big\{ \Big\| \sum\nolimits_i \phi(a_i)\psi(b_i) \Big\| : \phi, \psi \text{ are representations with commuting ranges} \Big\}$$

for every $\sum_i a_i \otimes b_i \in A \odot B$.

Proof. We saw in Corollary B.22 that $(\bar{\pi} \circ i_A, \bar{\pi} \circ i_B)$ is always a commuting pair of nondegenerate representations. On the other hand, if (ϕ, ψ) is such a pair, the representation $\pi := \phi \otimes_{\max} \psi$ is also nondegenerate, and satisfies $\phi = \bar{\pi} \circ i_A$, $\psi = \bar{\pi} \circ i_B$ by part (b) of the Theorem. Since the norm in any C^*-algebra is determined by the representations, and any pair of (possibly degenerate) representations gives a representation of $A \otimes_{\max} B$ by Theorem B.27(b), the last part follows.† □

Corollary B.29. *Suppose that A, B and C are C^*-algebras. Then there is an isomorphism α of $(A \otimes_{\max} B) \otimes_{\max} C$ onto $A \otimes_{\max} (B \otimes_{\max} C)$ such that*

$$\alpha((a \otimes b) \otimes c) = a \otimes (b \otimes c).$$

Proof. The previous corollary implies that the maximal norms on $(A \odot B) \odot C$ and $A \odot (B \odot C)$ are both given by taking sups over commuting triples of representations of A, B and C, so the isomorphism of algebraic tensor products is isometric. □

It is important in applications to know how tensor products of C^*-algebras interact with ideals and quotients. We have already seen that if I is an ideal in A, then we can identify the spatial tensor product $I \otimes_\sigma D$ with a subalgebra of $A \otimes_\sigma D$ (Corollary B.14), and because $I \odot D$ is an ideal in $A \odot D$, the subalgebra $I \otimes_\sigma D$ is actually an ideal. The quotient map $q : A \to A/I$ induces a homomorphism $q \otimes_\sigma \mathrm{id} : A \otimes_\sigma D \to (A/I) \otimes_\sigma D$ (Proposition B.13), which is surjective because its range contains the algebraic tensor product. In general, however, $\ker(q \otimes_\sigma \mathrm{id})$ can strictly contain $I \otimes_\sigma D = (\ker q) \otimes_\sigma D$. We shall see that the maximal tensor product is better behaved in this respect, and then give a sufficient condition for the equality of $\ker(q \otimes_\sigma \mathrm{id})$ and $I \otimes_\sigma D$.

†Strictly speaking, the pairs of commuting representations do not form a set, so the least upper bound on the right does not make sense. However, there is no real problem here: the sup is attained by the pair $(\bar{\pi} \circ i_A, \bar{\pi} \circ i_B)$ corresponding to a faithful representation π of $A \otimes_{\max} B$.

We shall couch these results in the language of exact sequences. By saying that

$$0 \longrightarrow I \xrightarrow{\ i\ } A \xrightarrow{\ q\ } B \longrightarrow 0 \tag{B.8}$$

is a *short exact sequence of C^*-algebras*, we usually mean that I is a (closed two-sided) ideal in a C^*-algebra A, that $i : I \to A$ is the inclusion map, and that $q : A \to B$ is a surjective homomorphism with $\ker q = I$, so that q induces an isomorphism of the quotient A/I onto B; sometimes (as in the next Proposition, for example) i will be just an embedding of I onto an ideal in A, and the notation indicates that we can identify I with its image in A.

Proposition B.30. *For every short exact sequence of C^*-algebras* (B.8) *and every C^*-algebra D, there is a short exact sequence*

$$0 \longrightarrow I \otimes_{\max} D \xrightarrow{i\otimes_{\max}\mathrm{id}} A \otimes_{\max} D \xrightarrow{q\otimes_{\max}\mathrm{id}} B \otimes_{\max} D \longrightarrow 0.$$

Lemma B.31. *Suppose $\phi : A \to B$ and $\psi : C \to D$ are homomorphisms of C^*-algebras. Then there is a unique homomorphism $\phi \otimes_{\max} \psi : A \otimes_{\max} C \to B \otimes_{\max} D$ such that $\phi \otimes_{\max} \psi(a \otimes c) = \phi(a) \otimes \psi(c)$.*

Proof. Because i_B and i_D have commuting ranges, so do

$$i_B \circ \phi : A \to M(B \otimes_{\max} D) \quad \text{and} \quad i_D \circ \psi : B \to M(B \otimes_{\max} D).$$

Thus by Theorem B.27 there is a homomorphism $(i_B \circ \phi) \otimes_{\max} (i_D \circ \psi)$ of $A \otimes_{\max} C$ into $M(B \otimes_{\max} D)$ such that

$$\big((i_B \circ \phi) \otimes_{\max} (i_D \circ \psi)\big)(a \otimes c) = i_B(\phi(a))i_D(\psi(c)) = \phi(a) \otimes \psi(c);$$

this equation implies both that $\phi \otimes_{\max} \psi := (i_B \circ \phi) \otimes_{\max} (i_D \circ \psi)$ has range in $B \otimes_{\max} D$, and that it has the required property. □

Proof of Proposition B.30. It is easy to see that the image of $i \otimes_{\max} \mathrm{id}$ is an ideal in $A \otimes_{\max} D$. To see that $i \otimes_{\max} \mathrm{id}$ is injective, we show that every representation of $I \otimes_{\max} D$ factors through $i \otimes_{\max} \mathrm{id}$. So suppose π is a nondegenerate representation of $I \otimes_{\max} D$. Then $\pi_I : \bar{\pi} \circ i_I$ and $\pi_D := \bar{\pi} \circ i_D$ are nondegenerate representations with commuting ranges. The extension $\bar{\pi}_I$ of π_I to A also has range commuting with that of π_D (see Corollary B.22(b)), so there is a representation $\bar{\pi}_I \otimes_{\max} \pi_D$ of $A \otimes_{\max} D$, and we trivially have

$$(\bar{\pi}_I \otimes_{\max} \pi_D) \circ (i \otimes_{\max} \mathrm{id}) = \pi_I \otimes_{\max} \pi_D = \pi.$$

Thus $i \otimes_{\max} \mathrm{id}$ is injective.

Since the range of the homomorphism $q \otimes_{\max} \mathrm{id}$ is a C^*-algebra containing every elementary tensor, it is all of $B \otimes_{\max} D$. Thus it remains to verify exactness at $A \otimes_{\max} D$, or, in other words, to prove that $\ker(q \otimes_{\max} \mathrm{id}) = \mathrm{im}(i \otimes_{\max} \mathrm{id})$. Since $q \circ i = 0$, we trivially have $\mathrm{im}(i \otimes_{\max} \mathrm{id}) \subset \ker(q \otimes_{\max} \mathrm{id})$. For the converse, suppose that π is a nondegenerate representation of $A \otimes_{\max} D$ such that $\ker \pi = \mathrm{im}(i \otimes_{\max} \mathrm{id})$. Then $\pi_A := \bar{\pi} \circ i_A$ vanishes on I: if $c \in I$, then

$$\pi_A(c)\big(\pi_D(d)h\big) = \pi(c \otimes d)h = \pi\big(i \otimes_{\max} \mathrm{id}(c \otimes d)\big)h = 0$$

for all $h \in \mathcal{H}_\pi$, and the nondegeneracy of π_D implies $\pi_A(c) = 0$. Thus there is a representation ρ of B such that $\pi_A = \rho \circ q$. Since the ranges of π_A and π_D commute, and $\operatorname{im}\pi_A = \operatorname{im}\rho$, so do the ranges of ρ and π_D, and there is a representation $\rho \otimes_{\max} \pi_D$ of $B \otimes_{\max} D$ such that

$$\rho \otimes_{\max} \pi_D(q(a) \otimes d) = \rho(q(a))\pi_D(d) = \pi_A(a)\pi_D(d) = \pi(a \otimes d);$$

we deduce that π factors through $q \otimes_{\max} \mathrm{id}$, and

$$\ker(q \otimes_{\max} \mathrm{id}) \subset \ker \pi = \operatorname{im}(i \otimes_{\max} \mathrm{id}),$$

as required. □

Proposition B.32. *Suppose that* (B.8) *is a short exact sequence of C^*-algebras, and that D is a C^*-algebra for which the spatial and maximal norms coincide on $B \odot D$. Then*

$$0 \longrightarrow I \otimes_\sigma D \xrightarrow{i\otimes\mathrm{id}} A \otimes_\sigma D \xrightarrow{q\otimes\mathrm{id}} B \otimes_\sigma D \longrightarrow 0$$

is an exact sequence of C^*-algebras.

Proof. We saw in the preamble to Proposition B.30 that $i \otimes \mathrm{id}$ is an isomorphism onto an ideal in $A \otimes_\sigma D$, and $q \otimes \mathrm{id}$ is surjective because its image contains $q(A) \odot D = B \otimes D$. It remains to see that $\operatorname{im}(i \otimes \mathrm{id}) = \ker(q \otimes \mathrm{id})$. Since $\operatorname{im}(i \otimes \mathrm{id}) \subset \ker(q \otimes \mathrm{id})$, there is a homomorphism ρ of the quotient $Q := A \otimes_\sigma D / \operatorname{im}(i \otimes \mathrm{id})$ onto $B \otimes_\sigma D$, and it will suffice to show that ρ is injective. We do this by constructing a left inverse for ρ — that is, a homomorphism $\phi : B \otimes_\sigma D \to Q$ such that $\phi \circ \rho = \mathrm{id}_Q$.

Let $\psi : A \otimes_\sigma D \to Q$ be the quotient map, and notice that $\big(q(a), d\big) \mapsto \psi(a \otimes d)$ is a well-defined bilinear map which induces a homomorphism $\phi : B \odot D \to Q$ satisfying $\phi\big(q(a) \otimes d\big) = \psi(a \otimes d)$. Then

$$\|c\|_\gamma := \max\{\, \|\phi(c)\|, \|c\|_\sigma \,\}$$

is a C^*-norm on $B \odot D$ dominating the spatial norm; since it is certainly dominated by the maximal norm, our assumption implies that $\|\cdot\|_\gamma = \|\cdot\|_\sigma$. Thus ϕ is bounded with respect to the spatial norm and extends to a homomorphism $\phi : B \otimes_\sigma D \to Q$. Since

$$\phi\big(\rho\big(\psi(a \otimes d)\big)\big) = \phi\big(q \otimes \mathrm{id}(a \otimes d)\big) = \phi\big(q(a) \otimes d\big) = \psi(a \otimes d),$$

we deduce that $\phi \circ \rho = \mathrm{id}_Q$, as desired. □

Hooptedoodle B.33. A C^*-algebra D is *exact* if

$$0 \longrightarrow I \otimes_\sigma D \xrightarrow{i\otimes\mathrm{id}} A \otimes_\sigma D \xrightarrow{q\otimes\mathrm{id}} B \otimes_\sigma D \longrightarrow 0$$

is exact for every exact sequence (B.8). The Proposition says that if the spatial norm on $B \odot D$ coincides with the maximal norm for every C^*-algebra B, then D is exact; in particular, the nuclear C^*-algebras studied in Section B.5 are exact. There are also nonnuclear C^*-algebras which are exact: the standard example is the reduced group C^*-algebra $C^*_r(\mathbb{F}_2)$ of the free group [166, Chapter 2]. It has been conjectured that the reduced group C^*-algebra of every discrete group is exact. The class of exact C^*-algebras is closed under taking subalgebras, quotients, direct limits and spatial tensor products, but not under taking extensions. For a recent survey of exact C^*-algebras and their properties, and for proofs of these results, see [166].

B.4 C^*-Algebras with Hausdorff Spectrum

Recall that a C^*-algebra A is CCR if $\pi(A) = \mathcal{K}(\mathcal{H}_\pi)$ for every irreducible representation π (see page 220 and Remark 5.17). For CCR algebras with Hausdorff spectrum the general theory of tensor products simplifies, and this simplified version suffices in the main body of the book. In this section, we will work primarily with the spatial norm $\|\cdot\|_\sigma$.

Since CCR algebras are by definition associated with the algebra $\mathcal{K}(\mathcal{H})$ of compact operators, to study them we need to know about the representation theory of $\mathcal{K}(\mathcal{H})$. Suppose $\pi : A \to B(\mathcal{H})$ is a representation of a C^*-algebra A and I is a set. Then there is a representation $I \cdot \pi$ of A on the direct sum $\bigoplus_{i\in I} \mathcal{H}$ such that $I \cdot \pi(a)(h_i) = \big(\pi(a)h_i\big)$; we call $I \cdot \pi$ a *multiple* of π. The map $h \otimes (x_i) \mapsto (x_i h)$ extends to a unitary isomorphism U of $\mathcal{H} \otimes \ell^2(I)$ onto $\bigoplus_{i\in I} \mathcal{H}$ which intertwines $\pi \otimes 1$ with $I \cdot \pi$; indeed, a straightforward calculation shows that $(I \cdot \pi(a))U = U(\pi(a) \otimes 1)$. Our first lemma says that every representation of $\mathcal{K}(\mathcal{H})$ is a multiple of the identity representation $\mathrm{id} : \mathcal{K}(\mathcal{H}) \to B(\mathcal{H})$.

Lemma B.34. *Suppose that $\pi : \mathcal{K}(\mathcal{H}) \to B(\mathcal{H}_\pi)$ is a nondegenerate representation of the C^*-algebra of compact operators on some Hilbert space $\mathcal{H}$. Then there is a Hilbert space $\mathcal{H}_0$ such that π is equivalent to $\mathrm{id} \otimes 1$ on $\mathcal{H} \otimes \mathcal{H}_0$.*

Proof. Let e be a unit vector in $\mathcal{H}$, and consider the projection $\pi(e \otimes \overline{e})$. The range of $\pi(e \otimes \overline{e})$ is a Hilbert space $\mathcal{H}_0$; let $\{e_i : i \in I\}$ be an orthonormal basis for $\mathcal{H}_0$. As in Example A.15, the formula $U_i(h) := \pi(h \otimes \overline{e})e_i$ defines an isometry U_i of $\mathcal{H}$ onto the invariant subspace

$$\mathcal{H}_i := \mathrm{span}\{\pi(T)e_i : T \in \mathcal{K}(\mathcal{H})\},$$

and U_i intertwines the identity representation id and $\pi|_{\mathcal{H}_i}$. For $S, T \in \mathcal{K}(\mathcal{H})$, the operator $(e \otimes \overline{e})T^*S(e \otimes \overline{e})$ is the scalar multiple $(Se \mid Te)(e \otimes \overline{e})$ of $e \otimes \overline{e}$, and hence

$$\big(\pi(S)e_i \mid \pi(T)e_j\big) = \big(\pi(S)\pi(e \otimes \overline{e})e_i \mid \pi(T)\pi(e \otimes \overline{e})e_j\big) = (Se \mid Te)(e_i \mid e_j);$$

thus the subspaces $\mathcal{H}_i$ of $\mathcal{H}_\pi$ are mutually orthogonal. They span $\pi(\mathcal{K}(\mathcal{H}))\mathcal{H}_0$, and also $\mathcal{H}_\pi$: by nondegeneracy, we can approximate any vector h by one of the form

$$\sum\nolimits_i \pi(h_i \otimes \overline{k_i})h = \sum\nolimits_i \pi(h_i \otimes \overline{e})\pi(e \otimes \overline{e})\pi(e \otimes \overline{k_i})h,$$

which belongs to $\pi(\mathcal{K}(\mathcal{H}))\mathcal{H}_0$. We deduce that $U := \bigoplus U_i$ is a unitary isomorphism of $\bigoplus_{i\in I} \mathcal{H}$ onto $\mathcal{H}_\pi = \bigoplus_{i\in I} \mathcal{H}_i$ which intertwines $I \cdot \mathrm{id}$ and $\pi = \bigoplus_{i\in I} \pi|_{\mathcal{H}_i}$. Since $\mathcal{H}_0 = \ell^2(I)$, the identification of $\mathcal{H} \otimes \ell^2(I)$ with $\bigoplus_{i\in I} \mathcal{H}$ completes the proof. □

Proposition B.35. *Suppose that A and B are C^*-algebras and that A is CCR with Hausdorff spectrum. Let $\|\cdot\|_\gamma$ be a C^*-norm on $A \odot B$. Then every irreducible representation π of $A \otimes_\gamma B$ is equivalent to a representation of the form $\rho \otimes \eta$ for some $\rho \in \hat{A}$ and $\eta \in \hat{B}$. Furthermore, $\|c\|_\sigma \geq \|c\|_\gamma$ for every $c \in A \odot B$, and $A \otimes_\sigma B$ coincides with $A \otimes_{\max} B$.*

Proof. Let π be an irreducible representation of $A \otimes_\gamma B$ on $\mathcal{H}_\pi$, and let π_A and π_B be the commuting representations of A and B such that $\pi(a \otimes b) = \pi_A(a)\pi_B(b)$, as in Corollary B.22. By the Dauns-Hofmann Theorem, each $f \in C_0(\hat{A})$ determines a multiplier in $ZM(A)$ such that $\tau(f \cdot a) = f(\tau)\tau(a)$ for $\tau \in \hat{A}$ and $a \in A$. Since the extensions $\bar{\pi}_A$ and $\bar{\pi}_B$ have commuting ranges by Corollary B.22, and since $f \in ZM(A)$, the operator $\bar{\pi}_A(f)$ commutes with every $\pi(a \otimes b) = \pi_A(a)\pi_B(b)$, and hence belongs to $\mathbb{C}1_{\mathcal{H}_\pi} = \pi(A \otimes_\gamma B)'$. Thus $\bar{\pi}_A$ determines a complex homomorphism of $C_0(\hat{A})$: there exists $\rho \in \hat{A}$ such that $\bar{\pi}_A(f) = f(\rho)1_{\mathcal{H}_\pi}$ for $f \in C_0(\hat{A})$. If $\rho(a) = 0$, then because $\hat{A}$ is Hausdorff we can factor $a = f \cdot b$ for some $f \in C_0(\hat{A})$ satisfying $f(\rho) = 0$ (we can do this exactly by Corollary 5.11, or approximately by choosing $f \equiv 1$ where $\|a\| \geq \epsilon$), and deduce that $\pi(a) = f(\rho)\pi(b) = 0$. Thus π_A factors through the quotient $A/\ker\rho$, which is isomorphic to $\mathcal{K}(\mathcal{H}_\rho)$ because A is CCR. By Lemma B.34, we can assume that $\mathcal{H}_\pi = \mathcal{H}_\rho \otimes \mathcal{H}_0$ and that $\pi_A = \rho \otimes 1$.

If $h \in \mathcal{H}_\rho$ is a fixed unit vector, then $V_h : k \mapsto h \otimes k$ embeds $\mathcal{H}_0$ onto the closed subspace $\big((h \otimes \overline{h}) \otimes 1\big)(\mathcal{H}_\rho \otimes \mathcal{H}_0)$ of $\mathcal{H}_\pi$. Since $(h \otimes \overline{h}) \otimes 1$ belongs to $\mathcal{K}(\mathcal{H}_\rho) \otimes 1 = \pi_A(A)$, each $\pi_B(b)$ commutes with $(h \otimes \overline{h}) \otimes 1$. Thus the subspace $V_h(\mathcal{H}_0)$ is invariant for π_B, and $\eta : b \mapsto V_h^*\pi_B(b)V_h$ is a representation of B on $\mathcal{H}_0$. Since the operator $(h \otimes \overline{h_1}) \otimes 1$ also commutes with everything in the range of π_B and satisfies $\big((h \otimes \overline{h_1}) \otimes 1\big)V_{h_1} = V_h$, η is independent of the choice of unit vector h, and we have

$$(1 \otimes \eta(b))(h \otimes k) = h \otimes \eta(b)k = V_h(\eta(b)k) = (\pi_B(b)V_h)k = \pi_B(b)(h \otimes k)$$

for every k and every unit vector h. Thus $\pi_B(b) = 1 \otimes \eta(b)$, and

$$\pi(a \otimes b) = \pi_A(a)\pi_B(b) = (\rho(a) \otimes 1)(1 \otimes \eta(b)) = \rho(a) \otimes \eta(b) = \rho \otimes \eta(a \otimes b).$$

If $T\eta(b) = \eta(b)T$ for all $b \in B$, then $1 \otimes T$ commutes with $\pi = \rho \otimes \eta$, and hence is scalar; thus η is irreducible, and the first part is proved.

For the remaining statements, recall that if C is a C^*-algebra and $c \in C$, there is an irreducible representation π of C such that $\|c\| = \|\pi(c)\|$ (Theorem A.14). Thus if $c \in A \otimes_\gamma B$, there exist irreducible representations $\rho \in \hat{A}$ and $\eta \in \hat{B}$ such that $\|c\|_\gamma = \|\rho \otimes \eta(c)\|$. Since $\rho \otimes \eta$ is norm-decreasing for the spatial norm (Theorem B.9), we deduce that $\|c\|_\gamma \leq \|c\|_\sigma$ for all $c \in A \odot B$. Since this applies to every C^*-norm, we deduce that $\|c\|_{\max} \leq \|c\|_\sigma$, and hence by maximality that $\|c\|_{\max} = \|c\|_\sigma$. □

For the statement of the next lemma, recall that $\mathcal{S}'$ denotes the commutant of a subset $\mathcal{S}$ of $B(\mathcal{H})$.

Lemma B.36. *Suppose that $\pi : A \to B(\mathcal{H}_\pi)$ and $\eta : B \to B(\mathcal{H}_\eta)$ are nondegenerate representations of C^*-algebras. If π is irreducible, then every $T \in \pi \otimes \eta(A \otimes_\sigma B)'$ has the form $T = 1_{\mathcal{H}_\pi} \otimes R$ for some $R \in \eta(B)'$. In particular, if both π and η are irreducible, then $\pi \otimes \eta$ is an irreducible representation of $A \otimes_\sigma B$.*

Proof. For any T in $B(\mathcal{H}_\pi \otimes \mathcal{H}_\eta)$ and $k, l \in \mathcal{H}_\eta$, $(g, h) \mapsto \big(T(g \otimes k) \mid h \otimes l\big)$ is a bounded sesquilinear form on $\mathcal{H}_\pi$, so there is a bounded operator $S_{k,l}$ on $\mathcal{H}_\pi$ such

that

$$\big(S_{k,l}(g) \mid h\big) = \big(T(g \otimes k) \mid h \otimes l\big) \quad \text{for all } g, h \in \mathcal{H}_\pi.$$

If $T \in \pi \otimes \eta(A \otimes_\sigma B)'$, then T commutes with $\pi(a) \otimes 1_{\mathcal{H}_\eta}$, because T commutes with each $\pi(a) \otimes \eta(u_j)$ as u_j runs through an approximate identity for B. Therefore for $g, h \in \mathcal{H}_\pi$ we have

$$\begin{aligned}\big(\pi(a)S_{k,l}(g) \mid h\big) &= \big(\big(\pi(a) \otimes 1\big)\big(T(g \otimes k)\big) \mid h \otimes l\big) \\ &= \big(T(\pi(a)g \otimes k) \mid h \otimes l\big) = \big(S_{k,l}(\pi(a)g) \mid h\big),\end{aligned}$$

and hence $S_{k,l} \in \pi(A)'$. Since π is irreducible, so that $\pi(A)' = \mathbb{C}1_{\mathcal{H}_\pi}$, there is a bounded sesquilinear form $B(\cdot,\cdot)$ on $\mathcal{H}_\eta$ such that $S_{k,l} = B(k,l)1_{\mathcal{H}_\pi}$, which in turn is given by a bounded operator R: $B(k,l) = (Rk \mid l)$. But then

$$\big(T(g \otimes k) \mid h \otimes l\big) = \big(g \otimes Rk \mid h \otimes l\big)$$

for all $g, h \in \mathcal{H}_\pi$ and $k, l \in \mathcal{H}_\eta$. Since $T = 1_{\mathcal{H}_\pi} \otimes R$ commutes with $1_{\mathcal{H}_\pi} \otimes \eta(b)$ for all $b \in B$, this implies that $R \in \eta(B)'$. □

Theorem B.37. *If A and B are CCR C^*-algebras with Hausdorff spectrum, then $(\pi, \eta) \mapsto \pi \otimes \eta$ induces a homeomorphism of $\hat{A} \times \hat{B}$ onto $(A \otimes_\sigma B)^\wedge$.*

Proof. If $\pi = \operatorname{Ad} U \circ \pi'$ and $\eta = \operatorname{Ad} V \circ \eta'$, then the unitary operator $U \otimes V$ implements a unitary equivalence between $\pi \otimes \eta$ and $\pi' \otimes \eta'$. Therefore $(\pi, \eta) \mapsto \pi \otimes \eta$ gives a well-defined map Φ on equivalence classes, and Lemma B.36 implies that Φ has range in $(A \otimes_\sigma B)^\wedge$.

On the other hand, the uniqueness in Corollary B.22 implies that $(\pi \otimes \eta)_A = \pi \otimes 1_{\mathcal{H}_\eta}$ and $(\pi \otimes \eta)_B = 1_{\mathcal{H}_\pi} \otimes \eta$. If $\pi \otimes \eta$ is equivalent to $\pi' \otimes \eta'$, then $\pi \otimes 1_{\mathcal{H}_\eta}$ is equivalent to $\pi' \otimes 1_{\mathcal{H}_\eta}$ and $\ker \pi = \ker \pi'$; similarly, $\ker \eta = \ker \eta'$. Since irreducible representations of algebras with Hausdorff spectrum are equivalent if and only if they have the same kernel, it follows that Φ is injective.

At the level of states, we have a map $\Psi : (\rho, \tau) \mapsto \rho \otimes \tau$ of $S(A) \times S(B)$ into $S(A \otimes_\sigma B)$. Since all these functionals have norm 1, a calculation on elementary tensors is enough to show that Ψ is continuous for the weak-$*$ topologies. Another straightforward calculation shows that, if π_ρ and π_τ are the GNS-representations with cyclic vectors h_ρ and h_τ, then

$$\rho \otimes \tau(a \otimes b) = \big(\pi_\rho \otimes \pi_\tau(a \otimes b)(h_\rho \otimes h_\tau) \mid h_\rho \otimes h_\tau\big). \tag{B.9}$$

If ρ and τ are pure, the representations π_ρ and π_τ are irreducible, and hence so is $\pi_\rho \otimes \pi_\tau$. Thus (B.9) implies that $\pi_\rho \otimes \pi_\tau$ is equivalent to the GNS-representation $\pi_{\rho\otimes\tau}$ (Corollary A.8); since $\pi_\rho \otimes \pi_\tau$ is irreducible, this implies both that $\rho \otimes \tau$ is pure (Lemma A.12), so that Ψ maps $P(A) \times P(B)$ into $P(A \otimes_\sigma B)$, and that we have a commutative diagram

$$\begin{array}{ccc} P(A) \times P(B) & \xrightarrow{\quad\Psi\quad} & P(A \otimes_\sigma B) \\ {\scriptstyle \Lambda_A \times \Lambda_B}\big\downarrow & & \big\downarrow{\scriptstyle \Lambda_{A\otimes_\sigma B}} \\ \hat{A} \times \hat{B} & \xrightarrow[\quad\Phi\quad]{} & (A \otimes_\sigma B)^\wedge \end{array}$$

in which Λ_A, Λ_B and $\Lambda_{A\otimes_\sigma B}$, denote the GNS-maps. Recall that the maps Λ_C are continuous, open surjections for the weak-$*$ topology on $P(C)$ and the usual topology on $\hat{C}$ (Theorem A.38). Thus if U is open in $(A\otimes_\sigma B)^\wedge$, then

$$\Phi^{-1}(U) = \Lambda_A \times \Lambda_B\big(\Psi^{-1}\big(\Lambda^{-1}_{A\otimes_\sigma B}(U)\big)\big)$$

is open, and Φ is continuous. Proposition B.35 implies that Φ is surjective, and it remains only to see that the inverse Φ^{-1} is continuous.

Suppose that $\Phi(\pi_i,\eta_i) = \pi_i\otimes\eta_i \to \Phi(\pi,\eta) = \pi\otimes\eta$ in $(A\otimes_\sigma B)^\wedge$; by symmetry, it will be enough to show that $\pi_i \to \pi$ in $\hat{A}$. Since the open neighbourhoods of π have the form

$$\mathcal{O}_J = \{\rho\in\hat{A} : J\not\subset \ker\rho\}$$

for some ideal J of A such that $\pi(J)\neq 0$ (Corollary A.28), it is enough to prove that eventually $\pi_i(J)\neq 0$. Since $J\otimes_\sigma B$ is an ideal in $A\otimes_\sigma B$ (see the discussion preceding Proposition B.30), the set

$$U := \{\tau\in(A\otimes_\sigma B)^\wedge : \tau(J\otimes_\sigma B)\neq 0\}$$

is an open neighbourhood of $\pi\otimes\eta$. Thus $\pi_i\otimes\eta_i\in U$ for large i, and this is only possible if $\pi_i(J)\neq 0$. This completes the proof. □

B.5 Tensor Products of General C^*-Algebras

In the previous section, we identified the spectrum of the spatial tensor product $A\otimes_\sigma B$ when A and B themselves have Hausdorff spectrum, and much of what we have proved holds more generally. It is not easy to dig detailed proofs of the resulting theorems out of the literature, so we shall provide them here. Since we have not used these results in the book and our goals are archival, we have settled for quoting some highly nontrivial but relatively well-archived material. Thus the pace in this section is a little faster.

The first main theorem says that the spatial norm is minimal among all C^*-norms on the tensor product. For this reason, the spatial tensor product is often called the minimal tensor product.

Theorem B.38. *Suppose that A and B are C^*-algebras and that $\|\cdot\|_\alpha$ is a C^*-norm on $A\odot B$. Then $\|\cdot\|_\alpha$ is a cross-norm which dominates the spatial norm: $\|t\|_\alpha \geq \|t\|_\sigma$ for $t\in A\odot B$.*

Recall that if $\phi\in S(A)$ and $\psi\in S(B)$, then there is a linear functional $\phi\odot\psi$ on $A\odot B$ which satisfies $\phi\odot\psi(t^*t)\geq 0$ for all $t\in A\odot B$ (Theorem B.9(a)). If $\|\cdot\|_\alpha$ is a C^*-norm on $A\odot B$, it is not *a priori* clear that $\phi\odot\psi$ is *α-continuous*; that is, continuous for the norm $\|\cdot\|_\alpha$ on $A\odot B$. If $\phi\odot\psi$ is α-continuous, then it extends to a bounded functional $\phi\otimes_\alpha\psi$ on $A\otimes_\alpha B$; this functional is easily seen to be positive, and is a state because $\phi\otimes_\alpha\psi(e_i\otimes f_j)\to 1$ when $\{e_i\}$ and $\{f_j\}$ are approximate identities for A and B. We shall show that every product functional is α-continuous, and minimality of the spatial norm will follow from the next proposition.

Proposition B.39. *Suppose that A and B are C^*-algebras and that $\|\cdot\|_\alpha$ is a C^*-norm on $A \odot B$. Then $\phi \odot \psi$ is α-continuous for every $\phi \in S(A)$ and every $\psi \in S(B)$ if and only if $\|t\|_\alpha \geq \|t\|_\sigma$ for all $t \in A \odot B$.*

Proof. The ($\Longleftarrow$) direction is provided by Corollary B.12. For the other direction, notice that if $s, t \in A \otimes_\alpha B$, then

$$s^*t^*ts \leq \|t\|_\alpha^2 s^*s \qquad \text{in } A \otimes_\alpha B. \tag{B.10}$$

Thus if $s, t \in A \odot B$, we can apply $\phi \odot \psi$ to (B.10) to conclude that

$$\|t\|_\alpha^2 \geq \frac{\phi \odot \psi(s^*t^*ts)}{\phi \odot \psi(s^*s)}$$

whenever $\phi \odot \psi(s^*s) > 0$. Now the result follows from Theorem B.9(b). □

So we want to prove that product functionals are α-continuous. The proof involves the concept of nuclearity. A C^*-algebra A is *nuclear* if there is only one C^*-norm on $A \odot B$ for every C^*-algebra B. In the same vein, we call an element b of a C^*-algebra B with identity *A-nuclear* if there is a unique C^*-norm on $A \odot C^*(b, 1)$, where $C^*(b, 1)$ is the C^*-subalgebra of B generated by b and the identity of B. If b is normal, the algebra $C^*(b, 1)$ is commutative, and we shall establish A-nuclearity of normal elements by proving that every commutative C^*-algebra is nuclear (Proposition B.43).

If $\|\cdot\|_\alpha$ is a C^*-norm on $A \odot B$ and if $\phi \in S(A)$, then we define

$$S_\phi^\alpha := \{\, \psi \in S(B) : \phi \odot \psi \text{ is } \alpha\text{-continuous} \,\}.$$

For each ϕ, S_ϕ^α is a convex, weak-$*$ compact subset of $S(B)$, and consists of those $\psi \in S(B)$ such that $\phi \odot \psi$ extends to a state $\phi \otimes_\alpha \psi$ on $A \otimes_\alpha B$.

Lemma B.40. *Suppose that A and B are C^*-algebras with identities, and that $\|\cdot\|_\alpha$ is a C^*-norm on $A \odot B$.*

(a) *Suppose that b is a self-adjoint A-nuclear element of B. Then for every pure state ϕ of A, we have*

$$\|b\| = \sup\{\, |\psi(b)| : \psi \in S_\phi^\alpha \,\}.$$

(b) *If every self-adjoint element b of B is A-nuclear, then $\phi \odot \psi$ is α-continuous for every $\phi \in S(A)$ and every $\psi \in S(B)$.*

Proof. Suppose that $b \in B$ is self-adjoint and A-nuclear. Then $C^*(b, 1)$ is commutative, and there is a complex homomorphism h on $C^*(b, 1)$ such that $|h(b)| = \|b\|$. Let $\phi \in P(A)$. Then $\phi \odot h$ is σ-continuous: the easy part of Proposition B.39 says that every product state is continuous with respect to the spatial norm. By assumption on b, $\phi \odot h$ is also α-continuous on $A \odot C^*(b, 1)$, and extends to a state $\phi \otimes_\alpha h$ on the closure $A \otimes_\alpha C^*(b, 1)$ of $A \odot C^*(b, 1)$ in $A \otimes_\alpha B$; this in turn

extends to a state $\xi \in S(A \otimes_\alpha B)$ by [111, Theorem 3.3.8]. Define $\psi_1 \in S(B)$ by $\psi_1(c) := \xi(1 \otimes c)$. We claim that

$$\xi(a \otimes c) = \phi(a)\psi_1(c) \qquad \text{for all } a \in A \text{ and } c \in B. \tag{B.11}$$

(This assertion would be obvious if ξ were multiplicative; since it need not be, we need to use that ϕ is pure.) For $0 \leq c \leq 1$, define $\phi_c(a) := \xi(a \otimes c)$. Then for any fixed c, ϕ_c and ϕ_{1-c} are positive linear functionals, and since $\phi(a) = \xi(a \otimes 1)$, we have $\phi = \phi_c + \phi_{1-c}$. Because ϕ is pure, and therefore an extreme point of $S(A)$, and

$$\|\phi_c\| + \|\phi_{1-c}\| = \phi_c(1) + \phi_{1-c}(1) = \phi(1) = 1,$$

we must have $\phi_c = \phi_c(1)\phi$. Thus

$$\xi(a \otimes c) := \phi_c(a) = \phi(a)\phi_c(1) = \phi(a)\xi(1 \otimes c) = \phi(a)\psi_1(c),$$

justifying the claim. We deduce that $\phi \odot \psi_1$ is the restriction of a state ξ to $A \odot B$, and hence that $\psi_1 \in S^\alpha_\phi$. Since

$$|\psi_1(b)| = |\xi(1 \otimes b)| = |\phi(1)h(b)| = \|b\|,$$

part (a) follows.

Since S^α_ϕ is convex and weak-$*$ closed, and determines the norm on self-adjoint elements of B by part (a), Lemma B.6 implies that $S^\alpha_\phi = S(B)$ for all $\phi \in P(A)$. For each $\psi \in S(B)$, define

$$S^\psi := \{\, \phi \in S(A) : \phi \odot \psi \text{ is } \alpha\text{-continuous} \,\}.$$

Again, S^ψ is convex and weak-$*$ compact. Since the statement $S^\alpha_\phi = S(B)$ means $\phi \in S^\psi$ for all $\psi \in S(B)$, we have just proved that $P(A) \subset S^\psi$ for every $\psi \in S(B)$. Thus it follows from the Krein-Milman Theorem that $S^\psi = S(A)$ for every ψ, and this is what we wanted to prove. □

To apply this lemma, we need to show that every self-adjoint element $b \in B$ is A-nuclear with respect to every C^*-algebra A. Since $C^*(b, 1)$ is commutative, it is enough to prove that $A \odot C$ always has a unique C^*-norm when C is commutative — or, in other words, that every commutative C^*-algebra C is nuclear. Recall that the irreducible representations of a commutative C^*-algebra C are all 1-dimensional, and the spectrum $\hat{C}$ is the space of non-zero complex homomorphisms on C with the weak-$*$ topology; if C is already of the form $C_0(T)$ for some locally compact space T, then $t \mapsto \epsilon_t$ is a homeomorphism of T onto $\hat{C}$. Since the only state associated to a 1-dimensional representation is the representation itself, $t \mapsto \epsilon_t$ is also a homeomorphism of T onto $P(A)$.

Lemma B.41. *Suppose that S and T are locally compact Hausdorff spaces. Then there is a unique C^*-norm on $C_0(S) \odot C_0(T)$.*

Proof. Suppose that $\|\cdot\|_\alpha$ is a C^*-norm on $C_0(S) \odot C_0(T)$. Corollary B.22 implies that every irreducible representation of $C_0(S) \otimes_\alpha C_0(T)$ has the form $\epsilon_s \otimes \epsilon_t$; thus

$$(C_0(S) \otimes_\alpha C_0(T))^\wedge = E := \{\, (s,t) \in S \times T : \epsilon_s \odot \epsilon_t \text{ is } \alpha\text{-continuous} \,\},$$

and for every $b \in C_0(S) \otimes_\alpha C_0(T)$ we have

$$\|b\|_\alpha = \sup\{\, |\epsilon_s \otimes_\alpha \epsilon_t(b)| : (s,t) \in E \,\}. \tag{B.12}$$

We claim that E is a closed subset of $S \times T$. Suppose that $\{\, (s_i,t_i) \,\} \subset E$ and that $(s_i,t_i) \to (s,t)$. We need to verify that $\epsilon_s \odot \epsilon_t$ is α-continuous. But $\epsilon_{s_i} \to \epsilon_s$ and $\epsilon_{t_i} \to \epsilon_t$ in the weak-$*$ topologies. Thus $\epsilon_{s_i} \odot \epsilon_{t_i}(v) \to \epsilon_s \odot \epsilon_t(v)$ for all $v \in C_0(S) \odot C_0(T)$, and by assumption

$$|\epsilon_{s_i} \odot \epsilon_{t_i}(v)| \leq \|v\|_\alpha.$$

It follows that $\epsilon_s \odot \epsilon_t$ is α-continuous. This establishes the claim.

If there is a point $(s_0,t_0) \in S \times T \setminus E$, then there is a neighbourhood $U \times V$ of (s_0,t_0) which does not meet E. There are non-zero functions $f \in C_0(S)$ and $g \in C_0(T)$ such that $f(s) = 0$ if $s \notin U$ and $g(t) = 0$ if $t \notin V$, and then

$$\|f \otimes g\|_\alpha = \sup\{\, |\epsilon_s \otimes \epsilon_t(f \otimes g)| : (s,t) \in E \,\} = 0;$$

since $f \otimes g \neq 0$ in $C_0(S) \odot C_0(T)$, this is impossible, and E must be all of $S \times T$. Thus the right-hand side of (B.12) is independent of $\|\cdot\|_\alpha$, and all the C^*-norms on $C_0(S) \odot C_0(T)$ coincide. □

Our next lemma allows us to reduce questions about nuclearity to algebras with identity. Recall that if A is a C^*-algebra, then we write $\widetilde{A}$ for the C^*-subalgebra of $M(A)$ generated by A and 1; thus $\widetilde{A} = A$ if A has an identity, and $\widetilde{A} = A^1$ otherwise.

Lemma B.42. *If A and B are C^*-algebras, then every C^*-norm on $A \odot B$ extends to C^*-norms on $\widetilde{A} \odot B$, $A \odot \widetilde{B}$, and $\widetilde{A} \odot \widetilde{B}$.*

Proof. It is enough by symmetry to prove that if A has no identity and $\|\cdot\|_\alpha$ is a C^*-norm on $A \odot B$, then $\|\cdot\|_\alpha$ extends to $A^1 \odot B$. Let $\pi : A \otimes_\alpha B \to B(\mathcal{H})$ be a faithful nondegenerate representation. By Corollary B.22 there are faithful nondegenerate representations $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ such that $\pi(a \otimes b) = \pi_A(a)\pi_B(b)$ for all $a \in A$ and $b \in B$. Since A does not have an identity, A^1 embeds naturally in $M(A)$ (see Example 2.41), and hence $\bar{\pi}_A$ is faithful on A^1. Thus $\bar{\pi}_A \otimes \pi_B$ is a faithful representation of $A^1 \odot B$ (see Remark B.4), and we can define $\|t\|_\alpha := \|\bar{\pi}_A \otimes \pi_B(t)\|$ for $t \in A^1 \odot B$. □

Proposition B.43. *Every commutative C^*-algebra is nuclear.*

Proof. By the previous Lemma, it is enough to prove that if X is a compact space and B is a C^*-algebra with identity, then all C^*-norms on $C(X) \odot B$ coincide. We shall do this by finding a formula for the norm of an arbitrary element which is independent of the given C^*-norm. So suppose that $\|\cdot\|_\alpha$ is a C^*-norm on $C(X) \odot B$.

We first claim that if ξ is a pure state on $C(X) \otimes_\alpha B$, and $\phi(f) := \xi(f \otimes 1)$ and $\psi(b) := \xi(1 \otimes b)$, then ϕ and ψ are states such that $\xi = \phi \otimes_\alpha \psi$. They are positive functionals satisfying $\phi(1_{C(X)}) = 1 = \psi(1_B)$, and they are bounded with norm 1 because the norm $\|\cdot\|_\alpha$ is subcross (Corollary B.24); thus they are certainly states. Since self-adjoint elements generate commutative C^*-algebras, it follows from Lemma B.41 that every self-adjoint element of B is $C(X)$-nuclear, and from Lemma B.40 that $\phi \odot \psi$ is α-continuous. Thus to verify the claim, it is enough to check that

$$\xi(f \otimes b) = \xi(f \otimes 1)\xi(1 \otimes b) \qquad \text{for all } f \in C(X) \text{ and } b \in B. \tag{B.13}$$

(This would be trivial if ξ were multiplicative.) However, if ω is a pure state on a C^*-algebra A with identity, then the GNS-representation π_ω is irreducible. Thus for z in the center of A, $\pi_\omega(z)$ belongs to $\pi_\omega(A)'$ and must be scalar; say $\pi_\omega(z) = \lambda 1_{\mathcal{H}_\omega}$. If h_ω is the cyclic vector for π_ω, we then have

$$\lambda = \lambda(h_\omega \mid h_\omega) = (\lambda h_\omega \mid h_\omega) = (\pi_\omega(z)h_\omega \mid h_\omega) = \omega(z),$$

and hence for any $a \in A$,

$$\omega(za) = (\pi_\omega(z)\pi_\omega(a)h_\omega \mid h_\omega) = \omega(z)(\pi_\omega(a)h_\omega \mid h_\omega) = \omega(z)\omega(a). \tag{B.14}$$

Since $f \otimes 1$ belongs to the center of $C(X) \otimes_\alpha B$ and ξ is pure, (B.13) follows from (B.14), and we have verified the claim.

If $t \in C(X) \otimes_\alpha B$ is self-adjoint, then there is a pure state ξ such that $\|t\|_\alpha = |\xi(t)|$ (Lemma A.10); in other words,

$$\|t\|_\alpha = \sup\{\, |\xi(t)| : \xi \in P(C(X) \otimes_\alpha B) \,\}. \tag{B.15}$$

Thus the claim in the previous paragraph implies that

$$\|t\|_\alpha \le \sup\{\, |\phi \otimes_\alpha \psi(t)| : \phi \in S(C(X)) \text{ and } \psi \in S(B) \,\} \le \|t\|_\alpha. \tag{B.16}$$

For any $t \in C(X) \odot B$, we can apply this to the self-adjoint element t^*t, and deduce that

$$\|t\|_\alpha^2 = \|t^*t\|_\alpha = \sup\{\, \phi \odot \psi(t^*t) : \phi \in S(C(X)) \text{ and } \psi \in S(B) \,\}.$$

Since this last expression is independent of $\|\cdot\|_\alpha$, the Proposition follows. □

Proof of Theorem B.38. Once we see that the spatial norm is minimal, it will follow that $\|\cdot\|_\alpha$ is a cross-norm: since both the spatial norm and the maximal norm are cross-norms (Theorem B.9(b) and Proposition B.25), we have

$$\|a\|\,\|b\| = \|a \otimes b\|_\sigma \le \|a \otimes b\|_\alpha \le \|a \otimes b\|_{\max} = \|a\|\,\|b\|.$$

Therefore, in view of Proposition B.39, we only need to prove that $\phi \odot \psi$ is α-continuous for every $\phi \in S(A)$ and $\psi \in S(B)$. Lemma B.42 implies that $\|\cdot\|_\alpha$ extends to a C^*-norm on $\widetilde{A} \odot \widetilde{B}$, and Lemma A.7 that ϕ and ψ extend to states on $\widetilde{A}$ and $\widetilde{B}$; thus we may as well assume that A and B have identities. Now Proposition B.43 implies that each self-adjoint element of B is A-nuclear, and the result follows from Lemma B.40. □

Corollary B.44. *Every C^*-algebra with Hausdorff spectrum is nuclear.*

Proof. This follows immediately from Proposition B.35 and Theorem B.38. □

Theorem B.37 says that for CCR algebras with Hausdorff spectrum, the map $(\pi, \eta) \mapsto \pi \otimes \eta$ is a homeomorphism of $\hat{A} \times \hat{B}$ onto $(A \otimes_\sigma B)^\wedge$. For general C^*-algebras, the same proof shows that it is a homeomorphism of $\hat{A} \times \hat{B}$ *into* $(A \otimes_\sigma B)^\wedge$, and we will show that it is still surjective if either A or B is GCR. When A is GCR, $\pi \mapsto \ker \pi$ is a homeomorphism of $\hat{A}$ onto $\operatorname{Prim} A$, and it doesn't make any difference whether we work with spectra or primitive ideal spaces. For more general classes of algebras, it does make a difference, and it is often easier to describe $\operatorname{Prim}(A \otimes_\sigma B)$. We will show that, provided A or B is nuclear, we can identify $\ker(\pi \otimes \eta)$ with the ideal $(\ker \pi) \otimes_\sigma B + A \otimes_\sigma (\ker \eta)$, and deduce from the previous results that $(P, Q) \mapsto P \otimes_\sigma B + A \otimes_\sigma Q$ is a homeomorphism of $\operatorname{Prim} A \times \operatorname{Prim} B$ into $\operatorname{Prim}(A \otimes_\sigma B)$. Under the same nuclearity hypothesis, this homeomorphism is surjective, whereas the map on spectra need not be. Without the nuclearity hypothesis, the map on primitive ideal spaces need not be surjective either [165].

Theorem B.45. *Suppose that A and B are C^*-algebras.*

(a) *The map $(\pi, \eta) \mapsto \pi \otimes \eta$ induces a homeomorphism Φ of $\hat{A} \times \hat{B}$ onto its range in $(A \otimes_\sigma B)^\wedge$.*

(b) *If either A or B is GCR, then this homeomorphism is surjective.*

(c) *If A and B are separable and if either A or B is nuclear, there is a homeomorphism of $\operatorname{Prim} A \times \operatorname{Prim} B$ onto $\operatorname{Prim}(A \otimes_\sigma B)$ which takes (P, Q) to $P \otimes_\sigma B + A \otimes_\sigma Q$.*

Remark B.46. Separability is used in (c) to ensure that prime ideals are primitive (see Theorem A.49). Since this is the only way separability is used, the Theorem also applies to other algebras in which prime ideals are known to be primitive: for example, to algebras whose primitive ideal space is second countable or almost Hausdorff (Theorem A.50). To prove (c), we also need to know that quotients of nuclear C^*-algebras are nuclear; this is a deep result, but in view of the current relaxation of our policy on prerequisites, we refer to [17, Corollary 4] for the details.

For the proofs of the first two parts we need some facts about *von Neumann algebras*: C^*-subalgebras of $B(\mathcal{H})$ which are closed in the weak operator topology. The *commutant* of a subset $\mathcal{S} \subset B(\mathcal{H})$ is the set

$$\mathcal{S}' := \{ T \in B(\mathcal{H}) : TS = ST \text{ for all } S \in \mathcal{S} \};$$

if $\mathcal{S}$ is closed under taking adjoints, then $\mathcal{S}'$ is a von Neumann algebra. Von Neumann's Double Commutant Theorem [111, Theorem 4.1.5] says that if $\mathcal{S}$ is a nondegenerate self-adjoint subalgebra of $B(\mathcal{H})$, the *double commutant* $\mathcal{S}'' := (\mathcal{S}')'$ coincides with the weak operator closure of $\mathcal{S}$. The tensor product of von Neumann algebras $\mathcal{A} \subset B(\mathcal{H})$ and $\mathcal{B} \subset B(\mathcal{V})$ is the von Neumann algebra $\mathcal{A} \bar{\otimes} \mathcal{B}$ on $\mathcal{H} \otimes \mathcal{V}$ generated by $\mathcal{A} \odot \mathcal{B}$. If $\mathcal{A}$ is a von Neumann algebra on $\mathcal{H}_1$, then

$$\mathcal{A} \otimes 1_{\mathcal{H}_2} := \{ T \otimes 1 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2) : T \in \mathcal{A} \}$$

is a von Neumann algebra on $\mathcal{H}_1 \otimes \mathcal{H}_2$ with

$$(\mathcal{A} \otimes 1_{\mathcal{H}_2})' = \mathcal{A}' \bar{\otimes} B(\mathcal{H}_2) \tag{B.17}$$

(see, e.g., [29, I.2 Proposition 4]). More generally, we have $(\mathcal{A} \bar{\otimes} \mathcal{B})' = \mathcal{A}' \bar{\otimes} \mathcal{B}'$ for any von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ [159, Theorem 5.9].

A representation π of a C^*-algebra A is irreducible iff $\pi(A)' = \mathbb{C}1_{\mathcal{H}_\pi}$ (Lemma A.1), and hence it follows from the Double Commutant Theorem that π is irreducible iff $\pi(A)'' = B(\mathcal{H}_\pi)$. A representation $\pi : A \to B(\mathcal{H}_\pi)$ is called a *factor representation* if the von Neumann algebra $\pi(A)''$ generated by its range is a factor; that is, if the center $\pi(A)'' \cap \pi(A)'$ of $\pi(A)''$ contains only scalar multiples of the identity operator $1_{\mathcal{H}_\pi}$. We will need the nontrivial result that GCR algebras are Type I [28, Theorem 5.5.2], which implies that every factor representation of a GCR algebra is equivalent to one of the form $\pi \otimes 1_{\mathcal{H}}$ for some irreducible representation π [28, Proposition 5.4.11].

We proved parts (a) and (b) of Theorem B.45 for CCR algebras A and B with Hausdorff spectrum in Theorem B.37. We used the extra hypotheses in two ways: to show that if $\pi \otimes 1_{\mathcal{H}_1}$ is unitarily equivalent to $\eta \otimes 1_{\mathcal{H}_2}$, then π and η are equivalent, and to allow application of Proposition B.35 to deduce surjectivity of Φ. Therefore parts (a) and (b) will follow from the argument used in Theorem B.37 and the next two lemmas.

Lemma B.47. *Suppose that π and η are irreducible representations of a C^*-algebra A. If there are Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\pi \otimes 1_{\mathcal{H}_1}$ and $\eta \otimes 1_{\mathcal{H}_2}$ are unitarily equivalent, then π and η are unitarily equivalent.*

Proof. Let $U : \mathcal{H}_\pi \otimes \mathcal{H}_1 \to \mathcal{H}_\eta \otimes \mathcal{H}_2$ be a unitary intertwining $\pi \otimes 1_{\mathcal{H}_1}$ and $\eta \otimes 1_{\mathcal{H}_2}$. Let p be a rank-one projection on $\mathcal{H}_1$, and set $P := 1_{\mathcal{H}_\pi} \otimes p$, $Q := UPU^*$. Since π and η are irreducible, we have $\pi(A)'' = B(\mathcal{H}_\pi)$ and $\eta(B)'' = B(\mathcal{H}_\eta)$, and it follows from (B.17) that $(\pi \otimes 1_{\mathcal{H}_1})(A)' = 1_{\mathcal{H}_\pi} \otimes B(\mathcal{H}_1)$ and $(\eta \otimes 1_{\mathcal{H}_2})(A)' = 1_{\mathcal{H}_\eta} \otimes B(\mathcal{H}_2)$. Thus P is a minimal projection in $(\pi \otimes 1_{\mathcal{H}_1})(A)'$; conjugation by U is an isomorphism of $\pi \otimes 1(A)'$ onto $\eta \otimes 1(A)'$, so this implies that Q is a minimal projection in $(\eta \otimes 1_{\mathcal{H}_2})(A)'$. Thus $Q = 1_{\mathcal{H}_\eta} \otimes q$ for some rank-one projection q in $B(\mathcal{H}_2)$. Notice that UP implements an equivalence between $\pi \otimes 1_{\mathcal{H}_1}|_{P(\mathcal{H}_\pi \otimes \mathcal{H}_1)}$ and $\eta \otimes 1_{\mathcal{H}_2}|_{Q(\mathcal{H}_\eta \otimes \mathcal{H}_2)}$; since these representations are equivalent to π and η, the result follows. □

Lemma B.48. *Suppose that A and B are C^*-algebras and that either A or B is a GCR algebra. If $\|\cdot\|_\gamma$ is a C^*-norm on $A \odot B$, then every irreducible representation of $A \otimes_\gamma B$ is equivalent to one of the form $\pi \otimes \eta$ for some $\pi \in \hat{A}$ and $\eta \in \hat{B}$.*

Proof. Suppose that A is GCR and $\mu \in (A \otimes_\gamma B)^\wedge$; let μ_A and μ_B denote the corresponding commuting representations of A and B, as in Corollary B.22. Everything in $\mu_A(A)'' \cap \mu_A(A)'$ commutes with both $\mu_B(B)$ and $\mu_A(A)$, and therefore with everything in $\mu(A \otimes_\gamma B)'' = B(\mathcal{H}_\mu)$; thus μ_A is a factor representation of A, and is equivalent to $\pi \otimes 1_{\mathcal{H}_0}$ for some $\pi \in \hat{A}$ and some Hilbert space $\mathcal{H}_0$. Since

$\mu_B(B) \subset \mu_A(A)'$, this equivalence takes $\mu_B(B)$ into $(\pi(A) \otimes 1)' = 1_{\mathcal{H}_\pi} \otimes B(\mathcal{H}_0)$, and μ_B into $1_{\mathcal{H}_\pi} \otimes \eta$ for some representation η of B; since

$$1 \otimes \eta(B)' = (B(\mathcal{H}_\pi) \bar{\otimes} \eta(B))' \subset (1 \otimes \eta(B))',$$

η is a factor representation. Thus we can assume that $\mu = \pi \otimes \eta$ where π is irreducible and η is a factor representation. But then

$$\mu(A \otimes_\gamma B)'' = \big(\pi(A) \otimes \eta(B)\big)'' = \pi(A)'' \bar{\otimes} \eta(B)'' = B(\mathcal{H}_\pi) \bar{\otimes} \eta(B)''.$$

Since μ is irreducible,

$$\mu(A \otimes_\gamma B)'' = B(\mathcal{H}_\pi \otimes \mathcal{H}_0) = B(\mathcal{H}_\pi) \bar{\otimes} B(\mathcal{H}_0).$$

Thus $\eta(B)'' = B(\mathcal{H}_0)$, and η is irreducible. □

Corollary B.49. *Every GCR C^*-algebra is nuclear.*

Proof. Since the norm in a C^*-algebra is determined by its irreducible representations (Theorem A.14), the corollary follows from the Lemma, Theorem B.9(c) and Theorem B.38. □

It remains to prove part (c) of Theorem B.45. The idea is to prove that the homeomorphism of part (a) descends to a map on primitive ideals by identifying the kernel of $\pi \otimes \eta$ in terms of $\ker \pi$ and $\ker \eta$. This is a special case of the next lemma; for it, recall that the sum $I + J := \{a + b : a \in I, b \in J\}$ of two ideals is itself a closed ideal.

Lemma B.50. *Suppose that A or B is nuclear and that $\pi : A \otimes_\sigma B \to B(\mathcal{H}_\pi)$ is an irreducible representation. Let $\pi_A : A \to B(\mathcal{H}_\pi)$ and $\pi_B : B \to B(\mathcal{H}_\pi)$ be the commuting representations such that $\pi(a \otimes b) = \pi_A(a)\pi_B(b)$, as in Corollary B.22. Then*

$$\ker \pi = (\ker \pi_A) \otimes_\sigma B + A \otimes_\sigma (\ker \pi_B).$$

Proof. Let $I = (\ker \pi_A) \otimes_\sigma B + A \otimes_\sigma (\ker \pi_B)$. Then I is a closed ideal in $A \otimes_\sigma B$, and it is contained in $\ker \pi$. Let $w : A \otimes_\sigma B \to (A \otimes_\sigma B)/I$ be the quotient map. It will suffice to show that for each $\sum_{i=1}^k a_i \otimes b_i \in A \odot B$,

$$\Big\|w\Big(\sum\nolimits_{i=1}^k a_i \otimes b_i\Big)\Big\| \le \Big\|\pi\Big(\sum\nolimits_{i=1}^k a_i \otimes b_i\Big)\Big\| = \Big\|\sum\nolimits_{i=1}^k \pi_A(a_i)\pi_B(b_i)\Big\|.$$

The map $\big(\pi_A(a), \pi_B(b)\big) \mapsto w(a \otimes b)$ is well-defined and bilinear, so it induces a linear map $v : \pi_A(A) \odot \pi_B(B) \to (A \otimes_\sigma B)/I$, which is easily seen to be a $*$-homomorphism. Thus

$$\Big\|\sum\nolimits_i \pi_A(a_i) \otimes \pi_B(b_i)\Big\|_v := \Big\|w\Big(\sum\nolimits_i a_i \otimes b_i\Big)\Big\|$$

defines a C^*-seminorm $\|\cdot\|_\nu$ on $\pi_A(A) \odot \pi_B(B)$.

On the other hand, there is a $*$-homomorphism ϕ from $\pi_A(A) \odot \pi_B(B)$ into $B(\mathcal{H}_\pi)$ such that

$$\phi\Big(\sum\nolimits_{i=1}^k \pi_A(a_i) \otimes \pi_B(b_i)\Big) = \sum\nolimits_{i=1}^k \pi_A(a_i)\pi_B(b_i).$$

We claim that ϕ is injective. Suppose that $\sum_{i=1}^k \pi_A(a_i)\pi_B(b_i) = 0$. We can assume without changing the element $\sum \pi_A(a_i) \otimes \pi_B(b_i)$ that the $\pi_A(a_i)$ are linearly independent. Since $\pi_A(A)'' \cap \pi_B(B)'' = \mathbb{C}1_{\mathcal{H}_\pi}$, Proposition 7 of [29, I.2] shows that there are scalars $\{\lambda_{ij}\}_{i,j=1}^k$ such that

$$\begin{aligned} 0 &= \sum\nolimits_{i=1}^k \lambda_{ij}\pi_A(a_i) \quad \text{for } j = 1, \dots, k, \text{ and} \\ \pi_B(b_i) &= \sum\nolimits_{j=1}^k \lambda_{ij}\pi_B(b_j) \quad \text{for } i = 1, \dots, k. \end{aligned}$$

The first equation implies that $\lambda_{ij} = 0$ for all i, j, and the second that

$$\sum\nolimits_{i=1}^k \pi_A(a_i) \otimes \pi_B(b_i) = \sum\nolimits_{i,j=1}^k \lambda_{ij}\pi_A(a_i) \otimes \pi_B(b_j) = 0;$$

this proves the claim. Since ϕ is injective, we can use it to pull back the norm from $B(\mathcal{H}_\pi)$, and obtain a C^*-norm $\|\cdot\|_\beta$ on $\pi_A(A) \odot \pi_B(B)$ such that

$$\Big\|\sum\nolimits_{i=1}^k \pi_A(a_i) \otimes \pi_B(b_i)\Big\|_\beta := \Big\|\sum\nolimits_{i=1}^k \pi_A(a_i)\pi_B(b_i)\Big\|_{B(\mathcal{H}_\pi)}.$$

If A is nuclear, then $\pi_A(A) = A/(\ker \pi_A)$ is nuclear [17, Corollary 4], and $\|\cdot\|_\beta$ must coincide with the maximal norm. Since the maximal norm dominates every C^*-seminorm (Proposition B.25), we deduce that for $\sum a_i \otimes b_i \in A \odot B$,

$$\begin{aligned} \Big\|w\Big(\sum a_i \otimes b_i\Big)\Big\| &= \Big\|\sum\nolimits_i \pi_A(a_i) \otimes \pi_B(b_i)\Big\|_v \le \Big\|\sum\nolimits_i \pi_A(a_i) \otimes \pi_B(b_i)\Big\|_{\max} \\ &= \Big\|\sum\nolimits_i \pi_A(a_i) \otimes \pi_B(b_i)\Big\|_\beta = \Big\|\sum\nolimits_i \pi_A(a_i)\pi_B(b_i)\Big\|, \end{aligned}$$

as required. The same argument applies if B is nuclear, and hence the lemma is proved. □

Lemma B.51. *Suppose that $\pi : A \to B(\mathcal{H})$ is a factor representation of a C^*-algebra. Then the kernel of π is a prime ideal.*

Proof. Suppose that I and J are ideals in A with $IJ \subset \ker \pi$ and $I \not\subset \ker \pi$. Then $\mathcal{V} := \overline{\pi(I)\mathcal{H}}$ is a non-zero invariant subspace for π, and the orthogonal projection P of $\mathcal{H}$ onto $\mathcal{V}$ belongs to $\pi(A)'$. If $\{e_i\}$ is an approximate identity for I, then $\pi(e_i) \to P$ in the strong operator topology. Thus P belongs to $\pi(A)'' \cap \pi(A)' = \mathbb{C}1_{\mathcal{H}}$, and must be a scalar multiple of the identity. In particular, this implies that $\mathcal{V} = \mathcal{H}$. But then $JI = IJ \subset \ker \pi$ implies that $\{0\} = \overline{\pi(J)\pi(I)\mathcal{H}} = \overline{\pi(J)\mathcal{H}}$, and we have $J \subset \ker \pi$. □

Proof of Theorem B.45(c). If $\pi \in \hat{A}$ and $\eta \in \hat{B}$, then $(\pi \otimes \eta)_A = \pi \otimes 1$ and $(\pi \otimes \eta)_B = 1 \otimes \eta$. Thus we can apply Lemma B.50 to the representation $\pi \otimes \eta$ to deduce that

$$\ker(\pi \otimes \eta) = (\ker \pi) \otimes_\sigma B + A \otimes_\sigma (\ker \eta).$$

This equality shows that the map Φ of part (a) induces a well-defined map Ψ of $\operatorname{Prim} A \times \operatorname{Prim} B$ into $\operatorname{Prim}(A \otimes_\sigma B)$ such that

$$\Psi(P, Q) = P \otimes_\sigma B + A \otimes_\sigma Q$$

and

$$\begin{array}{ccc} \hat{A} \times \hat{B} & \xrightarrow{\quad\Phi\quad} & (A \otimes_\sigma B)^\wedge \\ \downarrow & & \downarrow \\ \operatorname{Prim} A \times \operatorname{Prim} B & \xrightarrow[\quad\Psi\quad]{} & \operatorname{Prim}(A \otimes_\sigma B) \end{array}$$

commutes. Because the vertical arrows are continuous open surjections, the continuity of Φ implies that Ψ is continuous.

To see that Ψ is injective, suppose that $\ker(\pi \otimes \eta) = \ker(\pi_1 \otimes \eta_1)$. If $\ker \pi \neq \ker \pi_1$, we may suppose there exists $a \in A$ such that $\pi(a) \neq 0$ and $\pi_1(a) = 0$ (otherwise swap π and π_1). But now if $\eta(b) \neq 0$, then $a \otimes b \in \ker(\pi_1 \otimes \eta_1)$ and $a \otimes b \notin \ker(\pi \otimes \eta)$, which is a contradiction. So we must have $\ker \pi = \ker \pi_1$, and similarly $\ker \eta = \ker \eta_1$. To establish surjectivity of Ψ, suppose that $\pi \in (A \otimes_\sigma B)^\wedge$, so that $\ker \pi$ is a typical element of $\operatorname{Prim}(A \otimes_\sigma B)$. Let π_A and π_B be the commuting representations such that $\pi(a \otimes b) = \pi_A(a)\pi_B(b)$. Then

$$\pi_A(A)'' \cap \pi_A(A)' \subset \pi_B(B)' \cap \pi_A(A)' = \pi(A \otimes_\sigma B)' = \mathbb{C}1_{\mathcal{H}_\pi}$$

and similarly for π_B, so π_A and π_B are factor representations. Thus $\ker \pi_A$ and $\ker \pi_B$ are prime by Lemma B.51, primitive by Theorem A.49, and the equality

$$\ker \pi = (\ker \pi_A) \otimes_\sigma B + A \otimes_\sigma (\ker \pi_B)$$

of Lemma B.50 says that Ψ is surjective.

We have now seen that Ψ is a continuous bijection. To see that its inverse is continuous, suppose $\Psi(P_i, Q_i) \to \Psi(P, Q)$ in $\operatorname{Prim}(A \otimes_\sigma B)$; it is enough to show that this implies $P_i \to P$ in $\operatorname{Prim} A$. Let J be a closed ideal in A which is not contained in P, so that

$$\mathcal{O}_J = \{K \in \operatorname{Prim} A : J \not\subset K\}$$

is a typical open neighbourhood of P. Then $\mathcal{O}_{J \otimes_\sigma B}$ is a neighbourhood of $\Psi(P, Q)$ in $\operatorname{Prim}(A \otimes_\sigma B)$, so eventually $\Psi(P_i, Q_i) \in \mathcal{O}_{J \otimes_\sigma B}$. But $J \subset P_i$ implies $J \otimes_\sigma B \subset P_i \otimes_\sigma B \subset \Psi(P_i, Q_i)$, so $\Psi(P_i, Q_i) \in \mathcal{O}_{J \otimes_\sigma B}$ implies $P_i \in \mathcal{O}_J$. Thus Ψ^{-1} is continuous, and Ψ is a homeomorphism. □

Hooptedoodle B.52. The first part of Theorem B.45 can be found in [172, 173]. (Wulfsohn has pointed out that his proof of Theorem B.45(c) in [174] is not valid in the generality claimed there, though it is valid for GCR algebras.) One can deduce from Theorem B.45 that $A \otimes_\sigma B$ is GCR (respectively, CCR) if and only if both A and B are GCR (respectively, CCR); see [172, 173], [162, Theorem 2], and the references given in [162]. It is also proved in [162, Theorem 2] that $A \otimes_\sigma B$ has continuous trace if and only if both A and B do; the "if" direction is contained in Corollary 3.37, Proposition 5.15, and Lemma 5.30.

Hooptedoodle B.53. The class of nuclear C^*-algebras occurs often in applications, partly because their tensor products are tractable, but also because the class is stable under standard constructions. Thus, for example, ideals and quotients of nuclear algebras are nuclear. Conversely, it is closed under taking extensions: if

$$0 \longrightarrow I \xrightarrow{\ i\ } A \xrightarrow{\ q\ } A/I \longrightarrow 0$$

is exact and both A/I and I are nuclear, then so is A. To see this, let B be a C^*-algebra. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I \otimes_{\max} B & \xrightarrow{i\otimes_{\max}\mathrm{id}} & A \otimes_{\max} B & \xrightarrow{q\otimes_{\max}\mathrm{id}} & A/I \otimes_{\max} B & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & I \otimes_\sigma B & \xrightarrow{i\otimes\mathrm{id}} & A \otimes_\sigma B & \xrightarrow{q\otimes\mathrm{id}} & A/I \otimes_\sigma B & \longrightarrow & 0
\end{array}$$

in which the top row is exact by Proposition B.30. A quick diagram chase shows that the middle vertical arrow is injective, and hence isometric.

Other constructions which preserve nuclearity include taking tensor products with nuclear algebras, forming inductive limits of nuclear algebras, and taking crossed products of nuclear algebras by actions of amenable locally compact groups; these stability properties are surveyed in [93]. More recently, it has been shown that crossed products of nuclear C^*-algebras by coactions of arbitrary locally compact groups are nuclear [133]. Theorems of Connes [19] and Haagerup [68] show that the nuclear C^*-algebras are precisely the C^*-algebras which are amenable in the sense of Johnson [80].

Appendix C

The Imprimitivity Theorem

In this appendix we shall give a more complete treatment of the fundamental examples which we discussed briefly at the ends of sections 2.1 and 2.4. Specifically, we shall show that if H is a closed subgroup of a locally compact group G, then $C_c(G)$ can be completed to form a Hilbert module $\mathsf{X}_{C^*(H)}$ over the group C^*-algebra of H, and identify $\mathcal{K}(\mathsf{X}_{C^*(H)})$ with the transformation group C^*-algebra $C_0(G/H) \rtimes_\tau G$. We shall then show how to recover the Mackey/Blattner process for inducing unitary representations from H to G, and derive the classical imprimitivity theorem characterizing these induced representations. This last result explains why $\mathcal{K}(\mathsf{X})$ is often referred to as an imprimitivity algebra and why the isomorphism of $\mathcal{K}(\mathsf{X}_{C^*(H)})$ with $C_0(G/H) \rtimes_\tau G$ is sometimes called the (abstract) Imprimitivity Theorem.

We postponed this discussion because we did not want to assume background knowledge of locally compact groups and vector-valued integration, and here we will begin with a brief review of this necessary background. In Section C.1 we discuss Haar measure and quasi-invariant measures on homogeneous spaces, and in Section C.2 we consider C^*-algebra-valued integrals of continuous functions with compact support. In Section C.3, we construct the group C^*-algebra $C^*(G)$ of a locally compact group G, and prove that there is a one-to-one correspondence between unitary representations of G and representations of $C^*(G)$. We prove the Imprimitivity Theorem in Section C.4, relate it to the Mackey/Blattner formulation in Section C.5, and prove the Stone-von Neumann Theorem in Section C.6.

C.1 Haar Measure and Measures on Homogeneous Spaces

A measure μ on a locally compact group for which every open set is measurable is called a Borel measure. If for each open set $V \subset G$, $\mu(V) = \sup\{\,\mu(C) : C \text{ is a compact subset of } V\,\}$ and if for each measurable set $A \subset G$, $\mu(A) = \inf\{\,\mu(V) : V \text{ is a open neighbourhood of } A\,\}$, then μ is called *regular*. A measure is said to be *left invariant* if $\mu(g \cdot A) = \mu(A)$ for all measurable sets A and all $g \in G$. A regular left-invariant Borel measure on G is called a *Haar*

measure. Every locally compact group G has a Haar measure, and it is unique up to multiplication by a positive scalar. This Haar measure assigns strictly positive measure to every nonempty open subset of G and finite measure to every compact subset. These properties of Haar measure guarantee that for every $f \in C_c(G)$,

$$\|f\|_1 := \int_G |f(s)|\, d\mu(s)$$

is finite.*

To see that Haar measure exists, one uses the Riesz Representation Theorem, and constructs instead a linear functional I on $C_c(G)$ which is positive in that $I(f) \geq 0$ whenever $f \in C_c^+(G)$, and left invariant in that $I\big(\tau_r(f)\big) = I(f)$ for all $f \in C_c(G)$ and $r \in G$. (Recall that $\tau_r(f)(s) := f(r^{-1}s)$ and $C_c^+(G)$ is the set of functions $f \in C_c(G)$ satisfying $f(s) \geq 0$ for all $s \in G$.) Such a functional is called a *Haar functional.* Details of this argument† and proofs of our assertions about Haar measure can be found in [55, §2.2], [54, Chap. III §7] or [71, §15].

Examples of Haar measure include Lebesgue measure on $\mathbb{R}^n$ and $\mathbb{T}^m$, as well as counting measure on a discrete group. In these examples the Haar measure μ is also right invariant: $\mu(A \cdot g) = \mu(A)$ for all $g \in G$. In general, a left-invariant Haar measure need not be right invariant, but the uniqueness of Haar measure implies that there is a continuous function $\Delta : G \to \mathbb{R}$ such that

$$\Delta(t) \int_G f(st)\, d\mu(s) = \int_G f(s)\, d\mu(s)$$

for each $f \in C_c(G)$. This *modular function* Δ is a continuous homomorphism [55, Proposition 2.24]. When $\Delta \equiv 1$, the left Haar measure is right invariant, and we say the group is *unimodular*; all abelian and compact groups are unimodular, but many important examples such as the $ax + b$ group are not [55, §2.4]. The need to decorate formulas with modular functions will cause us some pains later‡.

Suppose that H is a closed subgroup of a locally compact group G. The homogeneous space G/H will play a pivotal role. Even in simple examples, such as $G = \mathbb{R}$ and $H = \mathbb{Z}$, there need be no continuous cross-section $c : G/H \to G$ for the orbit map $q : G \to G/H$ satisfying $q \circ c = \mathrm{id}$. However, there is always a kind of approximate cross-section which will be prove useful. Here and later, we often write $\dot{s}$ for the coset $sH \in G/H$.

Proposition C.1. *Let G be a locally compact group and H a closed subgroup. Let ν denote Haar measure on H. Then there is a bounded continuous function*

*Of course, $\|\cdot\|_1$ is a norm on $C_c(G)$ and the completion of $\big(C_c(G), \|\cdot\|_1\big)$ is the Banach space $L^1(G)$ of (equivalence classes) of integrable functions on G. Since we wish to allow arbitrary locally compact groups G, but wish to avoid the associated measure-theoretic difficulties, we will work primarily with $C_c(G)$.

†Haar showed in 1933 that every second countable locally compact group had a left-invariant measure [69]. Weil and Kakutani independently extended his result to arbitrary locally compact groups, and uniqueness was proved by von Neumann. Further discussion and references can be found in the notes in [55, §2.7] or at the end of [71, §15].

‡We have elected to include modular functions so that our treatment will be useful for reference. Someone looking at these constructions for the first time might be well-advised to ignore the modular functions.

$b : G \to [0, \infty)$ *such that* $(\operatorname{supp} b) \cap CH$ *is compact for every compact set* C *in* G, *and such that*

$$\int_H b(st)\, d\nu(t) = 1 \quad \textit{for all } s \in G.$$

Such a function is called a Bruhat approximate cross-section for G over H.

Proof. We claim that it suffices to prove the result when G is σ-compact. Let G_0 be a σ-compact open subgroup (such as that generated by any compact neighbourhood of e in G). Then $H_0 = G_0 \cap H$ is a closed subgroup of G_0. Suppose there exists an approximate cross-section b_0 for G_0 over H_0. Since G/G_0 is discrete, any cross-section $c : G/G_0 \to G$ is continuous, and we can define a bounded continuous function b by $b(s) = b_0\big(c(\dot{s})^{-1}s\big)$. Then

$$\int_H b(st)\, d\nu(t) = \int_H b_0\big(c(\dot{s})st\big)\, d\nu(t) = 1 \quad \text{for all } s \in G.$$

If C is compact then there are $s_1, \ldots, s_N \in G$ such that $s_i = c(\dot{s}_i)$ and $C \subset \bigcup_{i=1}^N s_i G_0$. Therefore $\operatorname{supp}(b) \subset \bigcup_{i=1}^N s_i \operatorname{supp}(b_0)$, and it suffices to consider the case where G is σ-compact, as claimed.

Suppose that $G = \bigcup_{n=1}^\infty K_n$ with each K_n compact and contained in the interior K_{n+1}^o of K_{n+1}. (For notational convenience, let $K_0 = K_{-1} = \emptyset$.) Let $C_n = K_n \setminus K_{n-1}^o H$ and $U_n = K_{n+1}^o \setminus K_{n-2}H$. Note that C_n is compact, that U_n is a relatively compact open[§] neighbourhood of C_n, and that $C := \bigcup_{n=1}^\infty C_n$ is closed. Using a partition of unity[¶] on G subordinate to the cover $C^c \cup \bigcup_{n=1}^\infty U_n$, we can find functions $f_n \in C_c^+(G)$ such that $\operatorname{supp}(f_n) \subset U_n$, such that $f := \sum_n f_n$ is in $C_b^+(G)$, and such that $f(s) = 1$ for all $s \in C$.

Now suppose that $A \subset K_n$ and $AH \cap U_m \neq \emptyset$. Then $A \cap U_m H \neq \emptyset$ and $A \cap (K_{m-2}H)^c \neq \emptyset$. It follows that $m \leq n+1$. Therefore the saturation CH of any compact set C has compact intersection with the support of f. We claim that if $s \in G$, then there exists $h \in H$ such that $sh \in C$. For we certainly have $s \in K_n H$ for some n. If $n = 1$, we're done; if $s \notin C_n$, then $s \in K_{n-1}H$, and we proceed by induction.

It follows from the two assertions in the previous paragraph that

$$\tilde{b}(\dot{s}) := \int_H f(st)\, d\nu(t)$$

is finite and nonzero for all $\dot{s} \in G/H$. We obtain a function b with the required properties by defining

$$b(s) = \tilde{b}(\dot{s})^{-1} f(s). \quad \square$$

We would like to choose a measure on the quotient G/H which is compatible with Haar measures on G and H. If H is normal in G, then things are relatively

[§]The product of a compact set and a closed set is always closed.

[¶]Every locally compact group is paracompact by [71, Theorem 8.13]

straightforward. For then G/H is a locally compact group and has a Haar measure α; the map $f \mapsto \int_{G/H} \int_H f(st)\, d\nu(t)\, d\alpha(\dot{s})$ is a Haar functional associated to a Haar measure μ on G such that

$$\int_G f(s)\, d\mu(s) = \int_{G/H} \int_H f(st)\, d\nu(t)\, d\alpha(\dot{s}) \quad \text{for all } f \in C_c(G). \tag{C.1}$$

In general, G/H is not a group, and may not have an invariant measure. However, it always has a *quasi-invariant measure*: a measure α on G/H such that the measure α_r defined by $\alpha_r(E) := \alpha(r^{-1} \cdot E)$ is equivalent to α for each $r \in G$. (Two measures are equivalent when they have the same null sets.) Our goal is to produce such a measure which satisfies a formula as close to (C.1) as possible.

The obstruction to producing an invariant measure has to do with how the Haar measures μ and ν on G and H fail to be right invariant. If H is not normal in G, then the modular function δ on H may not coincide with the restriction of the modular function Δ of G to H.[‖] Our construction of a quasi-invariant measure requires a continuous function ρ such that $\rho(st) = \big(\delta(t)/\Delta(t)\big)\rho(s)$ for all $s \in G$ and $t \in H$; such a function ρ is sometimes called a *rho-function*. If $\delta = \Delta|_H$, then we can take $\rho \equiv 1$, and the measure constructed below will be invariant for the left action of G on G/H.

Lemma C.2. *Suppose that G is a locally compact group with a closed subgroup H, and let μ, ν be Haar measures on G, H, respectively. Given any continuous function $\rho : G \to (0, \infty)$ such that*

$$\rho(st) = \big(\delta(t)/\Delta(t)\big)\rho(s) \quad \textit{for all } s \in G \textit{ and } t \in H, \tag{C.2}$$

there is a quasi-invariant measure α on the Borel sets of G/H such that

$$\int_G f(s)\rho(s)\, d\mu(s) = \int_{G/H} \int_H f(st)\, d\nu(t)\, d\alpha(\dot{s}). \tag{C.3}$$

Furthermore, the Radon-Nikodym derivative of the translate α_r with respect to α is given by

$$\frac{d\alpha_r}{d\alpha}(\dot{s}) = \frac{\rho(r^{-1}s)}{\rho(s)}.$$

There always exist continuous functions ρ satisfying (C.2).

Proof. Suppose that ρ satisfies (C.2). Define $\Phi : C_c(G) \to C_c(G/H)$ by

$$\Phi(f)(\dot{s}) := \int_H f(st)\, d\nu(t).$$

An application of Proposition C.1 shows that Φ is surjective: if $\phi \in C_c(G/H)$, then $(\phi \circ q)b$ belongs to $C_c(G)$ and satisfies $\Phi((\phi \circ q)b) = \phi$. We will construct a positive

[‖] If H is normal, then evaluating both sides of (C.1) on ${}_hf : s \mapsto f(sh)$ shows that the modular functions on G and H coincide on H.

linear functional F on $C_c(G/H)$ and apply the Riesz representation theorem [125, Theorem 6.3.4] to obtain α. We would like to define F by

$$F\big(\Phi(f)\big) = \int_G f(s)\rho(s)\,d\mu(s); \tag{C.4}$$

since Φ is surjective, it is enough to show that the right-hand side of (C.4) depends only on $\Phi(f)$ rather than f. Equivalently, we need to show that if $\Phi(f)$ is identically zero, then the right-hand side of (C.4) is zero.

Suppose that for some $f \in C_c(G)$ we have

$$\int_H f(st)\,d\nu(t) = 0 \quad \text{for all } s \in G. \tag{C.5}$$

By multiplying a Bruhat approximate cross-section b by a compactly supported function which is identically one on $\operatorname{supp}(f)$, we obtain a function $b_1 \in C_c^+(G)$ such that

$$\int_H b_1(st)\,d\nu(t) = 1 \quad \text{for all } s \in \operatorname{supp}(f).$$

(Such a function is sometimes called a *cut-down Bruhat approximate cross-section.*) Define $g \in C_c(G)$ by $g(s) = b_1(s^{-1})\Delta(s^{-1})\rho(s^{-1})$. If (C.5) holds, then

$$\begin{aligned} I := \int_H g * f(t)\,d\nu(t) &= \int_H \int_G g(r) f(r^{-1}t)\,d\mu(r)\,d\nu(t) \\ &= \int_G g(r) \int_H f(r^{-1}t)\,d\nu(t)\,d\mu(r) = 0. \end{aligned}$$

Consequently,

$$\begin{aligned} 0 = I = \int_H g * f(t)\,d\nu(t) &= \int_H \int_G g(tr) f(r^{-1})\,d\mu(r)\,d\nu(t) \\ &= \int_H \int_G b_1(r^{-1}t^{-1})\Delta(r^{-1}t^{-1})\rho(r^{-1}t^{-1}) f(r^{-1})\,d\mu(r)\,d\nu(t) \\ &= \int_G \int_H b_1(rt)\big(\Delta(t)/\delta(t)\big)\rho(rt) f(r)\,d\nu(t)\,d\mu(r) \\ &= \int_G f(r)\rho(r)\,d\mu(r). \end{aligned}$$

Thus the functional F is well-defined on $C_c(G/H)$. To see that F is positive, note that if $\phi \in C_c(G/H)$ is positive, then $f := (\phi \circ q)b$ is a positive function satisfying $\Phi(f) = \phi$, and this choice of f makes the right-hand side of (C.4) positive. We deduce that there is a measure α satisfying (C.3).

Notice that if $\phi \in C_c(G/H)$, then

$$\int_{G/H} \phi(\dot{s})\,d\alpha_r(\dot{s}) = \int_{G/H} \phi(r \cdot \dot{s})\,d\alpha(\dot{s}).$$

Since we can write $\phi = \Phi(f)$, we have

$$\begin{aligned}\int_{G/H} \phi(r\cdot\dot{s})\,d\alpha(\dot{s}) &= \int_G f(rs)\rho(s)\,d\mu(s) = \int_G f(rs)\frac{\rho(s)}{\rho(rs)}\rho(rs)\,d\mu(s)\\ &= \int_G f(s)\frac{\rho(r^{-1}s)}{\rho(s)}\rho(s)\,d\mu(s)\\ &= \int_{G/H}\frac{\rho(r^{-1}s)}{\rho(s)}\int_H f(st)\,d\nu(t)\,d\alpha(\dot{s})\\ &= \int_{G/H}\frac{\rho(r^{-1}s)}{\rho(s)}\phi(\dot{s})\,d\alpha(\dot{s}),\end{aligned}$$

and the statement about Radon-Nikodym derivatives follows.

We have proved everything except the existence of a suitable function ρ. For this, choose $f \in C_c^+(G)$ with $f(e) \neq 0$ and $\int_H f(t)\,d\nu(t) = 1$, and define

$$\omega(s) = \int_H f(sts^{-1})\,d\nu(t) \quad \text{for all } s \in G.$$

We then have $\omega(st) = \delta(t)\omega(s)$, and we can take $\rho(s) := \Delta(s^{-1})\omega(s)$. □

C.2 Vector-Valued Integration on Groups

The theory of integration for functions with values in a Banach space can be quite complicated, but the theory simplifies considerably when one only cares about continuous functions of compact support with values in a C^*-algebra A. We shall further simplify things by only integrating functions with respect to a Haar measure μ on a locally compact group G. For the general theory, we refer to [54, Chap. II].

If $f \in C_c(G, A)$, then $s \mapsto \|f(s)\|$ belongs to $C_c(G)$, and

$$\|f\|_1 := \int_G \|f(s)\|\,d\mu(s)$$

is finite and satisfies $\|f\|_1 \leq \|f\|_\infty\,\mu\big(\operatorname{supp} f\big)$.

Lemma C.3. *Let A be a C^*-algebra. There is a unique linear map $I : C_c(G, A) \to A$ such that for any representation $\pi : A \to B(\mathcal{H}_\pi)$, we have*

$$\big(\pi\big(I(f)\big)h \mid k\big) = \int_G \big(\pi\big(f(s)\big)h \mid k\big)\,d\mu(s) \quad \textit{for all } h, k \in \mathcal{H}_\pi. \tag{C.6}$$

We write $\int_G f(s)\,d\mu(s) := I(f)$. Then

$$\left\|\int_G f(s)\,d\mu(s)\right\| \leq \|f\|_1, \tag{C.7}$$

$$\left(\int_G f(s)\,d\mu(s)\right)^* = \int_G f(s)^*\,d\mu(s), \tag{C.8}$$

and for all $a \in M(A)$ we have

$$\begin{aligned} \left(\int_G f(s)\,d\mu(s)\right) a &= \int_G f(s)a\,d\mu(s) \quad \textit{and} \\ a\left(\int_G f(s)\,d\mu(s)\right) &= \int_G af(s)\,d\mu(s). \end{aligned} \tag{C.9}$$

Finally, if $L : A \to B$ is a bounded linear map into a C^*-algebra B, then

$$L\left(\int_G f(s)\,d\mu(s)\right) = \int_G L\big(f(s)\big)\,d\mu(s). \tag{C.10}$$

Proof. The uniqueness is clear: if π is faithful and $b \in A$, the numbers $\big(\pi(b)h \mid k\big)$ determine b uniquely. Let $\pi : A \to B(\mathcal{H}_\pi)$ be a representation. The right-hand side of (C.6) is a sesquilinear form on $\mathcal{H}_\pi$ which is bounded by $\|f\|_1$. Therefore there is an operator $T_f^\pi \in B(\mathcal{H}_\pi)$ with $\|T_f^\pi\| \le \|f\|_1$ satisfying

$$(T_f^\pi h \mid k) = \int_G \big(\pi\big(f(s)\big)h \mid k\big). \tag{C.11}$$

Note that $f \mapsto T_f^\pi$ is linear in f. We claim that $T_f^\pi \in \pi(A)$.

Let $\epsilon > 0$, $K = \operatorname{supp}(f)$, and let C be a compact neighbourhood of K. Since K is compact, there is a symmetric open neighbourhood V of $e \in G$ such that $KV \subset C$ and $r^{-1}s \in V$ implies

$$\|f(r) - f(s)\| < \epsilon/\mu(C).$$

There are $r_1, \dots, r_n \in K$ such that $K \subset \bigcup_{i=1}^n r_i V$. Using a partition of unity subordinate to the finite cover $\{K^c\} \cup \{r_i V\}$, we can find $\phi_i \in C_c(G)$ such that $0 \le \phi_i \le 1$, $\operatorname{supp}(\phi_i) \subset r_i V$, $\sum_i \phi_i(s) = 1$ if $s \in K$, and $\sum_i \phi_i(s) \le 1$ for all s. Note that $\operatorname{supp}\big(\sum_i \phi_i\big) \subset C$. Then $g_\epsilon := \sum_{i=1}^n f(r_i)\phi_i$ satisfies

$$\begin{aligned} \|f - g_\epsilon\|_1 &= \int_G \|f(s) - g_\epsilon(s)\|\,d\mu(s) \\ &\le \int_G \sum\nolimits_{i=1}^n \|f(s) - f(r_i)\|\phi_i(s)\,d\mu(s) \\ &< \frac{\epsilon}{\mu(C)} \int_G \sum\nolimits_{i=1}^n \phi_i(s)\,d\mu(s) \le \epsilon. \end{aligned} \tag{C.12}$$

Thus $\|T_{g_\epsilon}^\pi - T_f^\pi\| < \epsilon$. But if $c_i := \int_G \phi_i(s)\,d\mu(s)$, then

$$T_{g_\epsilon}^\pi = \sum\nolimits_{i=1}^n c_i \pi\big(f(r_i)\big) \in \pi(A).$$

Since ϵ was arbitrary, it follows that $T_f^\pi \in \pi(A)$, as claimed.

Now let $\rho : A \to B(\mathcal{H}_\rho)$ be a faithful representation and define $I(f) := \rho^{-1}(T_f^\rho)$. Notice that $\rho^{-1}(T_{g_\epsilon}^\rho) = \sum_i c_i f(r_i)$, and since ρ is isometric, $\|I(f) - \sum_i c_i f(r_i)\| < \epsilon$. Therefore $\|\pi\big(I(f)\big) - T_f^\pi\| < 2\epsilon$. Since ϵ is arbitrary, this establishes (C.6).

Properties (C.7), (C.9) and (C.8) follow from the defining property (C.6). Finally, (C.10) follows from the estimates $\|L\big(I(f)\big) - \sum_i c_i L\big(f(r_i)\big)\| < \epsilon\|L\|$ and $\|L \circ g_\epsilon - L \circ f\|_1 < \epsilon\|L\|$. □

Unfortunately, not all the functions we need to consider are norm continuous. Important examples are the unitary representations $U : G \to U(\mathcal{H})$, which are by definition *strongly continuous* in the sense that $s \mapsto U_s h$ is continuous from G to $\mathcal{H}$ for every $h \in \mathcal{H}$.

Definition C.4. Suppose that A is a C^*-algebra and $a \in A$. Let $\|\cdot\|_a$ be the seminorm on $M(A)$ defined by $\|b\|_a = \|ab\| + \|ba\|$. The *strict topology* on $M(A)$ is the topology generated by the seminorms $\{\, \|\cdot\|_a : a \in A \,\}$.

Example C.5. Since $B(\mathcal{H}) \cong M\big(\mathcal{K}(\mathcal{H})\big)$, it has a strict topology. A net $\{\, T_i \,\}_{i\in A}$ in $B(\mathcal{H})$ converges to T strictly if and only if $T_i K \to TK$ and $KT_i \to KT$ for every $K \in \mathcal{K}(\mathcal{H})$. The $*$-strong topology on $B(\mathcal{H})$ has subbasic open sets

$$N(T, h, \epsilon) := \{\, S \in B(\mathcal{H}) : \|(S - T)h\| + \|(S^* - T^*)h\| < \epsilon \,\}$$

parameterized by $T \in B(\mathcal{H})$, $h \in \mathcal{H}$, and $\epsilon > 0$. Therefore a net $\{\, T_i \,\}_{i\in A}$ converges to T $*$-strongly if and only if both $T_i \to T$ and $T_i^* \to T^*$ in the strong operator topology. As the next result shows, these topologies agree on bounded subsets.

Lemma C.6. *On (norm) bounded subsets of $B(\mathcal{H})$, the strict topology coincides with the $*$-strong topology.*

We actually need a more general version of Lemma C.6 in which we replace the Hilbert space $\mathcal{H}$ with a Hilbert A-module X. Recall that $\mathcal{L}(\mathsf{X}) = M\big(\mathcal{K}(\mathsf{X})\big)$ (Corollary 2.54) so that $\mathcal{L}(\mathsf{X})$ has a strict topology. It also has a $*$-strong topology: by analogy with the above, we say that $\{\, T_i \,\}$ in $\mathcal{L}(\mathsf{X})$ converges to T $*$-strongly if and only if both $T_i(x) \to T(x)$ and $T_i^*(x) \to T^*(x)$ in X for all $x \in \mathsf{X}$. Lemma C.6 is a special case of:

Proposition C.7. *Suppose that X is a Hilbert A-module. Then strict convergence in $\mathcal{L}(\mathsf{X}) = M\big(\mathcal{K}(\mathsf{X})\big)$ implies $*$-strong convergence, and the strict topology coincides with the $*$-strong topology on (norm-) bounded subsets of $\mathcal{L}(\mathsf{X})$.*

Proof. Suppose that $T_i \to T$ strictly. Given $x \in \mathsf{X}$, Lemma 2.31 implies that there exists $y \in \mathsf{X}$ such that $x = y \cdot \langle y\ ,\ y\rangle_A = {}_{\mathcal{K}(\mathsf{X})}\langle y\ ,\ y\rangle(y)$. By assumption, both $T_i\,{}_{\mathcal{K}(\mathsf{X})}\langle y\ ,\ y\rangle \to T\,{}_{\mathcal{K}(\mathsf{X})}\langle y\ ,\ y\rangle$ and ${}_{\mathcal{K}(\mathsf{X})}\langle y\ ,\ y\rangle T_i \to {}_{\mathcal{K}(\mathsf{X})}\langle y\ ,\ y\rangle T$ in norm. In particular,

$$T_i(x) = T_i\big({}_{\mathcal{K}(\mathsf{X})}\langle y\ ,\ y\rangle(y)\big) \to T\big({}_{\mathcal{K}(\mathsf{X})}\langle y\ ,\ y\rangle(y)\big) = T(x).$$

Since the adjoint operation is norm continuous on $\mathcal{L}(\mathsf{X})$, the strict convergence also implies that $T_i^*\,{}_{\mathcal{K}(\mathsf{X})}\langle y\ ,\ y\rangle \to T^*\,{}_{\mathcal{K}(\mathsf{X})}\langle y\ ,\ y\rangle$, from which it follows that $T_i^*(x) \to T^*(x)$.

On the other hand, suppose that $T_i \to T$ $*$-strongly and that $\|T_i\| \le M$ for all i. For any fixed $x, y \in \mathsf{X}$, $T_i(x) \to T(x)$ implies that $T_i\,{}_{\mathcal{K}(\mathsf{X})}\langle x\ ,\ y\rangle \to T\,{}_{\mathcal{K}(\mathsf{X})}\langle x\ ,\ y\rangle$ in norm in $\mathcal{L}(\mathsf{X})$. Similarly, we have $T_i^*\,{}_{\mathcal{K}(\mathsf{X})}\langle x\ ,\ y\rangle \to T^*\,{}_{\mathcal{K}(\mathsf{X})}\langle x\ ,\ y\rangle$ in norm. It follows

that for any finite sum F of such basic operators, $T_iF \to TF$ and $T_i^*F \to T^*F$ in norm. If $K \in \mathcal{K}(\mathsf{X})$ and $\epsilon > 0$, there is a finite sum F such that $\|K - F\| < \epsilon/3M$. Choose i_0 such that $i \geq i_0$ implies $\|T_iF - TF\| < \epsilon/3$. Then for $i \geq i_0$, we have

$$\|T_iK - TK\| \leq \|T_iK - T_iF\| + \|T_iF - TF\| + \|TF - TK\| < \epsilon.$$

We can similarly deduce from $T_i^*F \to TF$ that $\|KT_i - KT\| = \|T_i^*K^* - T^*K^*\| \to 0$. □

For homomorphisms $u : G \to U\mathcal{L}(\mathsf{X})$, there are apparently several different different notions of continuity. We can ask that u be strictly continuous in the sense that $s \mapsto u_s$ is continuous from G into $\mathcal{L}(\mathsf{X})$ with the strict topology. Or we could, by analogy with unitary representations, ask that u be strongly continuous in the sense that $s \mapsto u_s(x)$ is continuous for all $x \in X$. Fortunately these notions coincide.

Corollary C.8. *If $u : G \to U\mathcal{L}(\mathsf{X})$ is a homomorphism into the unitary group of $\mathcal{L}(\mathsf{X})$, then u is strictly continuous if and only if it is strongly continuous.*

Proof. Since each u_s has norm one, Proposition C.7 implies that is is enough to prove that if $s \mapsto u_s(x)$ is continuous for all $x \in \mathsf{X}$, then $s \mapsto u_s^*(x)$ is continuous. The proof is similar to that of Lemma 1.3(a):

$$\begin{aligned}\|u_s^*(x) - u_t^*(x)\|_A^2 &= \langle u_s^*(x) - u_t^*(x)\ ,\ u_s^*(x) - u_t^*(x)\rangle_A \\ &= 2\|x\|_A^2 - \langle u_s u_t^*(x)\ ,\ x\rangle_A - \langle x\ ,\ u_s u_t^*(x)\rangle. \end{aligned} \tag{C.13}$$

If $s \to t$, then by assumption $u_s u_t^*(x) \to u_t u_t^*(x) = x$; therefore (C.13) converges to 0 as $s \to t$. □

Remark C.9. Proposition C.7 says that strictly convergent nets in $B(\mathcal{H})$ or $\mathcal{L}(\mathsf{X})$ always converge $*$-strongly. When $\mathcal{H}$ is infinite-dimensional, the next example shows that the converse can fail for unbounded nets. Since the principle of uniform boundedness implies that a strongly convergent *sequence* is bounded, and hence strictly convergent by Proposition C.7, the example also shows that even on separable Hilbert space the $*$-strong topology is not first countable, so that it is necessary to use nets instead of sequences.

Example C.10. Let $\{\, e_n \,\}_{n=1}^{\infty}$ be an orthonormal set in $\mathcal{H}$. For each $n \geq 1$, define $T_n \in B(\mathcal{H})$ by

$$T_n h = \sqrt{n}(e_n \otimes \overline{e_n})(h) = \sqrt{n}(h \mid e_n)e_n.$$

We will show that 0 belongs to the $*$-strong closure of $S := \{\, T_n : n \geq 1 \,\}$. Since the operators in S are self-adjoint, it is enough to prove that for every $\epsilon > 0$ and $h_1, \ldots, h_k \in \mathcal{H}$, S meets the neighbourhood $U = \{\, T \in B(\mathcal{H}) : \|Th_i\| < \epsilon \text{ for } i = 1, \ldots, k \,\}$ of 0. If $T_n \notin U$, then

$$\sum\nolimits_{i=1}^{k} \|T_n h_i\|^2 \geq \epsilon^2, \tag{C.14}$$

which implies that $\sum_{i=1}^{k} |(h_i \mid e_n)|^2 \geq \epsilon^2/n$. But

$$\sum\nolimits_{i=1}^{k} \|h_i\|^2 \geq \sum\nolimits_{i=1}^{k} \sum\nolimits_{n=1}^{\infty} |(h_i \mid e_n)|^2 = \sum\nolimits_{n=1}^{\infty} \sum\nolimits_{i=1}^{k} |(h_i \mid e_n)|^2,$$

so (C.14) can't hold for all n, and S must meet U.

Now define K to be zero on $\big(\{\, e_n \,\}_{n=1}^{\infty}\big)^{\perp}$, and satisfy $Ke_n = (1/\sqrt{n})e_n$. Then K is compact, but $\|T_n K\| = 1 = \|KT_n\|$, so the sequence $\{\, T_n K \,\}$ cannot converge in norm. Thus 0 is not in the strict closure of S.

We will write $M_s(A)$ for the space $M(A)$ with the strict topology. If $f \in C_c\big(G, M_s(A)\big)$, then for each $a \in A$ the function $s \mapsto f(s)a$ belongs to $C_c(G, A)$. In particular, $\{\, \|f(s)a\| : s \in G \,\}$ is bounded for each $a \in A$. The principle of uniform boundedness implies that $\{\, \|f(s)\| : s \in G \,\}$ is bounded, so that $C_c\big(G, M_s(A)\big)$ consists of norm-bounded functions.* Since any norm-decreasing nondegenerate representation π of $C_c(G)$ extends to a nondegenerate representation of $C^*(G)$, π has a canonical extension $\bar{\pi}$ to $M\big(C^*(G)\big)$ (Proposition 2.50).

Lemma C.11. *Let A be a C^*-algebra. There is a unique linear map $f \mapsto \int_G f(s)\, d\mu(s)$ from $C_c\big(G, M_s(A)\big)$ to $M(A)$ such that for any nondegenerate representation $\pi : A \to B(\mathcal{H}_\pi)$*

$$\Big(\bar{\pi}\Big(\int_G f(s)\, d\mu(s)\Big)h \mid k\Big) = \int_G \big(\bar{\pi}\big(f(s)\big)h \mid k\big) \quad \textit{for all } h, k \in \mathcal{H}_\pi. \tag{C.15}$$

We have

$$\left\| \int_G f(s)\, d\mu(s) \right\| \leq \|f\|_\infty \cdot \mu(\operatorname{supp} f).$$

Equations (C.8) *and* (C.9) *are valid in this context. If $L : A \to B$ is a nondegenerate homomorphism into a C^*-algebra B, then* (C.10) *holds as well.*

Proof. Since $\bar{\pi}\big(f(s)\big)\pi(a)h = \pi\big(f(s)a\big)h$, the function $s \mapsto \bar{\pi}\big(f(s)\big)k$ is continuous for all $k \in \pi(A)\mathcal{H}_\pi$. Since π is nondegenerate and f is norm-bounded, it follows that $s \mapsto \bar{\pi}\big(f(s)\big)$ is strongly continuous. Therefore $s \mapsto \big(\bar{\pi}\big(f(s)\big)h \mid k\big)$ is continuous and bounded by $\|f\|_\infty \cdot \mu(\operatorname{supp} f)\|h\|\,\|k\|$. Thus, as in the first part of the proof of Lemma C.3, there is an operator T_f^π in $B(\mathcal{H}_\pi)$ satisfying (C.11) (with $\bar{\pi}$ in place of π). If ρ is faithful as well as nondegenerate, then Proposition 2.53 implies that $\bar{\rho}$ is an isomorphism of $M(A)$ onto

$$C = \{\, T \in B(\mathcal{H}_\rho) : T\rho(A) \subset \rho(A) \text{ and } \rho(A)T \subset \rho(A) \,\}.$$

*It is not true that $s \mapsto \|f(s)\|$ need be continuous in this case. For example, consider the algebra $A = \{\, a \in C([0,1]) : a(0) = 0 \,\}$, and let $f(s)$ be the function which is 1 at 0, identically zero if $t \geq s$, and linear from 0 to s. Then multiplication by $f(s)$ is a multiplier on A, and $f(s)$ converges strictly to 0 as $s \to 0$, but $\|f(s)\| = 1$ if $s \neq 0$.

But

$$\begin{aligned}\big(T_f^\rho \rho(a)h \mid k\big) &= \int_G \big(\bar\rho(f(s))\rho(a)h \mid k\big)\, d\mu(s)\\ &= \int_G \big(\rho(f(s)a)h \mid k\big)\, d\mu(s)\\ &= \Big(\rho\Big(\int_G f(s)a\, d\mu(s)\Big)h \mid k\Big),\end{aligned}$$

where the last equality follows by applying Lemma C.3 to $g(s) := f(s)a$. Therefore $T_f^\rho \rho(a) \in \rho(A)$. Since a similar argument implies that $\rho(a)T_f^\rho \in \rho(A)$, we have $T_f^\rho \in C = \bar\rho\big(M(A)\big)$. Thus, we can define $\int_G f(s)\, d\mu(s) := (\bar\rho)^{-1}(T_f^\rho)$. Since $\bar\rho$ is an isomorphism of $M(A)$ onto C, the argument above shows that (C.9) holds. If π is any nondegenerate representation, then we can use Lemma C.3 to show that

$$\begin{aligned}\Big(\bar\pi\Big(\int_G f(s)\, d\mu(s)\Big)\pi(a)h \mid k\Big) &= \Big(\pi\Big(\int_G f(s)a\, d\mu(s)\Big)h \mid k\Big)\\ &= \int_G \big(\pi\big(f(s)a\big)h \mid k\big)\, d\mu(s)\\ &= \int_G \big(\bar\pi\big(f(s)\big)\pi(a)h \mid k\big)\, d\mu(s).\end{aligned}$$

Now (C.15) follows from the nondegeneracy of π, and (C.8) is straightforward to check.

Finally, if $L : A \to B$ is a nondegenerate homomorphism, then

$$\begin{aligned}\bar L\Big(\int_G f(s)\, d\mu(s)\Big)L(a)b &= L\Big(\int_G f(s)\, d\mu(s)a\Big)b\\ &= L\Big(\int_G f(s)a\, d\mu(s)\Big)b \quad \text{by (C.9)}\\ &= \Big(\int L\big(f(s)a\big)\, d\mu(s)\Big)b \quad \text{by (C.10)}\\ &= \int \bar L\big(f(s)\big)L(a)b\, d\mu(s)\\ &= \Big(\int_G \bar L\big(f(s)\big)\, d\mu(s)\Big)L(a)b.\end{aligned}$$

This suffices. □

As an application of the ideas in this section, we see that our C^*-algebra-valued integrals behave nicely with respect to Hilbert-module inner products.

Corollary C.12. *Let X be a Hilbert A-module and f a compactly supported function of G into $\mathcal{L}(\mathsf{X})$ which is continuous for the strict topology. Then for each $x, y \in \mathsf{X}$, $s \mapsto \langle x\,,\, f(s)(y)\rangle_A$ is in $C_c(G, A)$ and*

$$\Big\langle x\,,\, \Big(\int_G f(s)\, d\mu(s)\Big)(y)\Big\rangle_A = \int_G \langle x\,,\, f(s)(y)\rangle_A\, d\mu(s).$$

Remark C.13. Since $T \mapsto \langle x \ , \ T(y)\rangle_A$ is a bounded linear map from $\mathcal{L}(\mathsf{X})$ to A, this Corollary would be a straightforward consequence of property (C.10) of Lemma C.3 if f were continuous for the norm topology.

Proof. Since $s \mapsto f(s)(y)$ is continuous by Proposition C.7, the first assertion is a consequence of the Cauchy-Schwarz inequality (Lemma 2.5). Left-handed versions of Proposition 2.31 and Lemma 2.30 imply that $y = R(z)$ for some $R \in \mathcal{K}(\mathsf{X})$ and $z \in \mathsf{X}$. Since $T \mapsto \langle x \ , \ T(y)\rangle_A$ is a bounded linear operator from $\mathcal{K}(\mathsf{X})$ to A and $s \mapsto f(s)R$ is norm continuous, Lemma C.3 implies that

$$\begin{aligned}\Big\langle x \ , \Big(\int_G f(s)\,d\mu(s)\Big)(y)\Big\rangle_A &= \Big\langle x \ , \Big(\int_G f(s)R\,d\mu(s)\Big)(z)\Big\rangle_A \\ &= \int_G \big\langle x \ , f(s)R(z)\big\rangle_A \, ds \\ &= \int_G \big\langle x \ , f(s)(y)\big\rangle_A \, ds. \quad \square\end{aligned}$$

C.3 The Group C^*-Algebra

Let G be a locally compact group with left Haar measure μ and modular function Δ. The *group C^*-algebra* is meant to be a C^*-algebra whose representations correspond to the unitary representations of G in a natural way. We start with the space $C_c(G)$ of compactly supported continuous functions on G, which is a $*$-algebra with multiplication given by convolution:

$$z * w(s) = \int_G z(r)w(r^{-1}s)\,d\mu(r),$$

and involution given by $z^*(s) := \Delta(s^{-1})\overline{z(s^{-1})}$. Using Lemma C.11, each unitary representation U gives a $*$-representation $\pi_U : C_c(G) \to B(\mathcal{H})$ via

$$\pi_U(z) = \int_G z(s)U_s\,d\mu(s);$$

thus

$$\big(\pi_U(z)h \mid k\big) = \int_G z(r)\big(U_r h \mid k\big)\,d\mu(r) \quad \text{for all } z \in C_c(G) \text{ and } h, k \in \mathcal{H}.$$

Notice that $\|\pi_U(z)\| \leq \|z\|_1$. Since one of the standard properties of the modular function says

$$\int_G z(s)\,d\mu(s) = \int_G z(s^{-1})\Delta(s^{-1})\,d\mu(s)$$

for any $z \in C_c(G)$ [55, Proposition 2.24], we always have $\|z^*\|_1 = \|z\|_1$. A $*$-representation π of $C_c(G)$ is called *norm-decreasing* if $\|\pi(z)\| \leq \|z\|_1$, and *cyclic* if $\overline{\text{span}}\{\,\pi(z)h : z \in C_c(G)\,\} = \mathcal{H}_\pi$ for some $h \in \mathcal{H}_\pi$.

Our first task is to see that every norm-decreasing $*$-representation of $C_c(G)$ arises from a unitary representation U (Corollary C.18). We start by defining the *universal C^*-norm* of $z \in C_c(G)$:

$$\|z\| := \sup\{\, \|\pi(z)\| : \pi \text{ is a cyclic norm-decreasing } *\text{-representation of } C_c(G) \,\}.$$

Notice that $\|z\| \leq \|z\|_1$, and, since every $*$-representation is the direct sum of cyclic $*$-representations, $\|\pi(z)\| \leq \|z\|$ for every norm-decreasing $*$-representation of $C_c(G)$. (One takes cyclic representations in the definition of $\|\cdot\|$ merely to guarantee that we are taking the supremum over a set.*) It is not hard to see that $\|\cdot\|$ is a C^*-norm on $C_c(G)$, and the completion of $C_c(G)$ with respect to $\|\cdot\|$ is a C^*-algebra $C^*(G)$ called the *group cstar@group C^*-algebra* of G. Notice that any $\|\cdot\|_1$-norm-decreasing $*$-representation of $C_c(G)$ is necessarily $\|\cdot\|$-continuous and therefore extends to a representation of $C^*(G)$.

Hooptedoodle C.14. The norm $\|\cdot\|$ on $C_c(G)$ is called the universal norm to distinguish it from the *reduced C^*-norm* obtained from the left-regular representation of G. The left-regular representation is the unitary representation $\lambda : G \to U\big(L^2(G,\mu)\big)$ defined by $\lambda_r(h)(s) = h(r^{-1}s)$. The integrated form π_λ is a norm-decreasing injective $*$-representation of $C_c(G)$, and we can define the reduced norm $\|\cdot\|_r$ by

$$\|z\|_r := \|\pi_\lambda(z)\| \quad z \in C_c(G).$$

We always have $\|\cdot\|_r \leq \|\cdot\|$, but they do not always agree; indeed, the reduced norm equals the universal norm if and only if the group G is amenable [124, Theorem 7.3.9]. Using this characterization, one can see that all abelian and all compact groups are amenable. On the other hand, the free group $\mathbb{F}_2$ on two generators is not. In general, therefore, the completion $C_r^*(G)$ of $C_c(G)$ with respect to the reduced norm is a proper quotient of the group C^*-algebra $C^*(G)$; $C_r^*(G)$ is called the *reduced group C^*-algebra.*

Since the left-regular representation separates points of $C_c(G)$, the universal norm is actually a norm, and $C_c(G)$ embeds faithfully in the completion $C^*(G)$. Thus we can and will view $C_c(G)$ as a $*$-subalgebra of $C^*(G)$. We are going to use the *inductive limit topology* on $C_c(G)$, which is discussed in Appendix D.2. One of the reasons the inductive limit topology is important is that it implies convergence in the $\|\cdot\|_1$-norm, and therefore in the C^*-norm on $C_c(G)$ (Corollary D.6). In working with this topology, it is often enough to know that if $\{\, z_i \,\}_{i\in I}$ is a net of functions, $z_i \to z$ uniformly, and all the supports of the z_i are eventually in a fixed compact set, then $z_i \to z$ in the inductive limit topology.† For such nets it is clear that $\|z_i - z\|_1 \to 0$, and hence $\|z_i - z\| \to 0$ also.

One of the difficulties in working with group C^*-algebras is that not all the elements in $C^*(G)$ are represented by functions on G. This leads to interesting but

*Formally, we should be looking only at cyclic representations on subspaces of a fixed Hilbert space of suitably large dimension. Since every cyclic representation is unitarily equivalent to one of these, our formulation should not be misleading.

†Notice that we did not say that these are precisely the convergent nets. It is common in the literature to use the phrase "converges in the inductive limit topology" to describe nets which converge uniformly and are eventually supported in a fixed compact set, and it would be easy to come to the conclusion that such nets describe the topology. That this is not so is illustrated in Example D.9. (See also Remark D.11.)

counterintuitive situations, such as that described in Hooptedoodle C.14, where a faithful representation of $C_c(G)$ need not be faithful on $C^*(G)$. Another such subtlety is that "evaluation at $s \in G$" on elements of $C_c(G)$ is not continuous with respect to the universal C^*-norm.

If H is a locally compact group and $f : G \to C_c(H)$ is a sufficiently nice function, then we can view f as being either $C^*(H)$-valued or $C_c(H)$-valued, and we can try to make sense of $\int_G f\, d\mu$ both as an element of $C^*(H)$ (using Lemma C.3) and as an element of $C_c(H)$. Naturally, we'd then like to know that these elements coincide in $C^*(H)$. (If evaluation was a bounded operator on $C^*(H)$, we could see this by applying (C.10) of Lemma C.3.) Our next result accomplishes this in sufficient generality for our purposes.[‡]

Lemma C.15. *Let G and H be locally compact groups. Let $F \in C_c(G \times H)$ and define $f : G \to C_c(H)$ by $f(s)(t) = F(s,t)$. Then*

$$b(t) := \int_G f(s)(t)\, d\mu(s)$$

defines an element of $C_c(H)$. On the other hand, we can view f as an element of $C_c\big(G, C^*(H)\big)$ and obtain an element of $C^*(H)$ via Lemma C.3:

$$\tilde{b} = \int_G f(s)\, d\mu(s).$$

In $C^*(H)$, we have $b = \tilde{b}$.

Proof. Since the inductive limit topology is stronger than the C^*-norm topology, f is certainly in $C_c\big(G, C^*(H)\big)$ and $\tilde{b}$ is well-defined. On the other hand, it is not hard to see that $b \in C_c(H)$.

Let ι be the inclusion of $C_c(H)$ into $C^*(H)$. Then the composition $F \mapsto b \mapsto \iota(b)$ defines a linear map $\Phi : C_c(G \times H) \to C^*(H)$. Similarly, $F \mapsto \tilde{b}$ is a linear map $\Psi : C_c(G \times H) \to C^*(H)$. Since these maps clearly agree on functions of the form $F(s,t) = g(s)k(t)$ for $g \in C_c(G)$ and $k \in C_c(H)$, and since functions of this form span a dense subspace of $C_c(G \times H)$ in the inductive limit topology, it suffices to prove that Φ and Ψ are continuous for the inductive limit topology on $C_c(G \times H)$.

Let ν be a Haar measure on H. Suppose that $\operatorname{supp}(F) \subset C \times K$ with C compact in G and K compact in H. Then the continuity of Φ and Ψ follows from the following estimates:

$$\begin{aligned}\|\Psi(F)\| \le \|f\|_1 &= \int_G \|f(s)\|\, d\mu(s) \le \mu(C) \sup_{s \in G} \|f(s)\| \\ &\le \mu(C)\nu(K)\|F\|_\infty,\end{aligned}$$

and

$$\begin{aligned}\|\Phi(F)\| \le \|b\|_1 &= \int_H \|b(t)\|\, d\nu(t) \le \nu(K) \sup_{t \in H} \Big| \int_G f(s)(t)\, d\mu(s) \Big| \\ &\le \nu(K)\mu(C)\|F\|_\infty. \quad \square\end{aligned}$$

[‡]We thank Wojciech Szymański for simplifying a previous argument.

For $z \in C_c(G)$ and $r \in G$, define $\tau_r(z)(s) = z(r^{-1}s)$. Then using the left invariance of Haar measure, it is not hard to check that $\tau_s(z)^* * \tau_s(w) = z^* * w$, so that τ_r is isometric for the universal C^*-norm, and that $\tau_r(z) * w = \tau_r(z * w)$, so that τ_r extends to an adjointable operator on $C^*(G)_{C^*(G)}$ with adjoint $\tau_{r^{-1}} = \tau_r^{-1}$. Thus there is a unitary element $i_G(r)$ of $\mathcal{L}(C^*(G)) = M(C^*(G))$ characterized by $i_G(r)z = \tau_r(z)$ for $z \in C_c(G)$. Since $s \mapsto \tau_s(z)$ is continuous from G to $C_c(G)$ with the inductive limit topology, and since each $i_G(s)$ has norm 1, it is not hard to see that i_G is a strictly continuous homomorphism.

Corollary C.16. *If $z, w \in C_c(G)$, then as elements of $C^*(G)$ we have*

$$\int_G z(s) i_G(s) w \, d\mu(s) = z * w.$$

Proof. Apply Lemma C.15 with $H = G$ and $F(s,t) = z(s)w(s^{-1}t)$, so that $f(s) = z(s)i_G(s)w = z(s)\tau_s(w)$. □

Proposition C.17. *Suppose that X is a Hilbert A-module and that $u : G \to U\mathcal{L}(\mathsf{X})$ is a strongly continuous homomorphism of G into the unitary group of $\mathcal{L}(\mathsf{X})$. Then for each $z \in C_c(G)$, Lemma C.11 implies that*

$$\pi_u(z) = \int_G z(s) u_s \, d\mu(s)$$

is a well-defined operator in $\mathcal{L}(\mathsf{X})$, and π_u extends to a nondegenerate homomorphism of $C^*(G)$ into $\mathcal{L}(\mathsf{X})$, called the integrated form *of u.*

Conversely, if $\pi : C_c(G) \to \mathcal{L}(\mathsf{X})$ is a norm-decreasing nondegenerate $*$-homomorphism, then there is a strongly continuous homomorphism $u : G \to U\mathcal{L}(\mathsf{X})$ such that $\pi = \pi_u$. In fact, if $\bar{\pi}$ is the canonical extension of π to $M\big(C^*(G)\big)$, then we can take $u_s = \bar{\pi}\big(i_G(s)\big)$.

Proof. The map $s \mapsto u_s$ is strictly continuous by Corollary C.8; thus $\pi_u(z)$ is defined by Lemma C.11 as claimed. Using Corollary C.12 and a mild form of Fubini's Theorem[§],

$$\begin{aligned}
\big\langle x \,,\, \pi_u(z * w)(y) \big\rangle_A &= \int_G z * w(s) \langle x \,,\, u_s(y) \rangle_A \, d\mu(s) \\
&= \int_G \int_G z(r) w(r^{-1}s) \langle x \,,\, u_s(y) \rangle_A \, d\mu(r) \, d\mu(s) \\
&= \int_G \int_G z(r) w(s) \langle x \,,\, u_{rs}(y) \rangle_A \, d\mu(s) \, d\mu(r) \\
&= \int_G \int_G z(r) w(s) \langle u_r^*(x) \,,\, u_s(y) \rangle_A \, d\mu(s) \, d\mu(r) \\
&= \int_G z(r) \langle u_r^*(x) \,,\, \pi_u(w)(y) \rangle_A \, d\mu(s) \\
&= \big\langle x \,,\, \pi_u(z)\pi_u(w)(y) \big\rangle_A .
\end{aligned}$$

[§]A suitable version of Fubini can be obtained representing everything on Hilbert space, and then invoking the result for scalar-valued functions.

A similar computation shows that $\pi_u(z^*) = \pi_u(z)^*$.

Now let ρ be a state on A. Then $(x \mid y)_\rho := \rho\big(\langle y\ ,\ x\rangle_A\big)$ is a positive sesquilinear form on X. The completion of X with respect to the norm $\|\cdot\|_\rho = (\cdot \mid \cdot)_\rho^{1/2}$ is a Hilbert space $\mathcal{H}_\rho$. Each $\pi_u(z)$ defines an operator $\tilde{\pi}(z)$ in $B(\mathcal{H}_\rho)$ (bounded by $\|z\|_1$), and $\tilde{\pi} : C_c(G) \to B(\mathcal{H}_\rho)$ is a norm-decreasing $*$-representation of $C_c(G)$. Therefore $\|\tilde{\pi}(z)\| \le \|z\|$ by definition of the universal norm. Thus

$$\begin{aligned}\rho\big(\langle \pi_u(z)(x)\ ,\ \pi_u(z)(x)\rangle_A\big) &= \big(\tilde{\pi}(z)(x) \mid \tilde{\pi}(z)(x)\big)_\rho = \|\tilde{\pi}(z)(x)\|_\rho^2 \le \|z\|^2\|x\|_\rho^2 \\ &\le \|z\|^2\rho\big(\langle x\ ,\ x\rangle_A\big).\end{aligned}$$

Since ρ is arbitrary, it follows that $\|\pi_u(z)\| \le \|z\|$, and π_u can be extended to $C^*(G)$ as claimed. The nondegeneracy is straightforward — consider nonnegative functions $f \in C_c(G)$ with small support and integral one.¶

Now let π be a norm-decreasing nondegenerate $*$-homomorphism of $C_c(G)$ into $\mathcal{L}(\mathsf{X})$. Extend π to $C^*(G)$, let $\bar{\pi}$ be the canonical extension to $M\big(C^*(G)\big)$, and take $u_s := \bar{\pi}\big(i_G(s)\big)$. Using the nondegeneracy of π and the strict continuity of i_G, it is not hard to check that $s \mapsto u_s$ is strongly continuous. Since $z*w = \int_G z(s)i_G(s)w\,d\mu$ (Corollary C.16), we have

$$\begin{aligned}\big\langle y\ ,\ \pi(z*w)(x)\big\rangle_A &= \int_G z(s)\big\langle y\ ,\ \pi(i_G(s)w)(y)\big\rangle_A\,d\mu(s) \\ &= \int_G z(s)\big\langle y\ ,\ \big(u_s\pi(w)\big)(y)\big\rangle_A\,d\mu(s) \\ &= \big\langle x\ ,\ \pi_u(z)\pi(w)(y)\big\rangle_A.\end{aligned}$$

Since π is nondegenerate, it follows that $\pi = \pi_u$, as required. □

Since a Hilbert space is a Hilbert $\mathbb{C}$-module, the following result is immediate from Proposition C.17 and the definition of the norm on $C_c(G)$.

Corollary C.18. *Suppose that $\pi : C_c(G) \to B(H_\pi)$ is a norm-decreasing nondegenerate $*$-representation of $C_c(G)$. Define $U_s = \bar{\pi}\big(i_G(s)\big)$. Then U is a unitary representation of G and $\pi = \pi_U$. In particular, for $z \in C_c(G)$ we have*

$$\|z\| = \sup\{\,\|\pi_U(z)\| : U \text{ is a unitary representation of } G\,\}.$$

Since every representation of $C^*(G)$ defines a unique norm-decreasing $*$-representation of $C_c(G)$, the following proposition is a straightforward consequence of the previous corollary.

Proposition C.19. *The map sending a unitary representation U of G to its integrated form π_U establishes a one-to-one correspondence between the unitary representations of G and the nondegenerate representations of $C^*(G)$.*

¶The more sophisticated could use an approximate identity for $C_c(G)$ in the inductive limit topology.

Example C.20. Suppose that G is abelian and that $\widehat{G}$ is the *Pontryagin dual* of G; that is, $\widehat{G}$ is the locally compact abelian group of continuous homomorphisms $\gamma : G \to \mathbb{T}$ with the topology of uniform convergence on compact sets [55, §4.1]. Then $C^*(G)$ can be identified with $C_0(\widehat{G})$ via the Fourier transform.

To see this, note that by Theorem A.14, the C^*-norm of $z \in C_c(G)$ satisfies

$$\|z\| = \sup\{\, \|\pi(z)\| : \pi \in C^*(G)^\wedge \,\}. \tag{C.16}$$

Since $C^*(G)$ is commutative, each $\pi \in C^*(G)^\wedge$ is one-dimensional (Example A.16), and is therefore the integrated form of a one-dimensional unitary representation, which is a continuous homomorphism $\gamma \in \widehat{G}$. In other words,

$$\pi(z) = \hat{z}(\gamma) := \int_G z(s)\gamma(s)\, d\mu(s),$$

which is the integral defining the Fourier transform $\hat{z} : \widehat{G} \to \mathbb{C}$ of z. Equation (C.16) therefore says that $\|z\| = \|\hat{z}\|_\infty$. From abelian harmonic analysis we know that $\hat{z}$ is continuous and vanishes at infinity on $\widehat{G}$, and that the image of $C_c(G)$ is dense in $C_0(\widehat{G})$ [55, Theorem 4.13]. Since the Fourier transform converts convolution to pointwise multiplication and the involution to conjugation, the map $z \mapsto \hat{z}$ extends to an isomorphism of the C^*-algebra $C^*(G)$ onto $C_0(\widehat{G})$.

C.4 The Imprimitivity Theorem

Let H be a closed subgroup of a locally compact group G. We want to construct a Hilbert $C^*(H)$-module X from $C_c(G)$, and to identify the imprimitivity algebra $\mathcal{K}(\mathsf{X})$. The main result says that $\mathcal{K}(\mathsf{X})$ is isomorphic to an algebra built from $E_0 := C_c(G \times G/H)$ in much the same way as $C^*(G)$ is built from $C_c(G)$, where E_0 has multiplication and involution given by

$$\phi * \psi(r, \dot{s}) = \int_G \phi(v, \dot{s})\psi(v^{-1}r, v^{-1} \cdot \dot{s})\, d\mu(v), \quad \text{and}$$
$$\phi^*(r, \dot{s}) = \Delta(r^{-1})\overline{\phi(r^{-1}, r^{-1} \cdot \dot{s})},$$

where as usual $\dot{s}$ is short for the coset $sH \in G/H$. Routine calculations show that E_0 is indeed a $*$-algebra with these operations.

A pair (M, U) consisting of a unitary representation $U : G \to U(\mathcal{H})$ and a representation $M : C_0(G/H) \to B(\mathcal{H})$ is a *covariant representation* of $\big(C_0(G/H), G, \tau\big)$ if

$$M\big(\tau_r(F)\big) = U_r M(F) U_r^* \quad \text{for all } r \in G \text{ and } F \in C_0(G/H).$$

We will show in the next Proposition that each covariant pair (M, U) determines a representation $L = M \rtimes U$ of E_0 via the formula

$$L(\phi) := \int_G M\big(\phi(r, \cdot)\big) U_r\, d\mu(r).$$

Once we have this, we trivially have

$$\|L(\phi)\| \leq \|\phi\|_1 := \int_G \|\phi(r,\cdot)\|\, d\mu(r),$$

so we can define a norm on E_0 by

$$\|\phi\| := \sup\{\, \|M \rtimes U(\phi)\| : (M,U) \text{ is a covariant representation}\,\}. \tag{C.17}$$

The completion of E_0 in this norm is a C^*-algebra; we shall denote it by $C_0(G/H) \rtimes_\tau G$, because it is an example of the crossed products discussed in Section 7.1. Because the algebra is determined by the action of G on G/H, or equivalently by the transformation group $(G, G/H)$, it is sometimes called a transformation group C^*-algebra .

We now give the promised proof that $M \rtimes U$ is a $*$-homomorphism. The extra generality obtained by considering covariant representations with values in $\mathcal{L}(\mathsf{X})$ (called *covariant homomorphisms*) will be useful later.

Proposition C.21. *Let X be a Hilbert A-module. Suppose that $u : G \to U\mathcal{L}(\mathsf{X})$ is a strongly continuous homomorphism, and that $M : C_0(G/H) \to \mathcal{L}(\mathsf{X})$ is a homomorphism satisfying the covariance condition*

$$M\big(\tau_r(F)\big) = u_r M(F) u_r^*$$

for all $r \in G$ and $F \in C_0(G/H)$. Then the map $L : E_0 \to \mathcal{L}(\mathsf{X})$ defined by

$$L(\phi) := \int_G M\big(\phi(r,\cdot)\big) u_r \, d\mu(r)$$

extends to a homomorphism $M \rtimes u$ of $C_0(G/H) \rtimes_\tau G$ into $\mathcal{L}(\mathsf{X})$. If M is nondegenerate, then so is $M \rtimes u$.

Proof. It is not hard to see that $r \mapsto \phi(r,\cdot)$ is continuous from G into $C_0(G/H)$, so $r \mapsto M\big(\phi(r,\cdot)\big)$ is norm continuous. Since $r \mapsto u_r$ is strictly continuous, the function $r \mapsto M\big(\phi(r,\cdot)\big)u_r$ belongs to $C_c(G, M_s(\mathcal{K}(\mathsf{X})))$, and the integral defining $L(\phi)$ makes sense by Lemma C.11. To see that L is a homomorphism on E_0, let $\psi, \phi \in E_0$. Since point evaluation is a homomorphism on $C_0(G/H)$, it follows from Lemma C.3 that in $C_0(G/H)$

$$\phi * \psi(s,\cdot) = \int_G \phi(r,\cdot)\tau_r\big(\psi(r^{-1}s,\cdot)\big)\, d\mu(r). \tag{C.18}$$

Therefore

$$\begin{aligned}
\big\langle x\ ,\ L(\phi * \psi)(y)\big\rangle_A &= \int_G \big\langle x\ ,\ M\big(\phi * \psi(s,\cdot)\big)u_s(y)\big\rangle_A\, d\mu(s) \\
&= \int_G\int_G \big\langle x\ ,\ M\big(\phi(r,\cdot)\big)M\big(\tau_r\big(\psi(r^{-1}s,\cdot)\big)\big)u_s(y)\big\rangle_A\, d\mu(s)\, d\mu(r) \\
&= \int_G\int_G \big\langle x\ ,\ M\big(\phi(r,\cdot)\big)u_r M\big(\psi(s,\cdot)\big)u_s(y)\big\rangle_A\, d\mu(s)\, d\mu(r) \\
&= \int_G\int_G \big\langle u_r^* M\big(\phi(r,\cdot)\big)^*(x)\ ,\ M\big(\psi(s,\cdot)\big)u_s(y)\big\rangle_A\, d\mu(s)\, d\mu(r) \\
&= \int_G \big\langle x\ ,\ M\big(\phi(r,\cdot)\big)u_r\big(L(\psi)(y)\big)\big\rangle_A\, d\mu(r) \\
&= \big\langle x\ ,\ L(\phi)L(\psi)(y)\big\rangle_A .
\end{aligned}$$

A similar computation shows that $L(\phi^*) = L(\phi)^*$, so L is a $*$-homomorphism.

To see that L extends to $C_0(G/H) \rtimes_\tau G$, we prove that $\|L(\phi)\| \le \|\phi\|$. As in the proof of Proposition C.17, we fix a state ρ on A and consider the Hilbert space completion $\mathcal{H}_\rho$ of X with respect to $(x \mid y)_\rho = \rho\big(\langle y\ ,\ x\rangle_A\big)$. The pair (M,u) determines a covariant representation (M_1,u_1) on $\mathcal{H}_\rho$, and the integrated form $L_1 = M_1 \rtimes u_1$ is a $*$-representation of E_0. By definition of the C^*-norm on E_0,

$$\begin{aligned}
\rho\big(\big\langle L(\phi)(x)\ ,\ L(\phi)(x)\big\rangle_A\big) &= \big(L_1(\phi)(x) \mid L_1(\phi)(x)\big)_\rho \le \|\phi\|^2 (x \mid x)_\rho \\
&= \|\phi\|^2 \rho\big(\langle x\ ,\ x\rangle_A\big).
\end{aligned}$$

The required estimate follows because this holds for all states ρ.

Now suppose that M is nondegenerate. To show that $L = M \rtimes u$ is nondegenerate, it suffices to see that $M\big(C_0(G/H)\big)\cdot\mathsf{X} \subset \overline{L\big(C_c(G\times G/H)\big)\cdot\mathsf{X}}$. Fix $F \in C_0(G/H)$, $x \in \mathsf{X}$, and $\epsilon > 0$. Let N be a neighbourhood of e in G such that $\|u_r(x) - x\|_A < \epsilon\|F\|_\infty^{-1}$ for $r \in N$. Choose $g \in C_c^+(G)$ such that $\operatorname{supp}(g) \subset N$ and $\int_G g\, d\mu = 1$. Then if we define $\phi \in C_c(G \times G/H)$ by $\phi(r,\dot{s}) := g(r)F(\dot{s})$, we have

$$\begin{aligned}
\|L(\phi)(x) - M(F)(x)\| &= \Big\| \int_G M(F)g(r)\big(u_r(x) - x\big)\, d\mu(r)\Big\| \\
&\le \epsilon,
\end{aligned}$$

as required. □

Remark C.22. In Proposition C.30, we will show that every representation of $C_0(G/H)\rtimes_\tau G$ is the integrated form of a covariant representation, so that $(M,U) \mapsto M \rtimes U$ is a one-to-one correpondence between the covariant representations of $(C_0(G/H), G, \tau)$ and the nondegenerate representations of $C_0(G/H) \rtimes_\tau G$.

Our main theorem concerns a closed subgroup H of a locally compact group G. Recall that we are writing μ and ν for Haar measures on G and H, and Δ and δ for the modular functions. In addition, we define $\gamma(t) = \big(\Delta(t)/\delta(t)\big)^{\frac{1}{2}}$ for $t \in H$.

Theorem C.23. *View $E_0 = C_c(G \times G/H)$ and $B_0 = C_c(H)$ as $*$-subalgebras of the C^*-algebras $E = C_0(G/H) \rtimes_\tau G$ and $B = C^*(H)$, respectively. Let $\mathsf{X}_0 = C_c(G)$. For $f, g \in \mathsf{X}_0$, $b \in B_0$, and $\phi \in E_0$, define*

$$f \cdot b = \int_H f(st^{-1})\Delta(t^{-1})\gamma(t)b(t)\, d\nu(t), \tag{C.19}$$

$$\langle f\ ,\ g\rangle_{B_0}(t) = \gamma(t) \int_G \overline{f(r)}g(rt)\, d\mu(r) \tag{C.20}$$

$$\phi \cdot f(s) = \int_G \phi(r, \dot{s})f(r^{-1}s)\, d\mu(r), \text{ and} \tag{C.21}$$

$${}_{E_0}\langle f\ ,\ g\rangle(r, \dot{s}) = \int_H f(st)\overline{g(r^{-1}st)}\Delta(r^{-1}st)\, d\nu(t). \tag{C.22}$$

Then X_0 is an $E_0 - B_0$-pre-imprimitivity bimodule. The completion X is a $C(G/H) \rtimes_\tau G - C^*(H)$-imprimitivity bimodule, and $C(G/H) \rtimes_\tau G$ and $C^*(H)$ are Morita equivalent.

Just as for $C_c(G)$, convergence in the inductive limit topology in E_0 implies convergence in the $\|\cdot\|_1$-norm, and therefore in the C^*-norm. The following technical lemma on convergence in the inductive limit topology will be useful.

Lemma C.24. *Suppose that $\{\phi_{N,K}\}$ is a net of functions in E_0, indexed by decreasing neighbourhoods of $e \in G$ and increasing compact subsets K of G/H, satisfying*

$$\phi_{N,K}(r, \dot{s}) = 0 \quad \text{if } r \notin N, \text{ and} \tag{C.23}$$

$$\int_G \phi_{N,K}(r, \dot{s})\, d\mu(r) = 1 \quad \text{if } \dot{s} \in K. \tag{C.24}$$

Then for every $\phi \in E_0$, $\{\phi_{N,K} * \phi\}$ converges to ϕ in the inductive limit topology on E_0, and for every $f \in \mathsf{X}_0$, $\phi_{N,K} \cdot f$ converges to f in the inductive limit topology on $C_c(G) = \mathsf{X}_0$.

Proof. Fix $\phi \in E_0$ and $\epsilon > 0$. Let $K_1 \times K_2$ be a compact set in $G \times G/H$ containing $\mathrm{supp}(\phi)$. Fix a compact neighbourhood V_0 of e in G. A compactness argument implies that there is a neighbourhood $V \subset V_0$ such that $t \in V$ implies

$$|\phi(t^{-1}r, t^{-1} \cdot \dot{s}) - \phi(r, \dot{s})| < \epsilon \quad \text{for all } (r, \dot{s}) \in G \times G/H.$$

Then if $N \subset V$ and $K \supset N \cdot K_2$, we have $\mathrm{supp}(\phi_{N,K} * \phi) \subset V_0K_1 \times V_0 \cdot K_2$ and $|\phi_{N,K} * \phi(r, \dot{s}) - \phi(r, \dot{s})| < \epsilon$ everywhere. A similar argument gives the second part. □

Lemma C.25. *There is a net $\{\phi_m\}_{m \in M}$ of functions in E_0 such that each ϕ_m is a finite sum of the form $\sum_{i=1}^{n_m} {}_{E_0}\langle f_i^m\ ,\ f_i^m\rangle$ for $f_i^m \in \mathsf{X}_0$, and such that for each $f \in \mathsf{X}_0$ and $\phi \in E_0$*

$$\phi_m \cdot f \to f \text{ in the inductive limit topology on } C_c(G), \text{ and} \tag{C.25}$$

$$\phi_m * \phi \to \phi \text{ in the inductive limit topology on } E_0. \tag{C.26}$$

Similarly, there is a net $\{b_j\}_{j\in J}$ *of functions in* B_0 *of the form* $b_j = \langle g_j , g_j\rangle_{B_0}$ *such that for each* $f \in \mathsf{X}_0$ *and* $b \in B_0$, $f \cdot b_j \to f$ *and* $b_j * b \to b$ *in the appropriate inductive limit topologies.*

Proof. We construct a net $\{\phi_{N,K}\}$ in E_0 satisfying (C.23) and (C.24) consisting of functions of the required form; then the previous Lemma completes the proof of the first part. Fix a neighbourhood N of e and a compact subset K of G/H. Let $q : G \to G/H$ be the quotient map, and choose a compact subset C of G with $q(C) = K$. Using Proposition C.1, we can find a cut-down Bruhat approximate cross-section $h \in C_c^+(G)$ such that

$$\int_H h(st)\,d\nu(t) = 1 \quad \text{for all } s \in CH.$$

Let $D = \operatorname{supp}(h)$, and let V be a symmetric neighbourhood of $e \in G$ such that $V^2 \subset N$. Then there are $s_1, \dots, s_n \in G$ such that $D \subset \bigcup_{i=1}^n Vs_i$. Multiplying by a partition of unity subordinate to the finite cover $\{D^c\} \cup \{Vs_i\}$ gives functions $h_i \in C_c^+(G)$ such that $\operatorname{supp}(h_i) \subset Vs_i$ and

$$\sum_{i=1}^n \int_H h_i(st)\,d\nu(t) = 1 \quad \text{for all } s \in CH.$$

Let $c_i = \left(\int_G h_i(r)\,d\mu(r)\right)^{-1}$ and take $f_i := c_i^{1/2} h_i$. Then

$$\begin{aligned} {}_{E_0}\langle f_i , f_i\rangle(r,\dot{s}) &= \int_H f_i(st)\overline{f_i(r^{-1}st)}\Delta(r^{-1}st)\,d\nu(t) \\ &= c_i \int_H h_i(st)h_i(r^{-1}st)\Delta(r^{-1}st)\,d\nu(t). \end{aligned}$$

Therefore ${}_{E_0}\langle f_i , f_i\rangle(r,\dot{s}) = 0$ if $r \notin V^2$. Furthermore,

$$\begin{aligned} &\sum_{i=1}^n \int_G {}_{E_0}\langle f_i , f_i\rangle(r,\dot{s})\,d\mu(r) \\ &\quad = \sum_{i=1}^n \int_G \int_H h_i(st)c_i h_i(r^{-1}st)\Delta(r^{-1}st)\,d\nu(t)\,d\mu(r) \\ &\quad = \sum_{i=1}^n \int_H h_i(st) \int_G c_i h_i(r^{-1}st)\Delta(r^{-1}st)\,d\mu(r)\,d\nu(t) \\ &\quad = \sum_{i=1}^n \int_H h_i(st) \int_G c_i h_i(r)\,d\mu(r)\,d\nu(t) \\ &\quad = \sum_{i=1}^n \int_H h_i(st)\,d\nu(t), \end{aligned}$$

which equals 1 when $s \in CH$, or equivalently when $\dot{s} \in K$. Thus we can take $\phi_{N,K} := \sum_i {}_{E_0}\langle f_i , f_i\rangle$.

The construction of the functions b_j is similar, but considerably less messy. □

Proof of Theorem C.23. In view of Proposition 3.12, it will suffice to verify the axioms of Definition 3.9. Straightforward computations, using that Δ and δ are homomorphisms into an abelian group, show that

$$\begin{aligned} f \cdot (b * c) &= (f \cdot b) \cdot c, \\ (\psi * \phi) \cdot f &= \psi \cdot (\phi \cdot f), \text{ and} \\ \phi \cdot (f \cdot b) &= (\phi \cdot f) \cdot b \end{aligned}$$

for all $f \in \mathsf{X}_0$, $\phi, \psi \in E_0$, and $b, c \in B_0$. In other words, X_0 is an $E_0 - B_0$-bimodule. Similar computations show that condition (d) of Definition 3.9 holds.

To verify condition (a) of Definition 3.9, we do more calculations. It is relatively easy to check that $\langle \cdot\ ,\ \cdot \rangle_{B_0}$ and ${}_{E_0}\langle \cdot\ ,\ \cdot \rangle$ satisfy the requirements of Lemma 2.16, with the exception of the positivity (condition (d) of Definition 2.1), which is rather subtle. For it, we observe that ${}_{E_0}\langle \cdot\ ,\ \cdot \rangle$ is continuous in the inductive limit topology: if $f_i \to f$ in the inductive limit topology in X_0, then for any $g \in X_0$, ${}_{E_0}\langle f_i\ ,\ g \rangle \to {}_{E_0}\langle f\ ,\ g \rangle$ in the inductive limit topology, and therefore in the C^*-norm, on E_0. A similar statement holds for $\langle \cdot\ ,\ \cdot \rangle_{B_0}$. If we now fix $f \in \mathsf{X}_0$ and take $\{\phi_m\}$ as in Lemma C.25, then (C.25) implies that in the C^*-norm

$$\begin{aligned} \langle f\ ,\ f \rangle_{B_0} &= \lim_m \langle \phi_m \cdot f\ ,\ f \rangle_{B_0} = \lim_m \sum\nolimits_{i=1}^{n_m} \big\langle {}_{E_0}\langle f_i^m\ ,\ f_i^m \rangle \cdot f\ ,\ f \big\rangle_{B_0} \\ &= \lim_m \sum\nolimits_{i=1}^{n_m} \big\langle f_i^m \cdot \langle f_i^m\ ,\ f \rangle_{B_0}\ ,\ f \big\rangle_{B_0} = \lim_m \sum\nolimits_{i=1}^{n_m} \langle f_i^m\ ,\ f \rangle_{B_0}^* \langle f_i^m\ ,\ f \rangle_{B_0}. \end{aligned}$$

Since elements of the form a^*a are positive and $\langle f\ ,\ f \rangle_{B_0}$ is the norm limit of positive elements, it follows that $\langle f\ ,\ f \rangle_{B_0} \geq 0$ for all $f \in \mathsf{X}_0$. The proof of the positivity of ${}_{E_0}\langle \cdot\ ,\ \cdot \rangle$ is similar. This establishes condition (a) of Definition 3.9.

Condition (b) of Definition 3.9 is a consequence of (C.26) and its counterpart for B_0. Thus we need only verify condition (c): that the module actions are bounded. As in the previous paragraph, this will boil down to the continuity of the actions and inner products in the inductive limit topologies.

For the moment, let X_1 denote the completion of X_0 as a Hilbert $C^*(H)$-module, as in Lemma 2.16. We will usually suppress the map $q : \mathsf{X}_0 \to \mathsf{X}_1$. Let $F \in C_b(G/H)$, define

$$F \cdot f(s) = F(\dot{s}) f(s)$$

for $f \in \mathsf{X}_0$, and observe that $\langle F \cdot f\ ,\ g \rangle_{B_0} = \langle f\ ,\ \overline{F} \cdot g \rangle_{B_0}$. Thus if $F_0 := \big(\|F\|^2 1 - \overline{F}F\big)^{\frac{1}{2}}$ in $C_b(G/H)$, then

$$\|F\|_\infty \langle f\ ,\ f \rangle_{B_0} - \langle F \cdot f\ ,\ F \cdot f \rangle_{B_0} = \langle F_0 \cdot f\ ,\ F_0 \cdot f \rangle_{B_0} \geq 0.$$

Thus $C_b(G/H)$ acts by bounded operators on X_0, so the action extends to X_1, and we have a homomorphism $j_{C_0(G/H)} : C_0(G/H) \to \mathcal{L}(\mathsf{X}_1)$ defined by

$j_{C_0(G/H)}(F)(f) = F \cdot f$ (really, $j_{C_0(G/H)}(F)\big(q(f)\big) = q\big(F \cdot f\big)$, but we have agreed to suppress q). Since a computation shows that

$$\langle \tau_s(f) \, , \, \tau_s(g) \rangle_{B_0} = \langle f \, , \, g \rangle_{B_0} \quad \text{for all } s \in G \text{ and } f, g \in \mathsf{X}_0,$$

setting $j_G(s)(f) := \tau_s(f)$ gives a strongly continuous homomorphism $j_G : G \to U\mathcal{L}(\mathsf{X}_1)$. The pair $(j_{C_0(G/H)}, j_G)$ is covariant in the sense of Proposition C.21:

$$j_{C_0(G/H)}\big(\tau_s(F)\big) = j_G(s) j_{C_0(G/H)}(F) j_G(s)^* \quad \text{for all } s \in G \text{ and } F \in C_0(G/H).$$

Therefore there is a homomorphism $L := j_{C_0(G/H)} \rtimes j_G$ of $C_0(G/H) \rtimes_\tau G$ into $\mathcal{L}(\mathsf{X}_1)$, which is automatically norm-decreasing. Thus by Corollary 2.22 we have*

$$\big\langle L(\phi)(f) \, , \, L(\phi)(f) \big\rangle_B \leq \|L(\phi)\|^2 \langle f \, , \, f \rangle_B \leq \|\phi\|^2 \langle f \, , \, f \rangle_B.$$

To show that

$$\langle \phi \cdot f \, , \, \phi \cdot f \rangle_{B_0} \leq \|\phi\|^2 \langle f \, , \, f \rangle_{B_0}, \tag{C.27}$$

therefore, it will suffice to show that $L(\phi)(f) = \phi \cdot f$ for $\phi \in E_0$. However $r \mapsto j_{C_0(G/H)}\big(\phi(r, \cdot)\big) j_G(r)(f)$ is continuous and compactly supported as a function from G into X_1. Thus by Corollary C.12,

$$\big\langle g \, , \, L(\phi)(f) \big\rangle_B = \int_G \big\langle g \, , \, j_{C_0(G/H)}\big(\phi(r, \cdot)\big) j_G(r)(f) \big\rangle_B \, d\mu(r). \tag{C.28}$$

The integrand of (C.28) is $C_c(H)$-valued and continuous in r with respect to the inductive limit topology. By Lemma C.15, (C.28) is in $C_c(H)$ and for $t \in H$ we have

$$\begin{aligned} \big\langle g \, , \, L(\phi)(f) \big\rangle_B(t) &= \int_G \big\langle g \, , \, j_{C_0(G/H)}\big(\phi(r, \cdot)\big) j_G(r)(f) \big\rangle_{B_0}(t) \, d\mu(r) \\ &= \int_G \int_G \gamma(t) g^*(s) \phi(r, \dot{s}^{-1}) f(r^{-1} s^{-1} t) \, d\mu(s) \, d\mu(r) \\ &= \gamma(t) \int_G g^*(s) \phi \cdot f(s^{-1} t) \, d\mu(s) \\ &= \langle g \, , \, \phi \cdot f \rangle_{B_0}(t). \end{aligned} \tag{C.29}$$

Thus $\big\langle g \, , \, L(\phi)(f) \big\rangle_B = \langle g \, , \, \phi \cdot f \rangle_{B_0}$ as desired, and we have established (C.27).

It remains to verify that for $b \in B_0$ and $f \in \mathsf{X}_0$ we have

$${}_{E_0}\big\langle x \cdot b \, , \, x \cdot b \big\rangle \leq \|b\|^2_{E_0} \langle f \, , \, f \rangle. \tag{C.30}$$

*We've used $\langle \cdot \, , \, \cdot \rangle_B$ in place of $\langle \cdot \, , \, \cdot \rangle_{B_0}$ to emphasize that, *a priori*, $L(\phi)(f)$ lives in the completion X_1 rather than in $q(\mathsf{X}_0)$.

We proceed exactly as above. Let X_2 be the completion of X_0 as a Hilbert $C_0(G/H) \rtimes_\tau G$-module. (Once we verify (C.30), then X_1 and X_2 will coincide with X by Proposition 3.11.) We define a *right* action of H on X_0 by

$$f \cdot u_t(s) = f(st^{-1})\Delta(t^{-1})\gamma(t).$$

Note that $t \mapsto f \cdot u_t$ is continuous for the inductive limit topology, and

$$\begin{aligned} f \cdot u_{th}(s) &= f(sh^{-1}t^{-1})\Delta(h^{-1}t^{-1})\gamma(ht) \\ &= (f \cdot u_t)(sh^{-1})\Delta(h^{-1})\gamma(h) = (f \cdot u_t) \cdot u_h(s). \end{aligned}$$

Furthermore, for each $p \in H$ we have

$$\begin{aligned} {}_{E_0}\langle f \cdot u_p \, , \, g \cdot u_p \rangle(r, \dot{s}) &= \int_H f(stp^{-1})\overline{g(r^{-1}stp^{-1})}\delta(p^{-1})\Delta(r^{-1}stp^{-1})\, d\nu(t) \\ &= \int_H f(st)\overline{g(r^{-1}st)}\Delta(r^{-1}st)\, d\nu(t) \\ &= {}_{E_0}\langle f \, , \, g \rangle(r, \dot{s}). \end{aligned}$$

Thus if we view operators in $\mathcal{L}(\mathsf{X}_2)$ as operators on the right of X_2 (so that $x(TS) = (xT)S$), $p \mapsto u_p$ is a strongly continuous homomorphism of H into $U\mathcal{L}(\mathsf{X}_2)$. By Proposition C.17,

$$R(b) := \int_H b(t)u_t \, d\nu(t)$$

defines a homomorphism of $C^*(H)$ into $\mathcal{L}(\mathsf{X}_2)$. A computation similar to that in (C.29) shows that

$${}_E\langle fR(b) \, , \, g \rangle = {}_{E_0}\langle f \cdot b \, , \, g \rangle \quad \text{for all } f, g \in \mathsf{X}_0 \text{ and } b \in B_0,$$

and then (C.30) follows from $\|R(b)\| \leq \|b\|$. □

In the course of proving Theorem C.23, we established the following result which will be useful in the next section.

Corollary C.26. *If H is a closed subgroup of a locally compact group G, then there is a covariant homomorphism $(j_{C_0(G/H)}, j_G)$ of $\big(C_0(G/H), G\big)$ into $\mathcal{L}(\mathsf{X})$ characterized by*

$$j_G(r)(f)(s) = \tau_r(f)(s) = f(r^{-1}s), \quad \textit{and} \tag{C.31}$$

$$j_{C_0(G/H)}(F)(f)(s) = F(\dot{s})f(s) \tag{C.32}$$

for $f \in C_c(G)$, $r, s \in G$, and $F \in C_0(G/H)$.

C.5 Induced Representations of Groups

Given a unitary representation $V : H \to B(\mathcal{H}_V)$, we want to build a unitary representation of G, using the bimodule $\mathsf{X} = {}_E\mathsf{X}_{C^*(H)}$ constructed in the previous section and the machinery of Section 2.4. By Corollary C.26, $f \mapsto \tau_s(f)$ defines a strongly continuous homomorphism $j_G : G \to U\mathcal{L}(\mathsf{X})$. The representation of G induced from V is the representation $\operatorname{Ind}_H^G V$ obtained via this action of G on X, through the natural left action of $\mathcal{L}(\mathsf{X})$ on the Hilbert space $\mathsf{X} \otimes_{C^*(H)} \mathcal{H}_V$ of X–$\operatorname{Ind} \pi_V$. In other words,

$$\big(\operatorname{Ind}_H^G V\big)_s := \big(\mathsf{X}\text{–}\operatorname{Ind}_{C^*(H)}^{\mathcal{L}(\mathsf{X})} \pi_V\big)(j_G(s)). \tag{C.33}$$

Thus the induced representation is given on elementary tensors $f \otimes h \in \mathsf{X}_0 \odot \mathcal{H}_V$ by

$$\big(\operatorname{Ind}_H^G V\big)_s(f \otimes h) = \tau_s(f) \otimes h.$$

Our first goal in this section is to characterize the representations of G which are induced from a fixed subgroup H. The following version of Mackey's imprimitivity theorem was first stated by Rieffel [147], building on ideas of Glimm [58].

Theorem C.27 (Mackey's Imprimitivity Theorem). *Let H be a closed subgroup of a locally compact group G. Suppose that U is a unitary representation of G on $\mathcal{H}_U$. Then U is unitarily equivalent to a representation induced from H if and only if there is a representation M of $C_0(G/H)$ on $\mathcal{H}_U$ such that (M, U) is a covariant representation of $\big(C_0(G/H), G, \tau\big)$.*

Theorem C.27 will follow directly from Theorem C.23 and some basic results concerning representations of $C_0(G/H) \rtimes_\tau G$. The first of these is the following.

Lemma C.28. *If H is a closed subgroup of a locally compact group G, there is a covariant homomorphism $(i_{C_0(G/H)}, i_G)$ of $\big(C_0(G/H), G, \tau\big)$ into $M\big(C_0(G/H) \rtimes_\tau G\big)$ characterized by*

$$i_G(r)(\phi)(t, \dot{s}) = \phi(r^{-1}t, r^{-1} \cdot \dot{s}), \quad \text{and} \tag{C.34}$$

$$i_{C_0(G/H)}(F)(\phi)(t, \dot{s}) = F(\dot{s})\phi(t, \dot{s}) \tag{C.35}$$

for $r, t, s \in G$, $\phi \in C_c(G \times G/H)$, and $F \in C_0(G/H)$.

Proof. In view of Theorem C.23, the left action of $C_0(G/H) \rtimes_\tau G$ on X defines an isomorphism L of $C_0(G/H) \rtimes_\tau G$ into $\mathcal{L}(\mathsf{X})$ with range $\mathcal{K}(\mathsf{X})$ (Proposition 3.8). Therefore L extends to an isomorphism of $M\big(C_0(G/H) \rtimes_\tau G\big)$ onto $\mathcal{L}(\mathsf{X})$ (Corollary 2.54). Thus the covariant homomorphism $(j_{C_0(G/H)}, j_G)$ of Corollary C.26 gives us a covariant homomorphism $(i_{C_0(G/H)}, i_G)$ into $M(C_0(G/H) \rtimes_\tau G)$ such that

$$L\big(i_G(r)\phi\big) = j_G(r)L(\phi), \quad \text{and}$$
$$L\big(i_{C_0(G/H)}(F)\phi\big) = j_{C_0(G/H)}(F)L(\phi).$$

Since $L(\phi)(f) = \phi \cdot f$ for $\phi \in C_c(G \times G/H)$ and $f \in C_c(G)$, we have

$$j_G(r)L(\phi)(f) = \tau_r(\phi \cdot f).$$

Since

$$\begin{aligned}\tau_r(\phi \cdot f)(s) = (\phi \cdot f)(r^{-1}s) &= \int_G \phi(t, r^{-1} \cdot \dot{s}) f(t^{-1}r^{-1} \cdot \dot{s})\, d\mu(t) \\ &= \int \phi(r^{-1}t, r^{-1} \cdot \dot{s}) f(r^{-1} \cdot \dot{s})\, d\mu(t),\end{aligned}$$

(C.34) follows. A similar argument establishes (C.35). □

The next result is the analogue of Corollary C.16 for $C_0(G/H) \rtimes_\tau G$.

Lemma C.29. *Suppose that H is a closed subgroup of G and $\phi, \psi \in C_c(G \times G/H)$. Then as elements of $C_0(G/H) \rtimes_\tau G$,*

$$\int_G i_{C_0(G/H)}\big(\phi(s, \cdot)\big) i_G(s)(\psi)\, d\mu(s) = \phi * \psi. \tag{C.36}$$

Proof. On one hand, we can view $s \mapsto f(s) := i_{C_0(G/H)}\big(\phi(s, \cdot)\big) i_G(s)(\psi)$ as an element of $C_c\big(G, C_0(G/H) \rtimes_\tau G\big)$, and then

$$\tilde{\theta} := \int_G f(s)\, d\mu(s)$$

defines an element $\tilde{\theta}$ of $C_0(G/H) \rtimes_\tau G$ by Lemma C.3. On the other hand, we can also view f as taking values in $C_c(G \times G/H)$, and define a function $\theta \in C_c(G \times G/H)$ by

$$\begin{aligned}\theta(t, \dot{r}) : &= \int_G i_{C_0(G/H)}\big(\phi(s, \cdot)\big) i_G(s)(\psi)(t, \dot{r})\, d\mu(s) \\ &= \int_G \phi(s, \dot{r}) \psi(s^{-1}t, s^{-1} \cdot \dot{r})\, d\mu(s) \\ &= \phi * \psi(t, \dot{r}).\end{aligned}$$

To prove the lemma, we need to show that $\theta = \tilde{\theta}$ in $C_0(G/H) \rtimes_\tau G$. To do this, we use a variation of the proof of Lemma C.15. If $F \in C_c(G \times G \times G/H)$, then

$$(t, \dot{r}) \mapsto \int_G F(s, t, \dot{r})\, d\mu(s)$$

is an element of $C_c(G \times G/H)$. Composing with the inclusion of $C_c(G \times G/H)$ into $C_0(G/H) \rtimes_\tau G$ gives a linear map $\Phi : C_c(G \times G \times G/H) \to C_0(G/H) \rtimes_\tau G$. Viewing $s \mapsto F(s, \cdot, \cdot)$ as an element of $C_c(G, C_0(G/H) \rtimes_\tau G)$, we get a linear map $\Psi : C_c(G \times G \times G/H) \to C_0(G/H) \rtimes_\tau G$ via Lemma C.3. The maps Φ and Ψ agree on functions of the form $F(s, t, \dot{r}) = g(s)k(t, \dot{r})$ for $g \in C_c(G)$ and $k \in C_c(G \times G/H)$. Just as in the proof of Lemma C.15, Φ are Ψ can be shown to be continuous with respect to the inductive limit topology on $C_c(G \times G \times G/H)$; thus Φ and Ψ agree on all of $C_c(G \times G \times G/H)$. The equality of θ and $\tilde{\theta}$ now follows by taking $F(s, t, \dot{r}) = \phi(s, \dot{r})\psi(s^{-1}t, s^{-1} \cdot \dot{r})$. □

We can now prove that every representation of $C_0(G/H) \rtimes_\tau G$ is the integrated form of a covariant representation.

Proposition C.30. *Suppose that L is a nondegenerate representation of $C_0(G/H) \rtimes_\tau G$ on $\mathcal{H}_L$. Then there is a unique covariant representation (M^L, U^L) of $\big(C_0(G/H), G, \tau\big)$ on $\mathcal{H}_L$ such that $L = M^L \rtimes U^L$. Furthermore,*

$$M^L(F) := \bar{L}\big(i_{C_0(G/H)}(F)\big) \quad \textit{and} \tag{C.37}$$

$$U^L_s := \bar{L}\big(i_G(s)\big). \tag{C.38}$$

Proof. Define M^L and U^L by (C.37) and (C.38), respectively. Using the strong continuity of i_G and the nondegeneracy of L, it is not hard to verify that U^L is strongly continuous. Then the covariance of $(i_{C_0(G/H)}, i_G)$ implies that (M^L, U^L) is a covariant representation. Applying L to both sides of (C.36) shows that

$$M^L \rtimes U^L(\phi)L(\psi) = L(\phi)L(\psi) \quad \text{for all } \phi, \psi \in C_c(G \times G/H).$$

Since L is nondegenerate, it follows that $L = M^L \rtimes U^L$, as claimed.

On the other hand, if $L = M \rtimes U$ and $\phi \in C_c(G \times G/H)$, then

$$\begin{aligned}
\bar{L}\big(i_G(r)\big)L(\phi) = L\big(i_G(r)\phi\big) &= \int_G M\big(\phi(r^{-1}s, r^{-1}\cdot)\big)U_s \, d\mu(s) \\
&= \int_G M\big(\phi(s, r^{-1}\cdot)\big)U_r U_s \, d\mu(s) \\
&= \int_G U_r M\big(\phi(s,\cdot)\big)U_s \, d\mu(s) \\
&= U_r L(\phi).
\end{aligned}$$

Since a similar computation shows that $\bar{L}\big(i_{C_0(G/H)}(F)\big)L(\phi) = M(F)L(\phi)$, the uniqueness assertion follows from the nondegeneracy of L. □

Corollary C.31. *Suppose that V is a unitary representation of a closed subgroup H of a locally compact group G. Let $L = \mathsf{X}\text{–}\mathrm{Ind}\,\pi_V$ be the corresponding representation of $C_0(G/H) \rtimes_\tau G$. Then L is the integrated form of a covariant representation (M^L, U^L) with*

$$U^L = \mathrm{Ind}_H^G V.$$

Proof. This follows from (C.33) and (C.38), because the identification of $M(C_0(G/H) \rtimes_\tau G)$ with $\mathcal{L}(\mathsf{X})$ takes i_G to j_G. □

Mackey's Imprimitivity Theorem now follows easily:

Proof of Theorem C.27. If U is induced, then the existence of a representation M such that (M, U) is covariant follows from Corollary C.31. The "only if" part of the theorem follows.

On the other hand, if (M, U) is covariant, then $M \rtimes U$ is a nondegenerate representation of $C_0(G/H) \rtimes_\tau G$. Since the latter is Morita equivalent to $C^*(H)$

via X, $M \rtimes U$ is equivalent to X–Ind π for some nondegenerate representation π of $C^*(H)$ (Theorem 3.29). But $\pi = \pi_V$ for some unitary representation V of H (Proposition C.19), and U is equivalent to $\operatorname{Ind}_H^G V$ by Corollary C.31 and the uniqueness assertion in Proposition C.30. □

It is sometimes convenient to have a more concrete description of $\operatorname{Ind}_H^G V$ and the space on which it acts. This is provided by Mackey's original construction of induced representations of second countable locally compact groups, which was extended to general locally compact groups by Blattner. (Details and references can be found in the Introduction and Chapter XI of [54].) We will give their description and prove that it is equivalent to ours.

Suppose that H is a closed subgroup of a locally compact group G, and that V is a unitary representation of H on $\mathcal{H}_V$. Let

$$\mathcal{V}_1 := \{\, \xi \in C_b(G, \mathcal{H}_V) : \xi(rt) = V_t^{-1}\big(\xi(r)\big) \text{ for all } r \in G \text{ and } t \in H \,\}.$$

Notice that if $\xi \in \mathcal{V}_1$, then $\|\xi(rt)\| = \|\xi(r)\|$ for $t \in H$, and hence $\dot{r} \mapsto \|\xi(r)\|$ is a well-defined function on G/H. We set

$$\mathcal{V}_c := \{\, \xi \in \mathcal{V}_1 : \dot{r} \mapsto \|\xi(r)\| \text{ belongs to } C_c(G/H) \,\}.$$

Let α be a quasi-invariant measure on G/H such as that constructed in Lemma C.2. Then

$$(\xi \mid \eta) := \int_{G/H} \big(\xi(r) \mid \eta(r)\big)\, d\alpha(\dot{r}) \tag{C.39}$$

is a well-defined positive-definite sesquilinear form on $\mathcal{V}_c$. The completion $\mathcal{V} := L^2_V(G, \alpha)$ is a Hilbert space.

Hooptedoodle C.32. We use the notation $\mathcal{V} := L^2_V(G, \alpha)$ because $\mathcal{V}$ can be realized as a space of (equivalence classes) of functions on G. Precisely, each vector in $\mathcal{V}$ is a class of functions $\xi : G \to \mathcal{H}_V$ such that $\xi(rt) = V_t^{-1}\big(\xi(r)\big)$ almost everywhere and $\dot{r} \mapsto \|\xi(r)\|$ belongs to $L^2(G/H, \alpha)$.

Theorem C.33. *Suppose that H is a closed subgroup of a locally compact group G, and that V is a unitary representation of H on $\mathcal{H}_V$. Let $\rho : G \to (0, \infty)$ be a continuous function on G such that $\rho(rt) = \big(\delta(t)/\Delta(t)\big)\rho(r)$ for all $r \in G$ and $t \in H$. Let α be the quasi-invariant measure on G/H constructed from ρ as in Lemma C.2, and let $\mathcal{V} = L^2_V(G, \alpha)$ be the Hilbert space constructed above. Then*

$$(U_s\xi)(r) := \big(\rho(s^{-1}r)/\rho(r)\big)^{\frac{1}{2}}\xi(s^{-1}r) \quad \text{for } s, r \in G \text{ and } \xi \in \mathcal{V}_c$$

extends to a unitary representation of G on $\mathcal{V}$, which is unitarily equivalent to $\operatorname{Ind}_H^G V$.

Proof. We can assume that α has been chosen so that (C.3) holds with respect to Haar measures μ and ν on G and H, respectively. The left invariance of ν allows us to define $W : \mathsf{X}_0 \odot \mathcal{H}_V \to \mathcal{V}_c$ by

$$\big(W(f \otimes h)(r) \mid k\big) := \int_H \rho(rt)^{-\frac{1}{2}} f(rt)\big(V_t h \mid k\big)\, d\nu(t) \quad \text{for all } k \in \mathcal{H}_V.$$

Using that $\rho(rt) = \gamma(t)^{-2}\rho(r)$ for $t \in H$, we can compute the inner product in $\mathsf{X} \otimes_{C^*(H)} \mathcal{H}_V$:

$$\begin{aligned}
(f \otimes h \mid g \otimes k) &= \big(\pi_V\big(\langle g \,,\, f\rangle_{B_0}\big)h \mid k\big) \\
&= \int_H \langle g \,,\, f\rangle_{B_0}(t)(V_t h \mid k)\, d\nu(t) \\
&= \int_H \int_G \gamma(t)\overline{g(r)} f(rt)(V_t h \mid k)\, d\mu(r)\, d\nu(t) \\
&= \int_H \int_{G/H} \int_H \gamma(t)\overline{g(rs)} f(rst)\rho(rs)^{-1}(V_t h \mid k)\, d\nu(s)\, d\alpha(\dot{r})\, d\nu(t) \\
&= \int_{G/H} \int_H \int_H \gamma(s^{-1}t)\overline{g(rs)} f(rt)\rho(rs)^{-1}(V_{s^{-1}t} h \mid k)\, d\nu(s)\, d\nu(t)\, d\alpha(\dot{r}) \\
&= \int_{G/H} \int_H \int_H \gamma(s^{-1}t)\overline{g(rs)} f(rt)\rho(rs)^{-1}(V_t h \mid V_s k)\, d\nu(s)\, d\nu(t)\, d\alpha(\dot{r}) \\
&= \int_{G/H} \int_H \int_H \big(\rho(rt)^{-\frac{1}{2}} f(rt) V_t h \mid \rho(rs)^{-\frac{1}{2}} g(rs) V_s k\big)\, d\nu(s)\, d\nu(t)\, d\alpha(\dot{r}) \\
&= \int_{G/H} \big(W(f \otimes h)(r) \mid W(g \otimes k)(r)\big)\, d\alpha(\dot{r}).
\end{aligned}$$

Thus W is isometric, and extends to an isometry of $\mathsf{X} \otimes_{C^*(H)} \mathcal{H}_V$ into $\mathcal{V}$.

To see that W is surjective, and therefore unitary, fix $\xi \in \mathcal{V}_c$ and $\epsilon > 0$. It will suffice to produce η in the range of W such that $\|\xi - \eta\| < \epsilon$. Notice that the range of W is closed under multiplication by elements of $C_c(G/H)$. Let C be a compact set in G such that the support of ξ is contained in the interior of CH. We first claim that it will suffice to find for each $r \in \mathrm{supp}(\xi) \cap C$ a function ξ_r in the image of W such that

$$\|\xi_r(r) - \xi(r)\| < \frac{\epsilon}{\alpha\big(q(C)\big)^{\frac{1}{2}}}. \tag{C.40}$$

If (C.40) holds, then there is a neighbourhood N_r of r in G such that $N_r \subset C$ and

$$\|\xi_r(s) - \xi(s)\| < \frac{\epsilon}{\alpha\big(q(C)\big)^{\frac{1}{2}}} \quad \text{for all } s \in N_r; \tag{C.41}$$

since the norms of functions in $\mathcal{V}_c$ are constant on H-cosets, (C.41) holds for all $s \in N_r H$. By compactness there are $r_1, \ldots, r_n$ in C such that $C \subset \bigcup_{i=1}^n N_{r_i}$, and a partition of unity argument gives functions $\mu_i \in C_c^+(G/H)$ such that $\mathrm{supp}(\mu_i) \subset q(N_{r_i})$, $0 \le \mu_i \le 1$, $\sum_i \mu_i(\dot{s}) = 1$ for $s \in CH$ and $\sum_i \mu_i(\dot{s}) \le 1$ for all s. Then $\sum_i \mu_i \xi_{r_i}$ belongs to the range of W, and

$$\|\sum\nolimits_{i=1}^n \mu_i(\dot{s})\xi_{r_i}(s) - \xi(s)\| < \frac{\epsilon}{\alpha\big(q(C)\big)^{\frac{1}{2}}} \quad \text{for all } s \in G. \tag{C.42}$$

Since (C.42) vanishes off CH, we have $\|\sum_i \mu_i \xi_{r_i} - \xi\| < \epsilon$. This proves the claim.

Now let $r \in C \cap (\operatorname{supp}\xi)$. Since $s \mapsto V_s h$ is continuous for all $h \in \mathcal{H}_V$, there is a neighbourhood N of e in G such that $rN \subset C$ and $\|V_t(\xi(r)) - \xi(r)\| < \epsilon\alpha(q(C))^{-\frac{1}{2}}$ for $t \in N$. If $z \in C_c^+(G)$ has $\operatorname{supp}(z) \subset rN$ and

$$\int_H \rho(rt)^{-\frac{1}{2}} z(rt)\, d\nu(t) = 1,$$

then $\xi_r = W(z \otimes \xi(r))$ satisfies (C.40). This completes the proof that W is unitary.

To finish off, we compute

$$\begin{aligned}
\big(W(\operatorname{Ind}_H^G V)_s(f \otimes h)(r) \mid k\big) &= \int_H \rho(rt)^{-\frac{1}{2}} f(s^{-1}rt)(V_t h \mid k)\, d\nu(t) \\
&= \Big(\frac{\rho(s^{-1}r)}{\rho(r)}\Big)^{\frac{1}{2}} \int_H \rho(s^{-1}rt)^{-\frac{1}{2}} f(s^{-1}rt)(V_t h \mid k)\, d\nu(t) \\
&= \Big(\frac{\rho(s^{-1}r)}{\rho(r)}\Big)^{\frac{1}{2}} \big(W(f \otimes h)(s^{-1}r) \mid k\big) \\
&= \big(U_s W(f \otimes h) \mid k\big),
\end{aligned}$$

and the Theorem follows. □

C.6 The Stone-von Neumann Theorem

When H is trivial, Theorem C.23 is an abstract formulation of a classical result known as either the von Neumann uniqueness theorem [153, §10.4] or the Stone-von Neumann theorem. This theorem had its origins in reconciling the Heisenberg and Schrödinger formulations of quantum mechanics; further references and discussion can be found in [143].

Theorem C.34 (Stone-von Neumann Theorem). *Suppose that G is a locally compact group. Let λ be the left-regular representation of G on $L^2(G)$ given by $\lambda_s h(r) = h(s^{-1}r)$, and let M be the representation of $C_0(G)$ on $L^2(G)$ by pointwise multiplication: $M(f)h(r) = f(r)h(r)$. Then (M, λ) is a covariant representation of $(C_0(G), G)$ and $M \rtimes \lambda$ is a faithful irreducible representation of $C_0(G) \rtimes_\tau G$ with range $\mathcal{K}(L^2(G))$.*

After giving the proof of this result, we'll use it to derive something more closely approximating the classical formulation (Theorem C.38).

Proof of Theorem C.34. A quick calculation shows that (M, λ) is covariant, and hence there is a corresponding representation $M \rtimes \lambda$ of $C_0(G) \rtimes_\tau G$ on $L^2(G)$. Theorem C.23 implies that $C_0(G) \rtimes_\tau G$ is Morita equivalent to $\mathbb{C}$; thus Theorem 3.22 implies that $C_0(G) \rtimes_\tau G$ is simple, and $M \rtimes \lambda$ must be faithful. Therefore it suffices to see that $M \rtimes \lambda$ maps $C_c(G \times G)$ onto $\mathcal{K}(L^2(G))$; irreducibility will then follow because $\mathcal{K}(\mathcal{H})$ acts irreducibly on $\mathcal{H}$.

If $K \in C_c(G \times G)$, then

$$f_K(s,t) := K(t, s^{-1}t)\Delta(s^{-1}t) \tag{C.43}$$

defines an element of $C_c(G \times G)$, and

$$\begin{aligned} M \rtimes \lambda(f_K)h(s) &= \int_G f_K(r,s)h(r^{-1}s)\,d\mu(r) \\ &= \int_G f_K(sr^{-1},s)\Delta(r^{-1})h(r)\,d\mu(r) \\ &= \int_G K(s,r)h(r)\,d\mu(r). \end{aligned}$$

Thus $M \rtimes \lambda(f_K)$ is the Hilbert-Schmidt operator with kernel $K \in C_c(G \times G) \subset L^2(G \times G)$ [31, Chap. XI §6]. In particular, $M \rtimes \lambda(f_K)$ is compact, and since the Hilbert-Schmidt operators are dense in the compact operators, the range of $M \rtimes \lambda$ contains $\mathcal{K}\big(L^2(G)\big)$. On the other hand, given $f \in C_c(G \times G)$,

$$K(s,t) := f(st^{-1},s)\Delta(t^{-1})$$

satisfies (C.43) with $f = f_K$. Therefore the range of $M \rtimes \lambda$ is precisely $\mathcal{K}\big(L^2(G)\big)$. □

Remark C.35. Since every nondegenerate representation of the compact operators is equivalent to a direct sum of copies of the identity representation (Lemma B.34), it follows that every nondegenerate representation of $C_0(G) \rtimes_\tau G$ is a equivalent to a direct sum of copies of $M \rtimes \lambda$.

Hooptedoodle C.36. Theorem C.23 can also be used to prove that $C_0(G/H) \rtimes_\tau G$ is isomorphic to $C^*(H) \otimes_\sigma \mathcal{K}\big(L^2(G/H,\alpha)\big)$, where α is any quasi-invariant measure on G/H [63, 85]. However, the isomorphism is not natural when H is nontrivial.

When G is abelian, Theorem C.34 has a more classical form. To see this, we first want to remark that representations of $C_0(G) \rtimes_\alpha G$ are determined by two unitary representations satisfying a certain commutation relation (C.46). Let $\widehat{G}$ be the Pontryagin dual of G. After identifying the dual of $\widehat{G}$ with G via the Pontryagin Duality Theorem [55, Theorem 4.31], $C_0(G)$ can be identified with the the group C^*-algebra $C^*(\widehat{G})$ as in Example C.20: the Fourier transform $F : C_c(\widehat{G}) \to C_0(G)$ defined by

$$F(\phi)(s) = \int_{\widehat{G}} \phi(\gamma)\gamma(s)\,d\nu(\gamma),$$

extends to an isomorphism of $C^*(\widehat{G})$ onto $C_0(G)$. Thus the nondegenerate representations π of $C_0(G)$ are in one-to-one correspondence with unitary representations S^π of $\widehat{G}$:

$$\pi\big(F(\phi)\big) = \int_{\widehat{G}} \phi(\gamma)S^\pi_\gamma\,d\nu(\gamma), \text{ and} \tag{C.44}$$

$$S^\pi_\gamma = \bar{\pi}\big(\bar{F}(i_{\widehat{G}}(\gamma))\big), \tag{C.45}$$

where $\bar{F}$ is the extension of F to $M\big(C^*(\widehat{G})\big)$. (It is not hard to check that the extension $\bar{F}$ of F to $M(C^*(\widehat{G}))$ takes the generator $i_{\widehat{G}}(\gamma)$ to the function $\gamma \in C_b(G) = M(C_0(G))$.)

A pair (R, S) consisting of a unitary representation R of G on $\mathcal{H}$ and a unitary representation S of $\widehat{G}$ on H is called a *Heisenberg representation* of G if

$$S_\gamma R_s = \gamma(s) R_s S_\gamma \quad \text{for all } s \in G \text{ and } \gamma \in \widehat{G}. \tag{C.46}$$

Example C.37. Define $V : \widehat{G} \to UL^2(G)$ by $V_\gamma h(r) := \gamma(r)h(r)$. Then (λ, V) is a Heisenberg representation of G on $L^2(G)$ called the *Schrödinger representation.*

If $\phi \in C_c(\widehat{G})$ and $s \in G$, define $s \cdot \phi$ by $s \cdot \phi(\gamma) := \overline{\gamma(s)}\phi(\gamma)$. Then

$$F(s \cdot \phi)(r) = F(\phi)(s^{-1}r) = \tau_s\big(F(\phi)\big)(r).$$

Thus if (R, S) is a Heisenberg representation of G on $\mathcal{H}$ and if π_S is the integrated form of S, then it follows from (C.44) and Lemma C.11 that

$$\begin{aligned} R_s \pi\big(F(\phi)\big) &= \int_{\widehat{G}} \phi(\gamma) R_s S_\gamma \, d\nu(\gamma) \\ &= \int_{\widehat{G}} s \cdot \phi(\gamma) S_\gamma \, d\nu(\gamma) R_s \\ &= \pi\big(\tau_s(F(\phi))\big) R_s; \end{aligned}$$

in other words, (π_S, R) is a covariant representation of $\big(C_0(G), G, \tau\big)$.

Conversely, suppose that (π, R) is a covariant representation of $\big(C_0(G), G, \tau\big)$, and that π is the integrated form of $S = S^\pi$, so that S is given by (C.45). Then a computation shows that

$$F\big(i_{\widehat{G}}(\gamma)(s \cdot \phi)\big)(r) = \gamma(s) F\big(i_{\widehat{G}}(\gamma)\phi\big)(s^{-1}r) = \gamma(s)\tau_s\big(F\big(i_{\widehat{G}}(\gamma)\phi\big)\big)(r).$$

It follows that

$$\begin{aligned} S_\gamma R_s \pi\big(F(\phi)\big) &= S_\gamma \pi\big(\tau_s\big(F(\phi)\big)\big) R_s = S_\gamma \pi\big(F(s \cdot \phi)\big) R_s \\ &= \pi\big(F\big(i_{\widehat{G}}(\gamma) s \cdot \phi\big)\big) R_s \\ &= \gamma(s) \pi\big(\tau_s\big(F\big(i_{\widehat{G}}(\gamma)\phi\big)\big)\big) R_s \\ &= \gamma(s) R_s \pi\big(F\big(i_{\widehat{G}}(\gamma)\phi\big)\big) \\ &= \gamma(s) R_s S_\gamma \pi\big(F(\phi)\big). \end{aligned}$$

Since π is nondegenerate, this implies that (R, S) is a Heisenberg representation. We deduce that (R, S) is a Heisenberg representation if and only if (π_S, R) is a covariant representation of $\big(C_0(G), G, \tau\big)$.

Finally, notice that if $\mathcal{K} \subset \mathcal{H}$ is an invariant subspace for $\pi \rtimes R$, then $\mathcal{K}$ is invariant for both π and R. (This follows from Proposition C.30.) The space $\mathcal{K}$ is also invariant for the unitary representation S^π corresponding to π. Thus a direct-sum decomposition of $\pi \rtimes R$ gives a direct-sum decomposition of the corresponding Heisenberg representation. Since any nondegenerate representation $\pi \rtimes R$ is equivalent to a direct sum of copies of $M \rtimes \lambda$ (Remark C.35), and since it is easy to see that $S^M = V$, we have proved the classical Stone-von Neumann uniqueness theorem.

Theorem C.38 (von Neumann Uniqueness Theorem). *Suppose that G is a locally compact abelian group. Then every Heisenberg representation of G is equivalent to a direct sum of copies of the Schrödinger representation of G on $L^2(G)$.*

Appendix D

Miscellany

D.1 Direct Limits

We will need to use direct limits of groups to define sheaf cohomology groups. The definition itself is a bit messy, so we stress that a direct limit should be thought of as a generalized union. Indeed, the idea is to map all the groups to one large group, and then take the union of the images; the rub is that elements in different groups must be declared equal when their images coincide.

More precisely, let Λ be a directed set. A directed system of groups on Λ is a family of groups $\{ G_\lambda \}_{\lambda \in \Lambda}$ together with homomorphisms $\tau_{\lambda\beta} : G_\lambda \to G_\beta$ for each pair of indices such that $\lambda \leq \beta$, which satisfy $\tau_{\lambda\lambda} = \mathrm{id}$ for all λ and $\tau_{\beta\gamma} \circ \tau_{\lambda\beta} = \tau_{\lambda\gamma}$ whenever $\lambda \leq \beta \leq \gamma$. A family of homomorphisms $h_\lambda : G_\lambda \to H$ is said to be *compatible* if

$$\begin{array}{ccc} G_\lambda & \xrightarrow{\tau_{\lambda\beta}} & G_\beta \\ & {\scriptstyle h_\lambda} \searrow \quad \swarrow {\scriptstyle h_\beta} & \\ & H & \end{array}$$

commutes for every pair satisfying $\lambda \leq \beta$. An *inductive limit* of the system $\{ G_\lambda, \tau_{\lambda\beta} \}$ is a group G together with a compatible family of homomorphisms $\tau^\lambda : G_\lambda \to G$ such that, whenever $\{ h_\lambda \}$ is a compatible family of homomorphisms into H, there is a unique homomorphism $h : G \to H$ such that

$$\begin{array}{ccc} G_\lambda & \xrightarrow{\tau^\lambda} & G \\ & {\scriptstyle h_\lambda} \searrow \quad \swarrow {\scriptstyle h} & \\ & H & \end{array}$$

commutes.

Every directed system has a direct limit, and it is unique up to isomorphism. We denote it by

$$\varinjlim_{\lambda \in \Lambda} G_\lambda .$$

The direct limit can be directly constructed as the quotient of a subgroup of $\prod_\lambda G_\lambda$, as follows. Let G be the subgroup of those elements which eventually satisfy $\tau_{\lambda\beta}(g_\lambda) = g_\beta$ for $\lambda \le \beta$. Let

$$F := \{\, x \in G : x_\lambda = 0 \text{ for large } \lambda \,\}.$$

Then G/F is a direct limit for the system G_λ, with $\tau^\lambda : G_\lambda \to G \to G/H$ defined by

$$\tau^\lambda(g)_\beta = \begin{cases} \tau_{\lambda\beta}(g) & \text{if } \lambda \le \beta, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It is important to remember that every element of $\varinjlim G_\lambda$ has the form $\tau^\lambda(g)$ for some λ and $g \in G_\lambda$, and that $\tau^\lambda(g) = \tau^\beta(g')$ if and only if there exists γ such that $\lambda \le \gamma$, $\beta \le \gamma$ and $\tau_{\lambda\gamma}(g) = \tau_{\beta\gamma}(g')$.

A standard observation about direct limits is the following. Suppose that $\{\, H_\lambda, \rho_{\lambda\beta} \,\}$ is another directed system of groups indexed by the same directed set Λ, and there are homomorphisms $h_\lambda : G_\lambda \to H_\lambda$ such that

$$\begin{array}{ccc} G_\lambda & \xrightarrow{\tau_{\lambda\beta}} & G_\beta \\ {\scriptstyle h_\lambda}\downarrow & & \downarrow{\scriptstyle h_\beta} \\ H_\lambda & \xrightarrow[\rho_{\lambda\beta}]{} & H_\beta \end{array}$$

commutes for every $\lambda \le \beta$. Then we have a commutative diagram

$$\begin{array}{ccc} G_\lambda & \xrightarrow{\tau_{\lambda\beta}} & G_\beta \\ {\scriptstyle h_\lambda}\downarrow & & \downarrow{\scriptstyle h_\beta} \\ H_\lambda & \xrightarrow[\rho_{\lambda\beta}]{} & H_\beta \\ & {\scriptstyle \rho^\lambda}\searrow \quad \swarrow{\scriptstyle \rho^\beta} & \\ & \varinjlim H_\lambda & \end{array}$$

whenever $\lambda \le \beta$, so $h'_l := \rho^\lambda \circ h_l$ defines a compatible family of homomorphisms on $\{\, G_\lambda, \tau_{\lambda\beta} \,\}$, and there is a unique homomorphism $h : \varinjlim G_\lambda \to \varinjlim H_\lambda$ characterized by $h\big(\tau^\lambda(g)\big) = \rho^\lambda\big(h_\lambda(g)\big)$.

D.2 The Inductive Limit Topology

Throughout this section, X will be a fixed locally compact Hausdorff space. For $K \subset X$, we let

$$C_c(X; K) := \{\, f \in C_c(X) : \operatorname{supp} f \subset K \,\}.$$

equipped with the norm topology coming from the supremum norm. When K is compact, $C_c(X;K)$ is a Banach space. If $\mathcal{K}(X)$ denotes the collection of compact subsets of X, then

$$C_c(X) = \bigcup_{K \in \mathcal{K}(X)} C_c(X;K).$$

Proposition D.1 (see [54, Proposition II.14.3]). *There is a largest topology on $C_c(X)$ with respect to which (1) $C_c(X)$ is a locally convex linear space, and (2) the inclusions $\iota_K : C_c(X;K) \hookrightarrow C_c(X)$ are continuous for all $K \in \mathcal{K}(X)$.*

This topology has the following universal property (P): a linear map F from $C_c(X)$ into a locally convex space M is continuous if and only if $F \circ \iota_K$ is continuous for every compact subset K of X.

Remark D.2. Notice that it is not immediate that there is a unique topology which contains all the topologies satisfying (1) and (2): the existence of such a topology is part of the result.

Definition D.3. If X is a locally compact Hausdorff space, then the topology on $C_c(X)$ given in Proposition D.1 is called the *inductive limit topology* on $C_c(X)$.

Remark D.4. The *final topology* for the family of inclusion maps ι_K is the unique largest topology making the inclusion maps continuous [125, §1.4.9]; in this topology, $A \subset C_c(X)$ is open if and only if $\iota_K^{-1}(A)$ is open in $C_c(X;K)$ for all $K \in \mathcal{K}(X)$. This is the "inductive limit topology" in the category of topological spaces and continuous maps. It is not clear under which circumstances this topology coincides with the inductive limit topology defined above. However, it will follow from Proposition D.7 that the final topology always contains the inductive limit topology.

Property (P) implies that, if $\{ f_\lambda \}_{\lambda \in \Lambda}$ is a net which converges uniformly to f, and if the supports $\operatorname{supp} f_\lambda$ are eventually contained in a fixed compact set K, then $f_\lambda \to f$ in the inductive limit topology. Regrettably, the converse fails for nets even when $X = \mathbb{Z}$ (see Example D.9), and only holds for sequences when X is σ-compact. (See the discussion in [41, §6.6], and note that a convergent sequence is always bounded.)

The following observation is a direct consequence of property (P).

Lemma D.5. *Suppose that X is a locally compact Hausdorff space and that μ is any Borel measure on X assigning finite measure to all compact sets of X. Then the inclusion of $C_c(X)$ in $L^1(X,\mu)$ is continuous with respect to the inductive limit topology.*

Since the C^*-norm on $C_c(G)$ is dominated by the L^1-norm, we have an immediate corollary:

Corollary D.6. *The inclusion of $C_c(G)$ into $C^*(G)$ is continuous with respect to the inductive limit topology.*

The next proposition is based on Theorems 6.4 & 6.5 of [152], and requires some terminology. Recall that a subset S of a (complex) vector space V is *convex* if $tS + (1-t)S \subset S$ for all $0 \leq t \leq 1$. We say that S is *balanced* if $zS \subset S$ for all $z \in \mathbb{C}$ with $|z| \leq 1$. A collection of subsets β is a *local base* for a topology on V if $\{ v + W : v \in V, W \in \beta \}$ is a base for the topology on V. A vector space V with a T_1-topology is called a *topological vector space* if the operations of vector addition and scalar multiplication are continuous. Such a space V is an abelian topological group with respect to vector addition, so V is necessarily Hausdorff and any neighbourhood base at 0 is a local base for the topology. If V has a local base consisting of convex sets, then V is called *locally convex*. Thus normed vector spaces are are in particular locally convex topological vector spaces.

Proposition D.1 is an immediate corollary of the following.

Proposition D.7. *Suppose that X is a locally compact Hausdorff space.*

(a) *If β is the collection of balanced convex sets W in $C_c(X)$ such that $\iota_K^{-1}(W)$ is open in $C_c(X;K)$ for all compact subsets K of X, then β is a local base for a topology τ on $C_c(X)$.*

(b) *With respect to τ, $C_c(X)$ is a locally convex topological vector space.*

(c) *The inclusion $\iota_K : C_c(X;K) \hookrightarrow C_c(X)$ is a homeomorphism onto its range for each compact set K in X.*

(d) (Property (P)) *For any locally convex topological vector space M and any linear map $F : C_c(X) \to M$, F is continuous if and only if $F \circ \iota_K$ is continuous for all $K \in \mathcal{K}(X)$.*

(e) *Suppose σ is a topology on $C_c(X)$ for which $C_c(X)$ is a locally convex topological space and the inclusions ι_K are continuous for all compact subsets K of X. Then σ is contained in τ. In particular, τ is the unique strongest topology on $C_c(X)$ such that (1) and (2) of Proposition D.1 are satisfied.*

Proof. Let τ denote the collection of sets in $C_c(X)$ which are unions of sets of the form $f + W$ with $f \in C_c(X)$ and $W \in \beta$. We proceed by first showing that τ is a locally convex topology on $C_c(X)$ under which it is a topological vector space. We then show that $\big(C_c(X), \tau\big)$ satisfies (b)–(e).

To see that τ is a topology, it will suffice to see that if V_1 and V_2 are elements of τ and if $f \in V_1 \cap V_2$, then there is a $W \in \beta$ such that

$$f + W \subset V_1 \cap V_2. \tag{D.1}$$

But for $i = 1, 2$, there must be $f_i \in C_c(X)$ and $W_i \in \beta$ such that

$$f \in f_i + W_i \subset V_i.$$

Choose a compact set K containing the supports of f_1, f_2, and f. Then for $i = 1, 2$, we have $f - f_i \in W_i \cap C_c(X;K)$. Since the sets W_i are open, there exist $\delta_i > 0$ such that

$$f - f_i \in (1 - \delta_i) W_i,$$

and since each W_i is convex,

$$\begin{aligned} f + \delta_i W_i &= f_i + (f - f_i) + \delta_i W_i \subset f_i + (1 - \delta_i) W_i + \delta_i W_i \\ &\subset f_i + W_i \subset V_i. \end{aligned}$$

Therefore (D.1) holds with $W := \delta_1 W_1 \cap \delta_2 W_2$. This establishes (a).

The τ-continuity of addition is an easy consequence of the convexity of the W's:

$$(f + \tfrac{1}{2}W) + (g + \tfrac{1}{2}W) \subset (f + g) + W.$$

The τ-continuity of scalar multiplication is a bit messy and follows because each $W \in \beta$ is balanced. To see this, fix $z_0 \in \mathbb{C}$, $f_0 \in C_c(X)$ and $W \in \beta$. Since $f \in C_c(X; K)$ for some K and since $W \cap C_c(X; K)$ is a neighbourhood of 0 in $C_c(X; K)$, there exists $\delta > 0$ such that $\delta f_0 \in \frac{1}{2}W$. Choose c such that $2c\big(|z_0| + \delta\big) = 1$. Now if $f - f_0 \in W$ and $|z - z_0| < \delta$, then

$$\begin{aligned} zf - z_0 f_0 &= z(f - f_0) + (z - z_0) f_0 \in zcW + \tfrac{1}{2}W \\ &\subset c(\delta + |z_0|)W + \tfrac{1}{2}W \subset \tfrac{1}{2}W + \tfrac{1}{2}W \\ &\subset W. \end{aligned}$$

Thus scalar multiplication is continuous.

If $f, g \in C_c(X)$, then $W := \{\, h \in C_c(X) : \|h\|_\infty < \|f - g\|_\infty \,\}$ is an element of β and $f \notin g + W$. Thus the singleton $\{\, f \,\}$ is τ-closed, and τ is a T_1-topology. This proves (b).

Let τ_K denote the norm topology on $C_c(X; K)$. Suppose that $V \in \tau$ and that $f \in V \cap C_c(X; K)$. The definition of τ implies that there exists $W \in \beta$ such that $f + W \subset V$. Thus $(f + W) \cap C_c(X; K) \subset V \cap C_c(X; K)$. Since $W \cap C_c(X; K) \in \tau_K$, we deduce that $V \cap C_c(X; K) \in \tau_K$. This proves that each ι_K is continuous.

To see that ι_K is a homeomorphism, it suffices to see that the relative topology on $C_c(X; K)$ coincides with τ_K: if $E \in \tau_K$, we need to find $V \in \tau$ such that $E = V \cap C_c(X; K)$. But if $f \in E$, there exists $\delta^f > 0$ such that

$$B_{\delta^f}(f) := \{\, g \in C_c(X; K) : \|g - f\|_\infty < \delta^f \,\} \subset E.$$

Since $W_f := \{\, g \in C_c(X) : \|g\|_\infty < \delta^f \,\} \in \beta$, and since

$$(f + W_f) \cap C_c(X; K) = f + (W_f \cap C_c(X; K)) \subset E,$$

it follows that $V := \bigcup_{f \in E} f + W_f$ has the desired property. This proves (c).

Next we show that τ has property (P). Let M be a locally convex topological vector space and $F : C_c(X) \to M$ a linear map. Suppose that $F \circ \iota_K$ is continuous for all $K \in \mathcal{K}(X)$. Then $F^{-1}(V) \in \beta$ for every convex neighbourhood V of 0 in M. Therefore F is continuous with respect to τ. The converse is clear, and (d) follows.

Finally, suppose that σ is any other topology with the prescribed properties. Then property (P) implies that the identity map $\mathrm{id} : \big(C_c(X), \tau\big) \to \big(C_c(X), \sigma\big)$ is continuous. Thus $\sigma \subset \tau$, as required. □

Example D.8. Let $\mathfrak{F} := \mathbb{Z}^{\mathbb{R}^+}$ denote the collection of all functions from $\mathbb{Z}$ to $\mathbb{R}^+ := \{ x \in \mathbb{R} : x > 0 \}$. For each $p \in \mathfrak{F}$, let

$$W_p := \{ f \in C_c(\mathbb{Z}) : |f(n)| < p(n) \text{ for all } n \in \mathbb{Z} \}.$$

Each W_p is a nonempty element of the local base β for the inductive limit topology, and it is not hard to see that $\beta_0 := \{ W_p : p \in \mathfrak{F} \}$ is itself a local base for the inductive limit topology on $C_c(\mathbb{Z})$. In fact, if $W \in \beta$, the openness of $W \cap C_c(\mathbb{Z}; [-n, n])$ allows us to define $p \in \mathfrak{F}$ inductively such that $W_p \subset W$.

If $\{ f_\lambda \}_{l \in \Lambda}$ is a net in $C_c(X)$, then we say the net is *eventually compactly supported* if there is a compact set K and an index λ_0 such that $\operatorname{supp} f_\lambda \subset K$ for all $\lambda \geq \lambda_0$. The next example shows that a convergent net in $C_c(X)$ need not be eventually compactly supported even if $X = \mathbb{Z}$.

Example D.9. Let $\delta_n \in C_c(\mathbb{Z})$ be the indicator function of $\{ n \}$. Note that $\Lambda := \mathfrak{F} \times \mathbb{Z}^+$ is directed by $(p, n) \geq (q, m)$ iff $n \geq m$ and $p(k) \geq q(k)$ for all $k \in \mathbb{Z}$. If we define $f_{(p,n)} := p(n)^{-1} \delta_n \in C_c(\mathbb{Z})$, then it is not hard to see that $\{ f_\lambda \}_{\lambda \in \Lambda}$ is eventually in every W_p belonging to the local base β_0 as defined in Example D.8. In other words, $f_\lambda \to 0$ in the inductive limit topology on $C_c(\mathbb{Z})$. However, the net $\{ f_\lambda \}_{\lambda \in \Lambda}$ is not eventually compactly supported.

Suppose that Y and Z are topological spaces, and that $F : X \to Y$ is a function. If $\{ y_\lambda \}_{\lambda \in \Lambda}$ is a net in Y, then the image of $\{ y_\lambda \}_{\lambda \in \Lambda}$ is the net in Z given by $\{ F(y_\lambda) \}_{\lambda \in \Lambda}$. The function F is continuous if and only if the image of any convergent net in Y is convergent in Z. The following lemma means that despite pathologies like those exhibited in Example D.9, eventually compactly supported nets are often enough to study the inductive limit topology.

Lemma D.10. *Suppose that X is a locally compact Hausdorff space, that M is a locally convex linear space, and that $F : C_c(X) \to M$ is linear. Then F is continuous if and only if F maps eventually compactly supported convergent nets in $C_c(X)$ to convergent nets in M.*

Proof. Recall that F is continuous if and only if $F \circ \iota_K$ is continuous for all $K \in \mathcal{K}(X)$ (Property (P)). For the latter, it certainly suffices to check only nets in $C_c(X)$ which are eventually compactly supported. □

Remark D.11. One can still wonder if there is a topology on $C_c(X)$ for which the uniformly convergent eventually compactly supported nets are precisely the convergent nets. In fact, there are criteria — given in Theorem 9 of [86, Chap. 2] — on a collection of nets with proscribed limits (a *convergence class*), which characterize those which are the collection of convergent nets for some topology. It is not hard to see that the uniformly convergent eventually compactly supported nets fail condition (d) on page 74 of [86].

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