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A Tour of Subriemannian Geometries, Their Geodesics and Applications

Richard Montgomery

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A Tour of Subriemannian Geometries, Their Geodesics and Applications

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A Tour of Subriemannian Geometries, Their Geodesics and Applications

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ABSTRACT. A subriemannian, or Carnot-Carthéodory, geometry is a nonintegrable distribution, or subbundle of the tangent bundle of a manifold, which is endowed with an inner product. Part I presents the basic theory and examples, focussing on the geodesics. Chapters explaining the ideas of Cartan and Gromov are included. Part II presents applications to physics. These include Berry's quantum phase and an explanation of how a falling cat rights herself to land on her feet.

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Introduction

A *subriemannian geometry* is a manifold endowed with a distribution and a fiber inner product on the distribution. A *distribution* here means a family of k-planes, that is, a linear subbundle of the tangent bundle of the manifold. We refer to the distribution as the *horizontal space*, and objects tangent to it as *horizontal*. Given such a geometry we can define the distance between two points just as in Riemannian geometry, except that we are only allowed to travel along horizontal curves when joining the two points.

This book is a study of these geometries, focusing on their geodesics and their applications. The seeds for this book were planted in 1988 by Alex Pines, a physical chemist, and Shankar Sastry, an electrical engineer. Pines asked me, "What is the shortest loop with a given holonomy?" The data needed to pose this problem, which I call the *isoholonomic* problem, are a principal G-bundle with connection lying over a Riemannian manifold. A subriemannian geometry is constructed from this data by taking the distribution to be the horizontal space for the connection, and taking the inner product on this distribution to be the horizontal lift of the Riemannian inner product. The isoholonomic problem then becomes a special case of the problem of finding subriemannian geodesics.

Why would a physical chemist and electrical engineer be interested in the isoholonomic problem and subriemannian geometries? Pines and his postdocs, Joe Zwanziger and Maria Koenig, were designing NMR experiments to test the newly discovered quantum Berry's phase [Zwanziger et al. 1990]. Sastry and his students, Greg Walsh, Richard Murray, and others, were designing controllers for the orientation of robots and satellites using ideas from geometric mechanics. The theory underlying both the problem of mechanical reorientation and the phenomenon of Berry phase is clarified by putting them within the context of a principal G-bundle with connection.

For Berry's phase, the principal bundle is the Hopf fibration associated to the Hilbert space of the quantum theory. The total space is the sphere of normalized vectors in Hilbert space, the base is projective Hilbert space, and the group G is the circle group of phases. The connection is the unique connection invariant under the full unitary group. Pines wanted "efficient" or "short" ways of generating desired holonomies, or phases, and this was the genesis of his question.

For Sastry, the group G is the rotation group of Euclidean space. The base space is the space of all shapes the robot or satellite can take. The total space is the space of *located shapes* – robots or satellites located in physical space. I prefer to replace Sastry's robots and satellites by a cat. A falling cat, dropped with no spin, can change her shape at will while in free fall, yielding a curve in shape space. This shape curve results in a curve of motions of the physical cat in space, i.e. a curve in the total space. This curve is the horizontal lift of the base curve with respect to

the canonical connection defined by the condition that angular momentum is zero. The cat must choose her shape curve to right herself. In other words, her holonomy must be rotation by 180 degrees about her ventral axis. Presumably she wants to right herself "efficiently", or along a "short path" in shape space. The falling cat's problem thus becomes the isoholonomic problem.

Part 1 of this book explains the basics of subriemannian geometry. Part 2 concerns the subriemannian geometries of bundle type, meaning those arising on principal bundles, and their physical applications. I tell the story of Berry's phase (chapter 13) and the falling cat (chapter 14) in detail.

Chapter 1 begins by describing the simplest nontrivial subriemannian geometry and its relation to the classical isoperimetric problem. This geometry is called the Heisenberg group. It is a subriemannian geometry of bundle type, where the structure group G is the real line thought of as a group under addition, the base space is the Euclidean plane, and the curvature of the connection is the area form on the plane. The holonomy of a loop in the plane is its enclosed area, so the isoholonomic problem of Pines becomes the classical isoperimetric problem, whose solutions have been known for two millennia. The main result of chapter 1 is the normal geodesic equations, a system of Hamiltonian differential equations on the cotangent bundle of the underlying manifold, and the assertion that their solutions project to subriemannian geodesics.

The first theorem in subriemannian geometry is due to Carathéodory, and is related to Carnot's thermodynamics. For this reason Gromov and others refer to subriemannian geometry as Carnot-Carathéodory geometry. Carathéodory's theorem concerns corank-one distributions. Its generalization to distributions of arbitrary codimension is called Chow's theorem.

A distribution of corank one is defined locally by a single Pfaffian equation $\theta = 0$, where θ is a nonvanishing one-form. Recall that a corank-one distribution is called *integrable* if through each point there passes a hypersurface that is everywhere tangent to the distribution. Also recall that integrability is equivalent to the existence of local integrating factors for θ, that is, to the existence of locally defined functions T and S such that $\theta = TdS$. In this case, any horizontal path passing through a point A must lie within the hypersurface $\{S = S(A)\}$. Consequently, pairs of points A, B that lie on different hypersurfaces cannot be connected by a horizontal path. Carathéodory's theorem is the converse of this statement.

THEOREM (Carathéodory). *Let Q be a connected manifold endowed with an analytic corank-one distribution. If there exist two points that cannot be connected by a horizontal path then the distribution is integrable.*

Carathéodory developed this theorem at the urging of the physicist Max Born, in order to derive the second law of thermodynamics and the existence of the entropy function S [Born 1964, esp. p. 38–40]. From the work of Carnot, Joules, and others, it was known that there exist thermodynamic states A, B that cannot be connected to each other by adiabatic processes, meaning slow processes in which no heat is exchanged. (This impossibility is related to the impossibility of perpetual motion machines.) Translate "adiabatic process" to mean "horizontal curve". Carathéodory wrote out this horizontal constraint as a Pfaffian equation, $\theta = 0$. The integral of θ over a curve is interpreted as the net heat change undergone by the process represented by the curve. Carathéodory's theorem, combined with the

work of Carnot, Joules et al. implies the existence of integrating factors, so that $\theta = TdS$. The function S is the entropy and T is the temperature.

In this book we use Carathéodory's theorem in its contrapositive form: if a corank-one distribution is not integrable, then any two points can be connected by a horizontal path. This contrapositive form generalizes to distributions of arbitrary corank, where it is called the Chow-Rashevskii theorem, or simply Chow's theorem. It is the foundation stone of subriemannian geometry, and takes up chapter 2.

To understand Chow's theorem, first recall the Frobenius integrability theorem, valid for distributions of any corank. A distribution of rank k is called *integrable* if through every point passes a k-dimensional horizontal surface. It is called *involutive* if it is closed under Lie bracket, meaning that $[X, Y]$ is a horizontal vector field if X and Y are horizontal. The Frobenius theorem asserts that integrability is equivalent to involutivity. At the opposite extreme from the integrable distributions stand the *bracket generating* or *completely nonintegrable* distributions, for which any tangent vector can be written as the sum of iterated Lie brackets $[X_1, [X_2, [X_3, \ldots]]]$ of horizontal vector fields. Chow's theorem asserts that for a completely nonintegrable distribution on a connected manifold, any two points can be connected by a horizontal path. Carathéodory's theorem is the codimension-one version of Chow, and appears simpler because for analytic codimension-one distributions "nonintegrable" and "completely nonintegrable" are equivalent conditions. Chow's theorem yields the fact that on a connected subriemannian manifold whose underlying distribution is completely nonintegrable, the distance between any two points is finite, since there is at least one horizontal path connecting the points.

Chow's theorem gives us a license to search for minimizing geodesics, i.e. shortest horizontal curves. The basic theory of these geodesics is described in chapters 1 and 3, and in appendix D.

One of the main subtleties of our subject is that there exist subriemannian geodesics that do not satisfy the normal subriemannian geodesic equations. These strange geodesics are called *singular* geodesics. The fact that they exist, and that their existence is a topologically stable phenomenon, is my main contribution to the subject. My basic example of a singular geodesic, as elucidated by Liu and Sussmann, is detailed in chapter 3. This example lives on a distribution called the Martinet distribution.

In chapter 4 we define the curvature and the nilpotentization of a distribution. These tensorial objects do not depend on the choice of metric on the distribution. The curvature is a two-form on the distribution which measures its nonintegrability. This curvature agrees with the traditional curvature of a connection on a principal bundle when the distribution is the horizontal space of the connection. The nilpotentization is a nilpotent Lie algebra canonically associated to a "regular point" of a distribution. It was introduced by analysts studying subelliptic operators. The first term of the nilpotent Lie algebra structure is the curvature. The curvature and nilpotentization are the simplest algebraic invariants of distributions.

In chapter 5 we study the endpoint map and its singularities. The endpoint map maps a horizontal curve passing through a fixed point of a subriemannian manifold

to its endpoint. It is a map from an infinite-dimensional manifold to a finite-dimensional one. Its critical points are called *singular curves* for the distribution. They play a basic role in the theory of subriemannian geodesics.

Chapter 6 contains a host of examples of distributions. It is included to give the reader a sense of the vast array of different distributions. Mathematicians familiar with the Darboux theorem from contact geometry, or with the Frobenius theorem, often expect distributions to be finitely determined, meaning that they can be put into local normal forms not depending on functional moduli. This expectation is in general false, and I try to correct it forcefully here.

Chapter 7 is an introduction to Cartan's method of equivalence as it applies to distributions. Cartan's method is the most powerful method available for uncovering and classifying the invariants of distributions, but it is known to few researchers. I learned what I know of it through lectures by R. Bryant, many at CIMAT, in Guanajuato, Mexico. I have tried to give an indication of how the method proceeds, and of the fact that the tensors and invariants involved in the classification of typical distributions are enormously more complicated than those that occur in Riemannian geometry, such as the Riemannian curvature tensor. This chapter includes an account of the Duke University thesis of Keener Hughen [1995], which constructs the invariants of subriemannian geometries of contact type on three-manifolds. The chapter ends with an investigation of rank-four distributions in seven dimensions. This application of Cartan's method to $(4,7)$ distributions appears for the first time here. A surprise is the appearance of an intrinsically defined conformal subriemannian structure, intimately tied to the quaternions. The most symmetric of the $(4,7)$ distributions plays a central role in Pansu's extension of Mostow's rigidity theorem.

Chapter 8 contains an exposition of Mitchell's theorem, which asserts that the Gromov-Hausdorff tangent cone to a subriemannian manifold at a regular point is its nilpotentization. I include the necessary background material on the Gromov-Hausdorff topology on the space of all pointed metric spaces. Much of chapter 8 comes from Bellaiche's excellent article [Bellaiche 1996].

Chapter 9 is a pedestrian account of Gromov's amazing paper "Groups of polynomial growth and expanding maps" [Gromov 1981a]. Gromov applies subriemannian ideas in combination with the Gromov-Hausdorff topology to prove a difficult theorem in pure group theory. This is apparently the first appearance of Carnot-Carathéodory geometry in Gromov's work. The combination of ideas here is amazing, and worth knowing for anyone interested in subriemannian geometry.

Chapter 10 lists four open problems in subriemannian geometry. At the end of the chapter we present a construction due to Octavian Popp of a smooth measure defined near a regular point of a subriemannian manifold. This measure competes with the Hausdorff measure for the title of "the natural subriemannian measure".

Part 2 explores physical phenomena which are best understood in terms of subriemannian geometries on principal G-bundles. Chapter 11 describes how to pass back and forth between a G-invariant Riemannian metric and a G-invariant subriemannian metric on a principal bundle. A G-invariant Riemannian metric on such a bundle yields a Riemannian metric on the base space, a family of inner products on the Lie algebra of G, and a connection. The connection is the one whose horizontal spaces are orthogonal to the vertical spaces of the bundle. Restricting the metric to the horizontal spaces yields a subriemannian structure. The main theorem

of chapter 11 describes the relation between the Riemannian and the subriemannian geodesics which is valid when the family of inner products on the Lie algebra is constant. The theorem asserts that every normal subriemannian geodesic can be obtained from a Riemannian geodesic by projecting it down to the base space and horizontally lifting the result back to the total space. The chapter ends by applying this theorem to describe all of the normal subriemannian geodesics for a number of homogeneous subriemannian manifolds.

In chapter 12, I shift the point of view on subriemannian geodesics to the base space M of the principal bundle $Q \to M$. The normal subriemannian geodesic equations are a G-invariant system of Hamiltonian equations on the cotangent bundle T^*Q. Being invariant, the equations can be pushed down to the quotient space $(T^*Q)/G$, which is a Poisson manifold. In my Ph.D. thesis I studied this quotient. Using a connection I constructed an isomorphism of $(T^*Q)/G$ with $T^*M \oplus \mathrm{Ad}^*(Q)$. Here $\mathrm{Ad}^*(Q)$ is the co-adjoint bundle associated to $Q \to M$. It is a vector bundle over M whose typical fiber is the dual of the lie algebra of G. The Poisson bracket on the direct sum $T^*M \oplus \mathrm{Ad}^*(Q)$ is the sum of the Poisson brackets on the factors plus a coupling term that involves the curvature of the connection. Under this isomorphism, the reduced normal subriemannian geodesic equations become a system of equations known in physics as Wong's equations, the equations of a particle in a Yang-Mills field. The "nonabelian charge" of this nonabelian particle lives in the co-adjoint bundle. In the case of an Abelian group the co-adjoint bundle canonically trivializes and the charge does not evolve in this trivialization. When G is S^1 or $\mathbb{R}$, Wong's equations become the Lorentz equations describing a charged particle moving on the base space under the influence of the magnetic field which is the curvature of the connection. Besides providing an amusing connection to physics, this reformulation allows us to solve easily the subriemannian geodesic equations for a geometry not covered by the method of chapter 11. This geometry is that of a two-step nilpotent Lie group. The particular case of a free two-step nilpotent group had an important, somewhat confusing role in the study of singular geodesics, as we explain in section 12.3.5.

In chapters 13 and 14, I move to the physical examples. Chapter 13 describes the quantum Berry phase. This phase is the holonomy of the canonical connection for the Hopf fibration $S^{2n+1} \to \mathbb{C}\mathrm{P}^n$. The chapter connects the geometry to the physics, describing the relevance of the phase in interpreting experimental data. I discuss the relationships between the uncertainty principal and the Fubini-Study metric on $\mathbb{C}\mathrm{P}^n$ and between quantum statistical mechanics and nonabelian Berry phases. I present a brief overview of the mathematical foundations of quantum mechanics in the hope that this chapter might be useful to geometers unfamiliar with quantum mechanics.

Chapter 14 describes three instances of subriemannian geometries of bundle type which arise in classical mechanics. An alternative title of this chapter could be "Mundane gauge theory". In these instances the total space is the configuration space of a mechanical system, the base space is the space of shapes, and the group G is the group of rigid motions of space, or its subgroup of rotations. The first instance is the problem of a falling cat. The connection is defined by the condition that the angular momentum equals zero. The holonomy is the re-orientation the cat is trying to achieve: upside down to right-side up. The second instance is the motion of a swimming microorganism. The connection is defined by the equations

of fluid mechanics with low Reynolds number. The third instance is the classical N-body problem. The connection is again the "cat connection" – angular momentum equals zero – except that the cat consists of N point particles connected to each other by the universal law of gravitation. The examples are not new. Shapere and Wilczek, near the beginning of the Berry phase craze in the 1980s, pointed out that the cat and the microorganism can be viewed as instances of gauge theory. Guichardet realized the same thing even earlier for the N-body problem. What is new in this chapter is that we show how all three examples flow directly from the geometry of a principal bundle with a G-invariant metric. For the falling cat and the N-body problem the kinetic energy provides this metric, while for the microorganism the metric comes from the measure of the power output during a shape deformation.

Four appendices are included. Appendix A, on geometric mechanics, was written to aid the geometer not familiar with the language of mechanics and the mechanician not familiar with the language of differential geometry. Appendix B concerns line bundles and circle bundles and is included for those unfamiliar with bundle theory. Appendix C shows that the Ambrose-Singer theorem is essentially Chow's theorem applied to the bundle case. In order to prove the full Ambrose-Singer theorem we require somewhat more than Chow's theorem. We need a deep extension of Chow's theorem due to Sussmann, which we include. Finally, appendix D concerns the analysis of the endpoint map, one of the central objects of the geodesic theory. This appendix includes proofs of the geodesic existence theorems.

What this book skips. I have omitted a number of important topics in and around subriemannian geometry. To help orient the reader, I briefly describe some of these ignored topics.

I have completely ignored *nonholonomic mechanics.* Nonholonomic mechanics is the study of mechanical systems, such as pennies rolling on tables, ice skates, bicycles on roads, in which velocities are constrained by frictional forces to lie within a nonholonomic distribution. Vershik and Faddeev [1981] clearly delineate the difference between the equations of nonholonomic mechanics and those for subriemannian geodesics. (See also [Arnol'd et al. 1988], where the term *vakanomic mechanics* is used for subriemannian geometry.) Suppose our manifold Q is endowed with a Riemannian metric and a distribution. To obtain the equations of nonholonomic mechanics for a free particle $q(t)$ whose velocity is constrained to the distribution, compute its acceleration $\nabla_{\dot{q}}\dot{q}$ and orthogonally project the result onto the distribution. Both the Levi-Civita connection and the orthogonal projection require the ambient Riemannian metric. On the other hand, to obtain the normal subriemannian geodesic equations, restrict the length or energy functional to the space of paths that are tangent to the distribution and that join two given points, and write out the critical-point equations for this restricted functional. The nonholonomic mechanics and subriemannian geodesic equations are the same if and only if the distribution is involutive. The normal subriemannian geodesic equations are Hamiltonian equations on T^*Q, whereas the nonholonomic mechanics equations are not. The nonholonomic mechanics are well posed in the sense that specifying an initial position and velocity uniquely determines the solution curve, while the normal subriemannian geodesic equations are not well posed in this sense. The subriemannian equations do not require knowledge of the Riemannian metric off

of the distribution, while the nonholonomic mechanics equations do require this knowledge.

Another notable omission is that, except for a brief discussion in chapter 10, we have skipped the interplay between subriemannian geometry and the analysis of subelliptic operators. The subriemannian analogue of the Riemannian Laplacian is Hörmander's sum of squares of vector fields operator. Let $X_a, a = 1, \ldots, k$, be an orthonormal basis of horizontal vector fields for a subriemannian manifold whose distribution has rank k. Hörmander's operator is $\Delta = \sum X_a^2$, where we think of the X_a as first-order differential operators. This is a second-order linear differential operator. Its principal symbol is twice the subriemannian Hamiltonian, the Hamiltonian that governs the normal geodesics. Hörmander proved that if the distribution is bracket generating, then the equation $\Delta f = u$ can be solved for u and f on a compact domain, u square-integrable, and u and f are related by estimates similar to the classical estimates of elliptic regularity theory. For this reason, the bracket-generating condition is often referred to as Hörmander's condition. There is a large and growing literature on these operators. Notable works are those of Malliavin (see the references in [Bismut 1984; Varopoulos et al. 1992; Rothschild and Stein 1976]). Essentially any problem arising for the Riemannian Laplacian can be phrased for these operators. Can you "hear" the subriemannian metric from the spectrum of its sublaplacian? Is there a Weyl-type formula for the growth of the eigenvalues of the sublaplacian? We have ignored the study of sublaplacians, except for the question of their well-definedness. A computation shows that our definition of the sublaplacian Δ gives different results if different horizontal orthonormal frames are chosen. The question arises: does there exist a canonical sublaplacian, in analogy with the Riemannian situation? As we will see in chapter 10, this is equivalent to the question of the existence of a canonical subriemannian measure, which remains open.

There have been several detailed studies of subriemannian balls in dimension three. The phenomena are varied and vast. Computer-generated pictures of these balls and their singularities are beautiful. One of the most surprising developments is that the subriemannian balls are typically not subanalytic in a neighborhood of a singular geodesic. We have not touched these developments and refer instead to [Agrachev et al. 1996; Bonnard and Chyba 1999; Chyba 1997].

A fourth topic which we will not touch is the interplay between subriemannian geometry and ideas stemming from the Mostow rigidity theorem. The Mostow rigidity theorem asserts that if two compact Riemannian manifolds have the same dimension $n \geq 3$, constant negative curvature -1, and isomorphic fundamental groups, then they are isometric, and moreover the isomorphism between fundamental groups can be realized by an isometry [Mostow 1973]. The theorem can be rephrased as a theorem about lattices in a noncompact semisimple Lie group. The group is the isometry group of the universal cover of the two manifolds. The lattices are the fundamental groups. They act isometrically on the universal cover by deck transformations. The universal cover itself can be taken to be the unit n-ball endowed with the Poincaré metric. The boundary of the ball is the sphere at infinity and plays a central role in Mostow's proof. The proof of Mostow's theorem boils down to showing that the abstract isomorphism between the two lattices Γ_1, Γ_2 can be used to construct a Γ_1-Γ_2-equivariant quasi-conformal map from the sphere at infinity to itself, and then showing that all such maps are induced by isometries.

This idea can be copied for other noncompact semisimple groups. Pansu [1989] considered the case of $Sp(n,1)$, the isometry group for a nonpositively-curved symmetric space of dimension $4n$ which is the quaternionic analogue of real hyperbolic space. The sphere at infinity for this symmetric space inherits a conformal subriemannian structure. (We consider this geometry in detail for the case $n = 2$ at the end of chapter 7.) Pansu's version of Mostow's theorem asserts that every quasiconformal subriemannian map (defined in a certain weak sense) is induced by an isometry of the symmetric space. W. Goldman calls it a "Mostow rigidity without the lattice". Pansu needed imaginative and deep subriemannian generalizations of results from Riemannian and Euclidean analysis to obtain his results. His work was taken up and generalized six years later by the two big names of rigidity theory, Margulis and Mostow [1995].

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