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ABSTRACT. Operads were originally studied as a tool in homotopy theory, specifically for iterated loop spaces. Recently the theory of operads has received new inspiration from and applications to homological algebra, category theory, algebraic geometry and mathematical physics. Many of the theoretical results and applications, scattered in the literature, are brought together here along with new results and insights as well as some history of the subject.

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10 9 8 7 6 5 4 3 2 1 12 11 10 09 08 07
# Contents

Preface ix

## Part I

### Chapter 1. Introduction and History

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A prehistory</td>
<td>3</td>
</tr>
<tr>
<td>1.1. Lazard’s formal group laws</td>
<td>3</td>
</tr>
<tr>
<td>1.2. PROPs and PACTs</td>
<td>4</td>
</tr>
<tr>
<td>1.3. Non-Σ operads and operads</td>
<td>5</td>
</tr>
<tr>
<td>1.4. Theories</td>
<td>7</td>
</tr>
<tr>
<td>1.5. Tree operads</td>
<td>8</td>
</tr>
<tr>
<td>1.6. $A_\infty$-spaces and loop spaces</td>
<td>9</td>
</tr>
<tr>
<td>1.7. $E_\infty$-spaces and iterated loop spaces</td>
<td>12</td>
</tr>
<tr>
<td>1.8. $A_\infty$-algebras</td>
<td>13</td>
</tr>
<tr>
<td>1.9. Partiality and $A_\infty$-categories</td>
<td>14</td>
</tr>
<tr>
<td>1.10. $L_\infty$-algebras</td>
<td>17</td>
</tr>
<tr>
<td>1.11. $C_\infty$-algebras</td>
<td>19</td>
</tr>
<tr>
<td>1.12. $n$-ary algebras</td>
<td>19</td>
</tr>
<tr>
<td>1.13. Operad bar construction and Koszul duality</td>
<td>20</td>
</tr>
<tr>
<td>1.14. Cyclic operads</td>
<td>21</td>
</tr>
<tr>
<td>1.15. Moduli spaces and modular operads</td>
<td>22</td>
</tr>
<tr>
<td>1.16. Operadic interpretation of closed string field theory</td>
<td>23</td>
</tr>
<tr>
<td>1.17. From topological operads to dg operads</td>
<td>26</td>
</tr>
<tr>
<td>1.18. Homotopy invariance in algebra and topology</td>
<td>27</td>
</tr>
<tr>
<td>1.19. Formality, quantization and Deligne’s conjecture</td>
<td>29</td>
</tr>
<tr>
<td>1.20. Insertion operads</td>
<td>31</td>
</tr>
</tbody>
</table>

## Part II

### Chapter 1. Operads in a Symmetric Monoidal Category

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1. Symmetric monoidal categories</td>
<td>37</td>
</tr>
<tr>
<td>1.2. Operads</td>
<td>40</td>
</tr>
<tr>
<td>1.3. Pseudo-operads</td>
<td>45</td>
</tr>
<tr>
<td>1.4. Operad algebras</td>
<td>46</td>
</tr>
<tr>
<td>1.5. The pseudo-operad of labeled rooted trees</td>
<td>50</td>
</tr>
<tr>
<td>1.6. The Stasheff associahedra</td>
<td>56</td>
</tr>
<tr>
<td>1.7. Operads defined in terms of arbitrary finite sets</td>
<td>60</td>
</tr>
<tr>
<td>1.8. Operads as monoids</td>
<td>67</td>
</tr>
<tr>
<td>1.9. Free operads and free pseudo-operads</td>
<td>71</td>
</tr>
</tbody>
</table>
1.10. Collections, $K$-collections and $K$-operads \hfill 84  
1.11. The GK-construction \hfill 86  
1.12. Triples \hfill 88  

Chapter 2. Topology – Review of Classical Results \hfill 93  
2.1. Iterated loop spaces \hfill 93  
2.2. Recognition \hfill 94  
2.3. The bar construction: theme and variations \hfill 96  
2.4. Approximation \hfill 97  
2.5. $\Gamma$-spaces \hfill 101  
2.6. Homology operations \hfill 102  
2.7. The linear isometries operad and infinite loop spaces \hfill 106  
2.8. $W$-construction \hfill 109  
2.9. Algebraic structures up to strong homotopy \hfill 112  

Chapter 3. Algebra \hfill 121  
3.1. The cobar complex of an operad \hfill 121  
3.2. Quadratic operads \hfill 137  
3.3. Koszul operads \hfill 145  
3.4. A complex relating the two conditions for a Koszul operad \hfill 149  
3.5. Trees with levels \hfill 154  
3.6. The spectral sequences relating $N(\mathcal{P})$ and $C(\mathcal{P})$ \hfill 158  
3.7. Coalgebras and coderivations \hfill 165  
3.8. The homology and cohomology of operad algebras \hfill 173  
3.9. The pre-Lie structure on $Coder(\mathcal{F}_D(X))$ \hfill 182  
3.10. Application: minimal models and homotopy algebras \hfill 186  

Chapter 4. Geometry \hfill 203  
4.1. Configuration spaces operads and modules \hfill 203  
4.2. Deligne-Knudsen-Mumford compactification of moduli spaces \hfill 212  
4.3. Compactification of configuration spaces of points in $\mathbb{R}^n$ \hfill 218  
4.4. Compactification of configurations of points in a manifold \hfill 234  

Chapter 5. Generalization of Operads \hfill 247  
5.1. Cyclic operads \hfill 247  
5.2. Application: cyclic (co)homology \hfill 258  
5.3. Modular operads \hfill 267  
5.4. The Feynman transform \hfill 279  
5.5. Application: graph complexes \hfill 290  
5.6. Application: moduli spaces of surfaces of arbitrary genera \hfill 304  
5.7. Application: closed string field theory \hfill 312  

Epilog \hfill 327  

Bibliography \hfill 329  

Glossary of notations \hfill 339  

Index \hfill 345
Preface

Operads are mathematical devices which describe algebraic structures of many varieties and in various categories. Operads are particularly important/useful in categories with a good notion of ‘homotopy’ where they play a key role in organizing hierarchies of higher homotopies. Significant examples first appeared in the 1960’s though the formal definition and appropriate generality waited for the 1970’s. These early occurrences were in algebraic topology in the study of (iterated) loop spaces and their chain algebras. In the 1990’s, there was a renaissance and further development of the theory inspired by the discovery of new relationships with graph cohomology, representation theory, algebraic geometry, derived categories, Morse theory, symplectic and contact geometry, combinatorics, knot theory, moduli spaces, cyclic cohomology and, not least, theoretical physics, especially string field theory and deformation quantization. The generalization of quadratic duality (e.g. Lie algebras as dual to commutative algebras) together with the property of Koszulness in an essentially operadic context provided an additional computational tool for studying homotopy properties outside of the topological setting.

The aim of this book is to exhibit operads as tools for this great variety of applications, rather than as a theory pursued for its own sake. Most of the results presented are scattered throughout the literature (some of them belonging to the current authors). At times the exposition goes beyond the original sources so that some results in the book are more general than the ones in the literature. Also a few gaps in the available proofs are filled. Some items, such as the construction of various free operads, are given with all the bells and whistles for the first time here.

In an extensive introduction, we review the history (and prehistory) and hope to provide some feeling as to what operads are good for, both in a topological context and a differential graded algebraic context. The basic examples of the endomorphism operad and tree operads are presented. Just as group theory without representations is rather sterile, so operads are best appreciated by their representations, known as (varieties of) algebras, especially ‘strong homotopy’ algebras. We introduce the most common types: $A_\infty$, $E_\infty$, $L_\infty$, $C_\infty$. We also consider generalizations such as cyclic operads, modular operads and partial operads.

Next we present a technical part, reviewing basic definitions in the full glory of the symmetric monoidal category setting and relating operads to associated structures: triples (monads).

We then review classical results (mostly in topology) without going into great detail in the proofs since most results exist in a well-established literature. We emphasize the guiding principles of ‘recognition,’ ‘approximation,’ homotopy invariance and computational consequences (homology operations).
In a more algebraic section, we establish certain key constructions and properties: bar and cobar constructions, free operads, Koszul duality and cohomology of operad algebras. An application is made to providing minimal models of homotopy algebras.

The remainder of the book is devoted to providing access to some of the myriad of results of the ‘renaissance of operads’ in which operads have proved their worth in contexts quite different from those of their birth. We emphasize algebraic constructions for operads, geometric examples related to configuration spaces and moduli spaces, generalizations such as cyclic and modular operads. Such generalizations are motivated by applications to deformation quantization, string field theory, quantum cohomology and Gromov-Witten invariants. We have had to be somewhat selective in our choice of topics influenced by our own personal tastes. We are confident that we have failed to include all the latest applications since the field is progressing so rapidly; see also the Epilog.

The book is intended for researchers and students as well as anyone who wishes to get the flavor of operads and their application. We have tried to provide overviews and introductions as well as technical machinery for the reader’s use. Particularly technical details have been sequestered to appendices; in Frank Adams’ language, to operate the machine, it is not necessary to raise the bonnet (look under the hood).

ACKNOWLEDGMENTS: We wish to thank the following people, with the standard reservation that although they helped us with this book in many ways, they are not responsible for the mistakes, which are solely our own sins. We thank J.M. Boardman, J. Bernstein, Y.-Z. Huang, M.M. Kapranov, T. Kimura, P. van der Laan, J.-L. Loday, J.P. May, D.H. Van Osdol, P. Salvatore, P. Somberg, R. Vogt and A.A. Voronov.

Many others who have contributed to our research in this field also deserve our appreciation. The senior author wishes to give particular thanks to John Coleman Moore and J. Frank Adams who started him on the road to operads. The junior author wishes to express his thanks to Jiří Bok who pried the door to mathematics open for him.

Our collaboration has benefited from facilities at the University of North Carolina at Chapel Hill, USA, at the University of Pennsylvania, Philadelphia, USA, at the Mathematical Institute of the Academy of Sciences of the Czech Republic at Prague and at Bar Ilan University, Ramat Gan, Israel.

The first author was supported by grant GA ČR 201/99/0675, grant ME 333 and grant DMS-9803435. The second author was partially supported by the Israel Science Foundation – Center of Excellence Program founded by the Israel Academy of Sciences and Humanities. The third author was supported by grant DMS-9803435.

CONVENTION: The book consists of two parts. When referring to theorems, formulas, etc., we write the roman numeral of the part explicitly only when the item we refer to is not in the current part of the book.
Epilog

We have tried to provide a reasonably complete survey of operads and their applications, although with our own viewpoints and emphases. As we were completing the present text, a search to FIND operad on the arXiv produced a list of 54 articles and a search for operad ANYWHERE in the listings on MathSciNet produced abstracts of 203 articles. Of these, 15 were posted to the arXiv in the past year and 31 have publication dates listed on MathSciNet of 2000 or 2001. Many of these do appear in the previous chapters (and we will not cite them here again); rather than add the rest to our bibliography, which would become out of date during the publication process, we recommend that the reader access those lists directly.

Several of the most recent, however, deserve further attention here; we group them loosely by topic.

**Geometric**: Operads are playing an increasing role in geometry, in particular, symplectic \([Xu99, Gin01]\) and also algebraic \([Kap98, Man99]\). Ginzburg \([Gin01]\) in fact generalizes to a ‘noncommutative geometry’ for an algebra over an arbitrary cyclic quadratic Koszul operad. Recent results of Salvatore \([Sal99, Sal01]\) on topology of configuration spaces and their completions use the presence of operadic structures on these spaces.

**Homotopy Theory**: Several papers devoted to homotopy invariant algebraic structures, homotopy theory of algebras over operads and homotopy theory in the category of operads appeared recently; see \([Smi99, Til00, Hin01, Smi01]\) or the proceedings \([MMW99]\). Also some classical topics received new attention; see \([May97, Ber99]\).

**Physics**: There continues to be strong interaction between operads and mathematical physics, and especially topological field theories and string theory.

In \([KVZ97]\), Kimura, Voronov and Zuckerman study the role of homotopy Gerstenhaber algebras and Tillmann \([Til99]\) teases out the operad algebra structure, particularly the BV-algebra, of TCFTs. Chas and Sullivan \([CS99]\) had a breakthrough development of an appropriate algebra of closed strings by inventing the algebra of cacti; see also Voronov \([Vor01]\). This algebraic structure has been developed further as homotopy theory by R. Cohen and J.D.S. Jones \([CJ01]\).

**Higher Category Theory**: The pentagon and hexagons of monoidal category coherence appeared also in the associahedra and the multiplihedra for \(A_\infty\)-spaces and, of course, also in \(A_\infty\)-categories. The higher dimensional cells appear in ‘higher category theory,’ manifestly, for example, in Gordon, Power and Street \([GPS95]\). Perhaps the most interesting cross-fertilization occurs in the work of Baez and Dolan on higher category theory \([BD98]\) using their consideration of operotopes; compare also the ‘globular’ approach of Batanin \([Bat98a, Bat98b]\).
Conversely, operads themselves are examined from a categorical point of view by Beke [Bek99].

**Polytopes:** Those early ‘hedra’ mentioned above continue to occur in a variety of contexts as do their progeny. Devadoss [Dev99] tessellates the moduli space and produces a *mosaic operad*. See also [LR01].

**Algebras and Groups:** Several generalizations of classical algebras, such as dialgebras of Loday and his ‘school’ [Lod95, Lod01] and various $k$-ary algebras [Gne95b, Gne95a] appeared recently; see also [GW00]. Formal groups and cogroups over operads are studied in [Fre98b, Fre98a]. Morita equivalence for modules over operads is studied by Kapranov and Manin in [KM01].

**Nonzero Characteristic:** In discussing dg operads, we have worked primarily in characteristic 0. For non-$\Sigma$ operads, this is unimportant but is crucial for many major operads, e.g the *Lie-Com* Koszul duality. Historically, in nonoperadic language, this is reflected in the difference between Harrison and André-Quillen cohomology for commutative algebras. As we saw in Section II.3.8, Harrison cohomology can be generalized straightforwardly for algebras over any Koszul quadratic operad. André and Quillen make essential use of simplicial techniques. For algebras over general operads, the simplicial point of view is utilized in [GH99, Fre97, Fre00].

Our selection of articles mentioned above necessarily depended on our personal inclinations and taste and certainly forms only a small portion of the rapidly growing literature on operads. We thus apologize to all whose work we omitted referencing here.

Bar Ilan, Chapel Hill and Praha, November 28, 2001
Bibliography


Mathematics and geometry, kategorii space of e-formulation of algebras, the, and manifolds, Firs for 0, R a Siberia, r, vertex homotopy geometry, convex I: curves in, d a curves, Trud cohesive for, algebras and values quantum "Div", operator 9 dasymp of, s and theorem theory coherence the Adv spaces, homotopy I, f the generalization, com n, or topology, Graduat classes, 4 t KZ Th rational, the, e Gruppe, in, n zweite n s, u The stable combinatorie in 2, operator, and Asterisque operads, stacks Mem, vertex and n­ in 9jn s, diagrams e of Lane's n cohomology, 3 y and, d, natural y uses, e, theorem (non)commutative 4, Lane's "det" operads, d koalgebr homology of und symplectic and f, s geometry, operations, d a n,­ Lie topology, J limit 3 product s, H des, t of, n permutoassociahedron, e associativities, t of their s Poisson - algebraic course

BIBLIOGRAPHY 333

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Highe Symposiu. 4. Jd s Acts, e r, solution. y f over stable d.

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D. Boardman, an, Illinoi, r, i associahedron, ii, f theory, y, algebras. homotopy and homology algebraic iterated, r, of complexes, a, t, r, Operads algebra, c, e y, and, p, r cyclohedron, y, loop categories, and, of construction f are Preprin, an, and, homological, s, n products, e, in quadratic. String algebraic J, s operadic point.

BIBLIOGRAPHY


Their open-closed homotopy algebras, operad, Batalin-Vilkovisky introduction and operad backgrounds in d.

Homotop complex operads, and printing, and Cambridge y operads - Ph.D algebras s Preprint l, secondar to Preprint geometry, on classical groups.

Action as string in work s, our Fifteenth algebra, manifolds, Eoo universal and student Gerstenhaber, minimal, and Mike Swiss, n Swiss-cheese the e to algebra, s Stasheff. operads, J, invariants t. 3


Glossary of notations

*a(v)* number of inputs (arity) of a vertex *v* of a rooted tree, 83, 217

*Aff* group of affine transformations of the complex plane $\mathbb{C}$, 213

*Aff*($V$) group of affine transformations of a vector space $V$, 220

*Ass* operad for associative algebras, 20, 44, 168

*Ass* non-$\Sigma$ operad for associative algebras, 14, 45

*Ass*$_\infty$ operad for $A_\infty$-algebras, 20

*Ass*$_\infty$ non-$\Sigma$ operad for $A_\infty$-algebras, 13, 26, 106

*B*($P$) bar construction on an operad $P$, 20

*Brace* operad for brace algebras, 31

*BV* operad for Batalin-Vilkovisky algebras, 206, 257

*BY* classifying space of an $A_\infty$-space $Y$, 11

*Bk* operad for $k$-braid algebras, 27

*B*$_\infty$ Baues operad, 30

*CH*$_{*}$($A; A$) Hochschild chains of $A$ with coefficients in itself, 259

*CH*$_{*}$($A; A$) Hochschild cochains of $A$ with coefficients in itself, 29, 104

*Cmon* operad for commutative associative monoids, 111

*CoEnd*$_K$ coendomorphism operad, 43

*Col* category of collections, 85

*Col*$_K$ category of $K$-collections, 85

*Com* operad for commutative associative algebras, 20, 44, 168

*Com*$_\infty$ operad for $C_\infty$-algebras, 20

*Con*($\mathbb{R}^k, n$) configuration space of $n$ distinct labeled points in $\mathbb{R}^k$, 26, 100

*Con*($M, n$) configuration space of $n$ distinct labeled points in a manifold $M$, 234

*Con*($M, n$) compactification of the configuration space $\text{Con}(M, n)$, 234, 239

*Con*($V, n$) configuration space of $n$ distinct labeled points in a vector space $V$, 220

*C*$_1$ little intervals operad, 95

*C*$_2$ little squares operad, 95

*C*$_k$ little $k$-cubes operad, 12, 94, 203

*C*(P) cobar complex of an operad $P$, 20, 121, 125

*$\mathbb{C}P^1$* complex projective line, 207

*C*$_P^\ast$($V$) $P$-algebra chain complex with trivial coefficients, 173

*C*$_P^\ast$($V; V$) $P$-algebra cochain complex with coefficients in itself, 177
<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\mathcal{D}}$</td>
<td>dual of a cocycle $\mathcal{D}$, 284</td>
</tr>
<tr>
<td>$\text{det}(S)$</td>
<td>determinant of a finite set $S$, 124</td>
</tr>
<tr>
<td>$\text{Det}(S)$</td>
<td>Determinant of a finite set $S$, 283</td>
</tr>
<tr>
<td>$\text{det}(T)$</td>
<td>determinant of a tree $T$, 124</td>
</tr>
<tr>
<td>$\text{Det}(T)$</td>
<td>Determinant of a tree $T$, 128</td>
</tr>
<tr>
<td>$\text{Det}(V)$</td>
<td>Determinant of a vector space $V$, 283</td>
</tr>
<tr>
<td>$\text{dgVec}$</td>
<td>category of differential graded vector spaces, 38, 121</td>
</tr>
<tr>
<td>$\text{dgOp}$</td>
<td>category of differential graded operads, 133</td>
</tr>
<tr>
<td>$\text{dg\PsiOp}$</td>
<td>category of differential graded pseudo-operads, 126</td>
</tr>
<tr>
<td>$D^k, B^k$</td>
<td>standard unit $k$-dimensional disk (ball), 204</td>
</tr>
<tr>
<td>$D_k$</td>
<td>little $k$-disks operad, 12, 203</td>
</tr>
<tr>
<td>$\mathcal{D}(\mathcal{P})$</td>
<td>dual dg operad of an operad $\mathcal{P}$, 20, 121, 127</td>
</tr>
<tr>
<td>$\mathcal{E}(\mathcal{P})$</td>
<td>$\Sigma$-module of decomposables of an operad $\mathcal{P}$, 187</td>
</tr>
<tr>
<td>$e_2$</td>
<td>operad for Gerstenhaber algebras, 30, 176</td>
</tr>
<tr>
<td>$\text{Edg}(T)$</td>
<td>set of all edges of a tree $T$ minus the root edge, 51, 128</td>
</tr>
<tr>
<td>$\text{Edge}(T)$</td>
<td>set of all edges of a tree $T$ including the root edge, 51</td>
</tr>
<tr>
<td>$\text{edge}(T)$</td>
<td>set of internal edges of a tree $T$, 51, 217, 123</td>
</tr>
<tr>
<td>$\text{edge}(\Gamma)$</td>
<td>set of edges of a graph $\Gamma$, 269</td>
</tr>
<tr>
<td>$\text{edge}(v)$</td>
<td>set of edges incident with a vertex $v$, 250</td>
</tr>
<tr>
<td>$\mathcal{E}(\text{nd})_X$</td>
<td>endomorphism operad, 6, 43, 254</td>
</tr>
<tr>
<td>$\mathcal{E}Z$</td>
<td>Eilenberg-Zilber operad, 105</td>
</tr>
<tr>
<td>$\mathcal{E}\text{tree}(n)$</td>
<td>set of of isomorphism classes of $n$-trees with a special vertex, 234</td>
</tr>
<tr>
<td>$E^#$</td>
<td>linear dual of a $\Sigma$-module, 141</td>
</tr>
<tr>
<td>$E^\vee$</td>
<td>Czech dual of a $\Sigma$-module $E$, 142</td>
</tr>
<tr>
<td>$\text{fCon}(M, n)$</td>
<td>framed configuration space of $n$ distinct labeled points in a manifold $M$, 210, 234</td>
</tr>
<tr>
<td>$\text{fCon}(M, n)$</td>
<td>compactification of the space $\text{fCon}(M, n)$, 234</td>
</tr>
<tr>
<td>$\text{fD}_k$</td>
<td>framed little $k$-disks operad, 13, 203</td>
</tr>
<tr>
<td>$\mathcal{F}_\mathcal{D}(\mathcal{A})$</td>
<td>Feynman transform of a $\mathcal{D}$-modular operad $\mathcal{A}$, 287</td>
</tr>
<tr>
<td>$\text{Flag}(\Gamma)$</td>
<td>set of flags (half-edges) of a graph $\Gamma$, 269</td>
</tr>
<tr>
<td>$\mathcal{F}(\mathcal{P})$</td>
<td>free $\mathcal{P}$-algebra functor, 47</td>
</tr>
<tr>
<td>$\mathcal{F}(\mathcal{P})$</td>
<td>cofree nilpotent $\mathcal{P}$-coalgebra functor, 166</td>
</tr>
<tr>
<td>$\tilde{\mathcal{F}}_V(n)$</td>
<td>moduli space $\tilde{\mathcal{F}}_V(n) := \text{Con}(V, n)/\text{Aff}(V)$, 220</td>
</tr>
<tr>
<td>$\mathcal{F}_V(n)$</td>
<td>compactification of the moduli space $\mathcal{F}_V(n)$, 225</td>
</tr>
<tr>
<td>$\mathcal{F}_V$</td>
<td>$\Sigma$-module ${\mathcal{F}<em>V(n)}</em>{n \geq 1}$, 220</td>
</tr>
<tr>
<td>$\mathcal{F}_V$</td>
<td>operad ${\mathcal{F}<em>V(n)}</em>{n \geq 1}$, 225</td>
</tr>
<tr>
<td>$\mathcal{F}_k(n)$</td>
<td>simplified notation for the space $\mathcal{F}_k(n)$, 220</td>
</tr>
<tr>
<td>$\mathcal{F}_k(n)$</td>
<td>simplified notation for the space $\mathcal{F}_k(n)$, 27, 225</td>
</tr>
<tr>
<td>$\mathcal{F}_k$</td>
<td>simplified notation for the space $\mathcal{F}_k$, 220</td>
</tr>
<tr>
<td>$\mathcal{F}_k$</td>
<td>simplified notation for the operad $\mathcal{F}_k$, 27, 226</td>
</tr>
<tr>
<td>$G_\infty$</td>
<td>minimal model of the operad $e_2$, 30</td>
</tr>
<tr>
<td>$G_{C^*_n}$</td>
<td>graph complex, 301</td>
</tr>
<tr>
<td>$\text{Gr}^{(n)}$</td>
<td>set of connected graphs with Euler characteristic $1 - n$, 301</td>
</tr>
</tbody>
</table>
GLOSSARY OF NOTATIONS

$Gr_{(n)}^{\text{net}}$ set of graphs $\Gamma \in Gr_{(n)}$ with a metric, 301
Grav gravity operad, 311
gVec category of graded vector spaces, 38
g($\Gamma$) genus of a labeled graph $\Gamma$, 270
$\mathcal{H} = (\mathcal{H}, Q)$ BRST complex, 24, 208
Harr$_*(A, M)$ Harrison homology of $A$ with coefficients in $M$, 266
$\mathcal{H}_G$ McClure-Smith operad, 30, 31
$HH^*(A; A)$ Hochschild cohomology of $A$ with coefficients in itself, 29
$HH_*(A; A)$ Hochschild homology of $A$ with coefficients in itself, 259
$\mathcal{H}_{\text{rel}} = (\mathcal{H}_{\text{rel}}, Q)$ relative BRST complex, 26
$hS(-| -)$ skew-symmetric shuffle product, 197
$H^*(A)$ cyclic homology of an associative algebra $A$, 259
$H^*_\text{DR}(A)$ de Rham cohomology of an algebra $A$, 260
$H^*_P(V; V)$ operadic cohomology of a $P$-algebra $V$ with coefficients in itself, 177
IFlag($\Gamma$) set of internal flags of a graph $\Gamma$, 285
In($v$) set of incoming edges of a vertex $v$ of a rooted tree, 51
Iso($D$) category of isomorphisms of a category $D$, 72, 271
JX James’ reduced product, 97
$K(\mathcal{P}^1)$ Koszul complex of an operad $\mathcal{P}$, 121, 145
$K(\mathcal{P}^1)^\#$ dual Koszul complex of an operad $\mathcal{P}$, 146
$K_n$ Stasheff’s associahedron, 9, 56
Leaf($T$) set of leaves of a tree $T$, 51
Leg($\Gamma$) set of legs (half-edges) of a graph $\Gamma$, 269
Leg($v$) set of legs (half-edges) adjacent to a vertex $v$, 269
Leib operad for Leibniz algebras, 257
Li linear isometries operad, 106, 107
LI category of real inner-product spaces, 107
Lie operad for Lie algebras, 20, 50
Lie$_\infty$ operad for $L_\infty$-algebras, 18
$L(T)$ number of leaves of a tree $T$, 128
$\mathcal{M}_{0,n}$ moduli space of $n$ distinct labeled points in $\mathbb{C}P^1$, 212
$\overline{\mathcal{M}}(n)$ compactification of the moduli space $\mathcal{M}_{0,n+1}$, 213, 305
$\mathcal{M}_{g,n}$ moduli space of genus $g$ Riemann surfaces with $n$ distinct labeled points, 292, 305
$\overline{\mathcal{M}}(g, n), \overline{\mathcal{M}}_{g,n+1}$ compactification of the moduli space $\mathcal{M}_{g,n+1}$, 304
$\overline{\mathcal{M}}$ configuration pseudo-operad $\{\overline{\mathcal{M}}(n)\}_{n \geq 2}$, 25, 216;
modular operad $\{\overline{\mathcal{M}}(g, n)\}_{(g, n) \in \mathcal{S}}$, 306
$\widetilde{\mathcal{M}}_0(n)$ moduli space of Riemann spheres with $n + 1$ labeled parametrized holes, 23, 207, 267
$\widetilde{\mathcal{M}}_0$ operad $\{\widetilde{\mathcal{M}}_0(n)\}_{n \geq 1}$, 23, 207, 247
$\widetilde{\mathcal{M}}(g, n)$ moduli space of genus $g$ Riemann surfaces with $n + 1$ labeled parametrized holes, 267, 304
$\widetilde{\mathcal{M}}$ unstable modular operad $\{\widetilde{\mathcal{M}}(g, n)\}_{g \geq 0, n \geq -1}$, 267
\( \mathcal{M}_{g,s} \) moduli space of genus \( g \) surfaces with \( s \) distinct unlabeled points, 292

\( \mathcal{M}^\text{dec}_{g,s} \) moduli space of genus \( g \) surfaces with \( s \) distinct unlabeled decorated points, 292

\( \mathcal{M}(-) \) triple for modular operads, 271

\( \mathcal{M}_{2n}(-) \) triple for \( \mathcal{O} \)-twisted modular operads, 281

\( MC_{g,s} \) mapping class group of a genus \( g \) surface with \( s \) distinct labeled points, 292

\( Met(T) \) space of metrics on a tree \( T \), 109, 221, 235

\( \mathcal{M}\text{Mod} \) category of stable modular \( \Sigma \)-modules, 268

\( \mathcal{M}\text{od}_k \) category of \( k \)-modules, 38

\( \mathcal{M}\text{od}(\mathcal{P}) \) modular completion of an operad \( \mathcal{P} \), 317

\( \mathcal{M}\text{od}_\mathcal{P} \) category of \( \mathcal{P} \)-modules, 139

\( \mathcal{M}\text{on} \) operad for associative topological monoids, 28, 99

\( \mathcal{M}\text{on}(C) \) non-\( \Sigma \) operad for associative topological monoids, 112

\( \mathcal{M}\text{on}(C) \) category of \( C \)-monoids, 40

\( MX \) rectification of a \( W/P \)-space \( X \), 116

\( \mathcal{N}(n) \) moduli space of Riemann spheres with \( n+1 \) marked decorated points, 25

\( \mathcal{N}(n) \) compactification of the space \( \mathcal{N}(n) \), 25

\( \mathcal{N} \) operad \( \{\mathcal{N}(n)\}_{n \geq 1} \), 23, 25

\( \text{Nerve}_*(C) \) nerve of a category \( C \), 272

\( \text{N}(\mathcal{P}) \) categorial cobar complex of an operad \( \mathcal{P} \), 121, 149

\( \text{Op} \) category of operads, 42

\( \text{Ord}(X) \) set of orderings of \( X \), 62

\( \mathcal{P} = \langle E; R \rangle \) presentation of an operad \( \mathcal{P} \), 139

\( \mathcal{P}^\dagger \) quadratic dual of a quadratic operad \( \mathcal{P} \), 21

\( \mathcal{P}^+ \) pseudo-operad associated to an operad \( \mathcal{P} \), 110;

augmentation ideal of an augmented operad \( \mathcal{P} \), 187

\( \mathcal{P}^\perp \) operad \( (s\mathcal{P}^\dagger)^\# \), 262

\( \mathcal{P} \times G \) semidirect product of an operad \( \mathcal{P} \) with a group \( G \), 204

\( \mathcal{P}(-) \) free \( \mathcal{P} \)-module functor, 139

\( \mathcal{P}_{B \to W} \) colored operad for homomorphisms of \( \mathcal{P} \)-spaces, 115

\( P_n \) permutahedron, 97

\( \text{Poiss} \) operad for Poisson algebras, 176, 256, 261

\( QP \) \( \Sigma \)-module of indecomposables of an operad \( \mathcal{P} \), 188

\( R \) Smith operad, 105

\( RGr_{g,s} \) set of ribbon graphs \( \Gamma \) such that \( \text{Surf}(\Gamma) \) has \( s \) holes and genus \( g \), 292

\( RGr^\text{met}_{g,s} \) set of ribbon graphs \( \Gamma \in RGr_{g,s} \) with a metric, 292

\( RGC_{g,s}^{\ast} \) ribbon graph complex, 295

\( R\text{mtree}(n) \) space of isomorphism classes of reduced rooted metric \( n \)-trees, 109, 221

\( R\text{mtree} \) pseudo-operad of reduced rooted metric trees, 110
\( \mathcal{R}_{\text{mtree}}(n) \)  
space of isomorphism classes of reduced rooted metric planar \( n \)-trees, 112

\( \mathcal{R}_{\text{mtree}} \)  
non-\( \Sigma \) pseudo-operad of reduced rooted planar metric trees, 112

\( \mathcal{R}_{\text{tree}}(n) \)  
set of isomorphism classes of reduced rooted \( n \)-trees, 55

\( \mathcal{R}_{\text{tree}} \)  
pseudo-operad of reduced rooted trees, 55

\( \mathcal{R}_{\text{tree}}(n) \)  
set of isomorphism classes of reduced planar rooted \( n \)-trees, 55

\( \mathcal{R}_{\text{tree}} \)  
non-\( \Sigma \) operad of reduced planar rooted trees, 55, 109

\( \mathcal{R}_{\text{tree}} \)  
category of reduced rooted \( n \)-trees, 87, 124

\( \mathcal{S}_A \)  
operadic suspension of a \( \Sigma \)-module \( A \), 127, 258

\( s^{-1}A \)  
operadic desuspension of a \( \Sigma \)-module \( A \), 127

\( \mathcal{S}_E \)  
modular suspension of a modular \( \Sigma \)-module \( E \), 283

\( \text{Set} \)  
category of sets, 38

\( \text{Set}_f \)  
category of finite sets, 40

\( \text{Set}_f\mathcal{-Mod} \)  
category of \( \text{Set}_f \)-modules, 40

\( \text{sgn}_n \)  
signum representation of the symmetric group \( \Sigma_n \), 264

\( \text{Sh}(-|-) \)  
shuffle product, 196

\( \text{Surj}[j,n] \)  
set of surjection sequences, 152

\( \text{Surj}_f[j,n] \)  
set of surjections \( f : [n] \to [j] \), 150

\( \text{Surj}_f \)  
category of surjections of finite sets, 60

\( \text{Surf}(\Gamma) \)  
surface associated to a ribbon graph \( \Gamma \), 292

\( T_{g,s} \)  
Teichmüller space of genus \( g \) surfaces with \( s \) distinct labeled points, 291

\( T_{g,s}^{\text{dec}} \)  
decorated Teichmüller space of genus \( g \) surfaces with \( s \) distinct labeled points, 291

\( T_{[f]} \)  
tree corresponding to a surjection \( f \), 151

\( \text{Tree}(n) \)  
set of isomorphism classes of rooted \( n \)-trees, 8

\( \text{Tree} \)  
operad of rooted trees, 8, 54

\( \text{Tree}(n) \)  
set of isomorphism classes of planar rooted \( n \)-trees, 8

\( \text{Tree} \)  
non-\( \Sigma \) operad of planar rooted trees, 8, 54

\( \text{Tree}(X) \)  
set of isomorphism classes of \( X \)-labeled trees, 53

\( \text{Tree} \)  
category of labeled rooted trees, 52

\( \text{Tree}_n \)  
category of labeled rooted \( n \)-trees, 52

\( \text{Tree}_X \)  
category of \( X \)-labeled rooted trees, 52

\( \text{Tree}_n^+ \)  
category of unrooted trees with legs labeled by \( \{0,\ldots,n\} \), 250

\( \text{UAss} \)  
operad for associative algebras with unit, 267

\( \text{UCom} \)  
operad for commutative associative algebras with unit, 267, 316

\( \text{unsh}(-,\ldots,-) \)  
set of unshuffles, 99

\( \text{UPOiss} \)  
operad for Poisson algebras with unit, 267

\( \text{Vec} \)  
category of vector spaces, 38

\( \text{Vert}(T) \)  
set of (internal) vertices of a tree \( T \), 51, 250

\( \overline{\text{Vert}}(T) \)  
set of all (including external) vertices of a tree \( T \), 51
GLOSSARY OF NOTATIONS

$\text{Vert}(\Gamma)$ set of vertices of a graph $\Gamma$, 269
$V^\# = (V^\#, d^\#)$ linear dual of a dg complex, 121
$\mathcal{W}(\mathcal{P})$ $W$-construction on an operad $\mathcal{P}$, 28, 111
$\mathcal{W}(\mathcal{\Sigma})$ non-$\Sigma$ $W$-construction on a non-$\Sigma$ operad $\mathcal{\Sigma}$, 111,
$\mathcal{W}_n$ cyclohedron, 241
$\mathbb{Z}(h)$ zero set of a metric $h \in \text{Met}(T)$, 221
$\mathbb{Z}_n$ cyclic group $\mathbb{Z}/n\mathbb{Z}$

$n$-star, 250
modular $n$-corolla of genus $g$, 271.
$n$-corolla, 251
trivial operad, 42, 187
set $\{1, \ldots, n\}$, 40
groupoid of a graph $\Gamma$, 269
number of internal edges of a tree $T$, 123
determinant operad, 128
box product on $\Sigma$-Mod, 68
box product on $\text{Coalg}_K$, 86
composition (Gerstenhaber) product, 6
dual graph of a stable curve $C$, 304
category of stable $S$-labeled graphs of genus $g$, 271
free cyclic operad functor, 251
free operad functor, 82
free $K$-operad functor, 88
free pseudo-operad functor, 81
free cyclic pseudo-operad functor, 251
universal bilinear form, 261
module of differentials of an algebra $A$, 260
$k$-fold loop space of a topological space $X$, 93
dg algebra of de Rham forms, 29
category of pseudo-operads, 67
category of $\Sigma$-modules, 40
category of cyclic $\Sigma$-modules, 249
permutation group of $\{1, \ldots, n\}$, 40
permutation group of $\{0, \ldots, n\}$, 264
symmetric groupoid, 40
suspension of a graded vector space $U$, 314
desuspension of a graded vector space $U$, 314
reflection of a graded vector space $U$, 314
cycle $(0, \ldots, n) \in \Sigma^+_n$, 247
Koszul sign, 195
skew-symmetric Koszul sign, 196
Index

(\(n+1\))-star, 250
\(1\)-brace, 183
\(2\)-brace, 183
\(A(\infty)\)-algebra, 195
\(A_\infty\)-category, 14
\(A_\infty\)-algebra, 13, 172, 195
\(A_\infty\)-map, 113
\(A_\infty\)-space, 10
\(B_\infty\)-algebra, 30
\(C_\infty\)-algebra, 19, 172, 196
\(E_\infty\)-operad, 12
\(G\)-operad, 204
\(H_n\)-space, 104
\(K\)-collection, 85
\(K\)-operad, 86
\(L_\infty\)-algebra, 17, 195
\(P\)-functor, 97
\(T\)-algebra, 88
\(V\)-manifold, 293
\(W\)-construction, 111
\(W/P\)-homomorphism, 116
\(X\)-labeled tree, 52
\(\square\)-degree, 149
\(\square\)-differential, 149
\(\square\)-product, 68
\(\square_K\)-product, 86
\(\Gamma\)-space, 101
\(\Sigma\)-module, 40
\(\Sigma^+\)-module, 249
\(\Theta\)-algebra, 7
\(\Theta\)-space, 7
\(\mathcal{P}\)-algebra, 46
\(\mathcal{P}\)-algebra chain complex, 173
\(\mathcal{P}\)-algebra cochain complex, 177
\(\mathcal{P}\)-algebra cohomology, 177
\(\mathcal{P}\)-algebra homology, 174
\(\mathcal{P}\)-coalgebra, 165
\(\mathcal{P}\)-degree, 177
\(\mathcal{P}\)-differential, 173
\(\mathcal{P}\)-homomorphism, 112
\(\delta\)-complex, 125
\(\mathcal{T}\)-twisted modular operad, 286
\(L(\infty)\)-algebra, 172, 195
k-braid algebra, 27
k-fold loop space, 93
n-Lie algebra, 19
n-labeled tree, 52
n-tree, 52
f-order, 156
\(\otimes\)-product of operads, 118
\(\mathcal{PACT}\), 5
\(\mathcal{PROP}\), 5
action of \(\mathcal{PROP}\), 5
additive algebraic \(K\)-theory, 260
algebra over a triple, 88
algebra over a \(\mathcal{PROP}\), 5
algebra over operad, 7
algebraic stack, 311
algebraic theory, 7
analyseur, 4
analyseur incomplet, 4
antibracket, 25
anticyclic cooperad, 262
anticyclic operad, 258
antifield, 25
antighost, 25
antighost operator, 24
approximation, 94
arity, 50, 268
associahedron, 9
associativity constraint, 37
associator, 37
atlas, 311
augmented operad, 187
balanced $A_{\infty}$-algebra, 19, 172, 196
bar construction on a $\mathcal{P}$-algebra, 173
bar construction on an operad, 20
Batalin-Vilkovisky algebra, 206
bicolored operad, 85, 115
bilinear space, 254
block permutation, 40
blowing down map, 240
brace, 183
braided monoidal category, 38
Browder operation, 103
Brownian bridge, 18
BRST cohomology, 24
BRST complex, 23, 24
BV-algebra, 206

canonical element, 319
canonical resolution, 179
categorical cobar complex, 150
category with multiplication, 37
center of a little disk, 205
CFT, 23
Chevalley-Eilenberg homology, 260
closed string field theory, 25
coarse moduli space, 311
cobar bicomplex of an operad, 124
cobar complex of an operad, 20, 125
coboundary, 283
cocycle, 282
coderivation of a $\mathcal{P}$-coalgebra, 169
coefficient system, 281
coenomorphism operad, 43
cofibrant model, 28
cofree nilpotent $\mathcal{P}$-coalgebra, 166
cohomology, 37
cohomological field theory, 306
collection, 85
color of a vertex, 72
colored operad, 85
commutative $A_{\infty}$-algebra, 172, 196
comp algebra, 6
complete algebraic system, 3
composition along a graph, 274
composition product, 105
configuration pseudo-operad, 216
conformal field theory, 22, 24
Connes-Tsygan exact sequence, 266
corolla, 10, 50
CSFT, 23
cup product, 102, 105
cyclic $\mathcal{P}$-algebra, 255
cyclic $\Sigma$-module, 249
cyclic $A_{\infty}$-algebra, 22
cyclic endomorphism operad, 254
cyclic homology of a $\mathcal{P}$-algebra, 261
cyclic homology of an assoc. algebra, 259
cyclic operad, 247
cyclic pseudo-operad, 253
cyclic quadratic operad, 255
cyclohedron, 241
Czech dual of a $\Sigma$-module, 142
decomposables, 187
decomposition into pairs, 298
decorated Teichmüller space, 291
definitions, 291
defining relations of an operad, 140
deligne's conjecture, 30
desuspension, 314
determinant operad, 128
de Rham cohomology, 260
diagonal, 197
differential graded $\Sigma$-module, 121
differential graded operad, 122
discriminant, 293
dual $\Sigma$-module, 141
dual complex, 121
dual dg operad, 20, 127
dual graph of a curve, 304
dual Koszul complex, 146
dual numbers, 169
dual of a cocycle, 284
dualizing cocycle, 284
Dyer-Lashof operations, 103
dromic transverse, 30
edge, 51, 269
Eilenberg-Zilber operad, 105
elementary contraction, 51
elementary equivalence, 187
elementary homotopy, 190
elementary morphism, 51
endomorphism operad, 6, 43
endomorphism $PROP$, 5
equivariant section, 199
extended cyclic operad, 258
extension of scalars, 190
external edge, 50
external vertex, 50
family of functions, 4
fat graph, 292
Feynman transform, 289
field, 18, 23
finitary algebraic theory, 101
Floer homotopy, 15
formal algebra, 29, 106
formal manifold, 29
formal operad, 30
frame, 203, 210
framed configuration space, 210
framed little $k$-disks operad, 13, 203
free $\mathcal{P}$-algebra, 47
free $\mathcal{P}$-module, 139
free $K$-operad, 88
free cyclic operad, 251
free operad, 82
free product, 200
free pseudo-operad, 81
fully-labeled tree, 52
fundamental identity, 20

G-algebra, 105
generating function of an operad, 176
genus, 268
genus of a labeled graph, 270
geometric realization, 269
Gerstenhaber algebra, 25, 27, 105, 206
Gerstenhaber bracket, 25
ghost, 25
ghost number, 24
graded Jacobi identity, 104
grafting of trees, 54
graph, 269
gravity operad, 311
group law, 4

H-map, 112
H-space, 112
Harrison homology, 266
height of a vertex, 51
Hochschild homology, 176, 259
homogeneity filtration, 158
homogeneity of a surface, 158
homotopy, 190
homotopy invariant algebraic structure, 115
Hopf operad, 197
horocycle, 291
hyperoperad, 281

ideal in an operad, 139
indecomposables, 188
infinite loop space, 12, 93
infinitesimal deformation, 182
injectivity radius, 210
internal degree, 121, 124, 128, 149, 173, 177
internal differential, 125, 150
internal edge, 50
internal vertex, 50
invariant bilinear form, 255
invariant equations of condition, 3
invertible graded vector space, 282
iterated loop space, 93

Jacobian, 20
James' reduced product, 97

Koszul complex, 145
Koszul operad, 21, 145
Koszul sign, 195
Kudo-Araki operations, 103

labeled graph, 270
leaf, 50
leaf relabeling action, 52
leaf vertex, 50
leaf-labeled tree, 52
left module over an operad, 138
leg, 50, 269
length function, 109
level function, 154
Lie-hedron, 17, 196
linear isometries operad, 106, 107
little $k$-cube, 94
little $k$-cubes operad, 12, 94
little $k$-disks operad, 12, 203
little disk, 209
little intervals operad, 95
little squares operad, 95
loop homotopy Lie algebra, 312, 314

M-construction, 116
macroscopic configuration, 233
main identity, 313, 315
marking, 291, 304
Massey product, 106
May model, 101
metric, 109, 292, 301
metric tree, 109, 220
microscopic configuration, 233
Milgram model, 101
minimal model, 28, 191
minimal operad, 28, 188
modular $\mathcal{D}$-operad, 286
modular $\Sigma$-module, 267
modular corolla, 271
modular operad, 22, 268
modular operadic completion, 318
modular suspension, 283
module of differentials, 260
module over an operad, 139
monad, 88
monoid, 39
monoidal category, 37
morphism of $\mathcal{P}$-algebras, 47
morphism of graphs, 270
Morse homology, 15
Morse theory, 15
multilineal string functions, 314
multiplihedra, 113

nerve of a category, 272
nilpotent coalgebra, 166
non-$\Sigma$ operad, 6, 45
non-$\Sigma$ pseudo-operad, 46
non-$\Sigma$ cyclic operad, 257
normal disk, 211
normal representative, 220
normal sequence, 220
normalization of a configuration, 220
normalization of a disk, 211

operad, 6, 41, 64, 69
operad of metric trees, 109
operad of type \((k, 2k - 1)\), 140
operad of virtual configurations, 220
operadic cohomology, 177
operadic desuspension, 127
operadic homology, 174
operadic suspension, 127
operators in standard form, 3
orbifold, 293
orientation cocycle, 284
orientation of a graph, 295
parallel transport, 239
parametrized hole, 207
partial algebra, 14
partial operad, 14, 219
permutahedron, 97
phase of a little disk, 205
phase parameter, 25
planar imbedding, 52
planar order, 83
planar tree, 52
pointed operad, 100
Poisson algebra, 27
pre-Lie algebra, 183
pre-Lie identity, 183
preoperad, 100
primitive element, 224
principal extension, 200
principal value, 307
pseudo-cooperad, 122
pseudo-operad, 45
puncture, 207
quadratic \(\Sigma\)-module, 140
quadratic differential, 291
quadratic dual, 21
quadratic dual of an operad, 142
quadratic operad, 140
quasi-isomorphism, 187
quism, 187
rational cell, 293
recognition principle, 9, 93
reduced suspension, 125
reduced tree, 50
reflection of a graded vector space, 314
relative \(\square\)-product, 138
resolution degree, 178
ribbon graph complex, 295
ribbon graph, 292
right module over an operad, 138
root, 50
root vertex, 50
rooted tree, 50
Schur functor, 47
semidirect product of an operad and group, 204
sewing, 24
sh \(\mathcal{S}\)-algebra, 194
shuffle product, 196, 320
skew-symmetric Koszul sign, 196
space of reduced metric trees, 109
special vertex, 234
species, 67
spine, 8
stability condition, 22
stable curve of arbitrary genus, 304
stable curve of genus 0, 213
stable graph, 271
stable labeled graph, 22
stable modular \(\Sigma\)-module, 267
Steenrod squares, 102
Strebel differential, 291
strict monoidal category, 37
strict surjection, 151
string background, 23
string field theory, 23
string product, 312
string theory, 23
string vertices, 26
strongly homotopy \(\mathcal{P}\)-map, 116
strongly homotopy \(\mathcal{P}\)-space, 115
strongly homotopy \(\mathcal{S}\)-algebra, 194
strongly homotopy associative algebra, 172, 195
strongly homotopy Lie algebra, 172, 195
strongly homotopy multiplicative map, 113
strongly homotopy \(\mathcal{P}\)-algebra, 28
subcategory of isomorphisms, 72
surjection algebra, 160
surjection sequence (surse), 152
suspension, 314
symmetric groupoid, 40
symmetric monoidal category, 37
symmetrization of a non-\(\Sigma\) operad, 45
T-ordering, 156
TCFT, 23
Teichmüller space, 291
topological \(A_\infty\)-category, 15
topological conformal field theory, 304
topological Feynman transform, 312
topological field theory, 22
trace, 257
tree corresponding to a surjection, 151
tree degree, 124, 128
tree differential, 124
tree level, 23
tree operad, 8
triple, 88
trivial tree, 50
twisting cochain, 262
two-sided bar construction, 96
unit object, 37
universal bilinear form, 261
universal differential, 260
unordered $\odot$-product, 64
unrooted tree, 250
unshuffle, 99

valence, 268
vertex, 51, 269
Virasoro algebra, 24
virtual configuration, 225

weak equivalence, 187
weak homotopy type, 187
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(Continued from the front of this publication)

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