Mathematical Surveys and Monographs
Volume 163

The Ricci Flow: Techniques and Applications

Part III: Geometric-Analytic Aspects

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http://dx.doi.org/10.1090/surv/163

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2010 *Mathematics Subject Classification.* Primary 53C44, 53C25, 58J35, 35K55, 35K05, 35K08, 35K10, 53C21.

For additional information and updates on this book, visit
www.ams.org/bookpages/surv-163

Library of Congress Cataloging-in-Publication Data

Chow, Bennett.
The Ricci flow : techniques and applications / Bennett Chow ... [et al.].
p. cm. — (Mathematical surveys and monographs, ISSN 0076-5376 ; v. 135)
Includes bibliographical references and indexes.
ISBN-13: 978-0-8218-3946-1 (pt. 1)
ISBN-10: 0-8218-3946-2 (pt. 1)
1. Global differential geometry. 2. Ricci flow. 3. Riemannian manifolds. I. Title.

QA670.R53 2007
516.3′62—dc22 2007275659

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10 9 8 7 6 5 4 3 2 1 15 14 13 12 11 10

Contents

Preface

I didn't have time to write you a short letter, so I wrote you a long one instead.
– Samuel Clemens

What Part III is about

I'm taking the time for a number of things
That weren't important yesterday.
– From "Fixing a Hole" by The Beatles

This is Part III (a.k.a. $\Delta R_{ijk\ell}$), a sequel to Part I ([**40**]; a.k.a. $R_{ijk\ell}$) and Part II ([**41**]; a.k.a. $\frac{\partial}{\partial t} R_{ijk\ell}$) of this volume (Volume Two) on techniques and applications of the Ricci flow (we shall refer to Volume One ([**42**]; a.k.a. g_{ij}) as Volume One).

In Part I we discussed various *geometric* topics in Ricci flow such as Ricci solitons, an introduction to the Kähler–Ricci flow, Hamilton's Cheeger–Gromov-type compactness theorem, Perelman's energy and entropy monotonicity, the foundations of Perelman's reduced distance function, the reduced volume, applications to the analysis of ancient solutions, and a primer on 3-manifold topology.

In Part II we discussed mostly *analytic* topics in Ricci flow including weak and strong maximum principles for scalar heat-type equations and systems on compact and noncompact manifolds, Böhm and Wilking's classification of closed manifolds with 2-positive curvature operator, Shi's local derivative estimates, Hamilton's matrix estimate, and Perelman's estimate for fundamental solutions of the adjoint heat equation.

Here, in Part III, we discuss mostly *geometric-analytic* topics in Ricci flow. In particular, we discuss properties of Perelman's entropy functional, point picking methods, aspects of Perelman's theory of κ-solutions including the κ-gap theorem, compactness theorem, and derivative estimates, Perelman's pseudolocality theorem, and aspects of the heat equation with respect to static and evolving metrics related to Ricci flow. In the appendices we review metric and Riemannian geometry including the space of points at infinity and Sharafutdinov retraction for complete noncompact manifolds with nonnegative sectional curvature. As in previous volumes, we have endeavored, as much as possible, to make the chapters independent of each other.

In Part IV we shall discuss some topics originally slated for Part III such as Hamilton's classification of nonsingular solutions, the linearized Ricci flow, stability of the Ricci flow, the space-time formulation of the Ricci flow, and Type II singularities from the numerical perspective.

Caveat: Many of the chapter numbers of references in Part II to Part III have changed and some of the referred chapters are in Part IV.

Acknowledgments

Now that your rose is in bloom, a light hits the gloom on the grey.

– From "Kiss from a Rose" by Seal

We would like to thank our colleagues, some of whom have been named in previous volumes, for their help, support, and encouragement. In addition, we would like to thank the following mathematicians for helpful discussions: Scot Adams, Jianguo Cao, Yu Ding, Patrick Eberlein, Joel Haas, Richard Hamilton, Emmanuel Hebey, Shengli Kong, John Lott, Kate Okikiolu, Anton Petrunin, Justin Roberts, Xiaochun Rong, Peter Scott, Peter Topping, Bing Wang, and Jiaping Wang. We are especially grateful to John Lott for a number of corrections and suggestions and to Jiaping Wang for help on technical issues.

We would like to especially thank Ed Dunne for his tireless efforts and patience in making the publication of our expository works on Ricci flow possible through the American Mathematical Society. We would like to thank the editors of the Mathematical Surveys and Monographs series. We would like to thank Cristin Zanella for her assistance. Special thanks to Arlene O'Sean for her expert copy editing.

We would like to thank Bo Yang and Shijin Zhang for proofreading parts of the manuscript.

During the preparation of this volume, Bennett Chow was partially supported by NSF grants DMS-9971891, DMS-020392, DMS-0354540, and DMS-0505507. David Glickenstein was partially supported by NSF grant DMS 0748283. Christine Guenther was partially supported by the Thomas J. and Joyce Holce Professorship in Science. Jim Isenberg was partially supported by NSF grants PHY-0354659 and PHY-0652903. Dan Knopf was partially supported by NSF grants DMS-0511184, DMS-0505920, and DMS-0545984. Peng Lu was partially supported by NSF grant DMS-0405255. Peng Lu was also partially supported by NSF funds for his visits to the UC San Diego mathematics department. Feng Luo was partially supported by NSF grant DMS-0103843. Lei Ni was partially supported by NSF grants DMS-0354540 and DMS-0504792. Bennett Chow and Lei Ni were partially supported by NSF FRG grant DMS-0354540 (joint with Gang Tian). We would like to thank the National Science Foundation, especially the Division of Mathematical Sciences and the Geometric Analysis subdivision. In particular, we would like to thank Christopher Stark, Helena Noronha, Alex

Freire, and Zhongmin Shen. We are grateful for Drs. Noronha's and Stark's encouragement in the early stages of this book project.

Bennett Chow would like to thank East China Normal University, the Mathematical Sciences Research Institute in Berkeley, Université de Cergy-Pontoise, and his home institution, University of California at San Diego, for providing a wonderful environment for writing this book. Ben would like to thank his parents, daughters, and friends for their help and encouragement. Ben is indebted to his parents, Yutze and Wanlin, for all of the nurturing, support, and encouragement they have given. Ben is especially grateful to Peng Lu for his persevering collaboration on this project.

Ben expresses extra special thanks to Classic Dimension for continued encouragement, support, guidance, understanding, patience, faith, forgiveness, and inspiration. Ben dedicates all of his expository works on Ricci flow and in particular this book to Classic Dimension.

Sun-Chin Chu would like to thank Nai-Chung Leung and Wei-Ming Ni for their encouragement and help over the years. Sun-Chin would like to thank his parents for their love and support throughout his life and dedicates this book to his family.

David Glickenstein would like to thank his wife, Tricia, and his parents, Helen and Harvey, for their love and support. Dave dedicates this book to his family.

Christine Guenther would like to thank Jim Isenberg as a friend and colleague for his guidance and encouragement. She thanks her family, in particular Manuel, for their constant support and dedicates this book to them.

Jim Isenberg would like to thank Mauro Carfora for introducing him to Ricci flow. He thanks Richard Hamilton for showing him how much fun it can be. He dedicates this book to Paul and Ruth Isenberg.

Tom Ivey would like to thank Robert Bryant and Andre Neves for helpful comments and suggestions.

Dan Knopf thanks his colleagues and friends in mathematics, with whom he is privileged to work and study. He is especially grateful to Kevin McLeod, whose mentorship and guidance have been invaluable. On a personal level, he thanks his family and friends for their love, especially Dan and Penny, as well as his parents, Frank and Mary Ann, and his wife, Stephanie.

Peng Lu would like to take this opportunity to thank all the people who helped him over the years.

Feng Luo would like to thank the NSF for partial support.

Lei Ni would like to thank Jiaxing Hong and Yuanlong Xin for initiating his interests in geometry and PDE. He also thanks Peter Li and Luen-Fai Tam for their teaching over the years and for collaborations. In particular, he would like to thank Richard Hamilton and Grisha Perelman, from whose papers he learned much of what he knows about Ricci flow.

Bennett Chow, UC San Diego and East China Normal University
Sun-Chin Chu, National Chung Cheng University
David Glickenstein, University of Arizona
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December 11, 2009

Contents of Part III of Volume Two

Well, you know, we're doing what we can.

– From "Revolution" by The Beatles

Chapter 17. Perelman's entropy $\mathcal{W}$ leads to the μ-invariant. We discuss qualitative properties of the μ-invariant such as lower and upper bounds and we give a proof of the fact that $\lim_{\tau\to 0^+} \mu(g,\tau) = 0$. We also discuss applications of the μ-invariant monotonicity formula. This includes the recent classification by Z.-L. Zhang of compact finite time singularity models as shrinking gradient Ricci solitons. We revisit the proof of the existence of a smooth minimizer for $\mathcal{W}$, providing more details than in Part I, and we also show that when the isometry group acts transitively, the minimizer is not unique for sufficiently small τ. Related to renormalization group considerations, some low-loop calculations are presented.

Chapter 18. We discuss some tools used in the study of the Ricci flow including the changing distances estimate for solutions of Ricci flow, point picking methods, rough monotonicity of the size of necks in complete noncompact manifolds with positive sectional curvature, and a local form of the weakened no local collapsing theorem.

Chapter 19. With the goal of understanding compactness in higher dimensions, we introduce the notion of 'κ-solution with Harnack', which is a variant of Perelman's notion of κ-solution. In dimensions 2 and 3 we show that κ-solutions with Harnack must have bounded curvature. We also discuss the construction of Perelman's rotationally symmetric ancient solution on $\mathcal{S}^n$, the result that κ-solutions with Harnack must have bounded curvature, the existence of an asymptotic shrinker in a κ-solution (correcting a gap (no pun intended) in Part I), and the κ-gap theorem.

Chapter 20. We show that noncompact κ-solutions have asymptotic scalar curvature ratio ASCR $= \infty$ and asymptotic volume ratio AVR $= 0$; the latter result does not require the κ-noncollapsed at all scales assumption. We show that solutions which are almost ancient and have bounded nonnegative curvature operator are collapsed at large scales and we obtain a curvature estimate in noncollapsed balls. We prove that the collection of κ-solutions with Harnack is compact modulo scaling. In dimension 3 this is equivalent to Perelman's compactness theorem and implies scaled derivative of curvature estimates.

Chapter 21. We discuss Perelman's pseudolocality theorem. Assuming an initial ball with scalar curvature bounded from below and which is almost Euclidean isoperimetrically, we obtain a curvature estimate in a smaller ball; this estimate gets worse as time approaches the initial time. One may consider this as sort of a pseudolocalization of the curvature doubling time estimate. One of the ideas in the proof is that one can localize the entropy monotonicity formula by multiplying the integrand by a suitable time-dependent cutoff function. In the setting of a proof by contradiction, a main idea is to use point picking methods to locate an infinite sequence of 'good' high curvature points and to study a local entropy in their neighborhoods via Perelman's Harnack-type estimate for fundamental solutions of the adjoint heat equation coupled to the Ricci flow.

Chapter 22. We discuss tools used in the proof of the pseudolocality theorem such as the point picking 'Claims 1 and 2', convergence of heat kernels under Cheeger–Gromov convergence, a uniform negative upper bound for the local entropies centered at the well-chosen bad points at time zero, and a sharp form of the logarithmic Sobolev inequality related to the isoperimetric inequality.

Chapter 23. We discuss existence and asymptotics for heat kernels with respect to *static* metrics. We follow the parametrix method of Levi and its Riemannian adaptation by Minakshisundaram and Pleijel. Starting with a good approximation to the heat kernel, we prove the existence of the heat kernel by establishing the convergence of the 'convolution series'. With this construction we compute some low-order asymptotics for the heat kernel.

Chapter 24. We adapt the methods of the previous chapter to study the existence and asymptotics for heat kernels with respect to *evolving* metrics. We consider aspects of the adjoint heat kernel for evolving metrics related to §9.6 of Perelman's paper [**152**]. We also discuss the existence of Dirichlet heat kernels on compact manifolds with boundary and heat kernels on noncompact manifolds with respect to evolving metrics.

Chapter 25. We discuss estimates for solutions to the heat equation with respect to evolving metrics including the parabolic mean value property for solutions to heat equations and the Li–Yau differential Harnack estimate for positive solutions to heat equations.

Chapter 26. Applying the estimates of the previous chapter, we discuss estimates for heat kernels with respect to evolving metrics including upper and lower bounds and the space-time mean value property. We also discuss the existence of distance-type functions on complete noncompact Riemannian manifolds with bounded gradient and Laplacian.

Appendix G. With Perelman's work, the space-time of a solution of the Ricci flow is given a quasi-length space structure. This geometric structure is foundational in the understanding of singularity formation under the

Ricci flow. We discuss notions of (quasi-)metric and (quasi-)length spaces, Gromov–Hausdorff convergence, and Aleksandrov spaces.

Appendix H. We discuss convex analysis on Euclidean spaces and on locally convex subsets in Riemannian manifolds.

Appendix I. We discuss the points at infinity for nonnegatively curved manifolds, the Sharafutdinov retraction theorem, and some consequences.

Appendix J. We provide solutions to some of the exercises in the book.

Notation and Symbols

Confusion never stops, closing walls and ticking clocks.
– From "Clocks" by Coldplay

The following is a list of some of the notation and symbols which we use in this book.

$\times$	multiplication, when a formula does not fit on one line
∇	covariant derivative
$\nabla_{\bigstar} r$	'set gradient' of the distance function to a point
$\square$	heat operator
$\square^*$	adjoint heat operator
$\doteqdot$	defined to be equal to
$\cdot$	dot product or multiplication
$\nabla\nabla f$	Hessian of f
$\alpha^\natural$	dual vector field to the 1-form α
ALE	asymptotically locally Euclidean
Area	area of a surface or volume of a hypersurface
ASCR	asymptotic scalar curvature ratio
AVR	asymptotic volume ratio
$B(p, r)$	ball of radius r centered at p
bounded curvature	bounded *sectional* curvature (for time-dependent metrics, the bound may depend on time)
$C_V \mathcal{J}$	tangent cone at V of a convex set $\mathcal{J} \subset \mathbb{R}^k$
const	constant
$\frac{d^+}{dt}$, $\frac{d^-}{dt}$, $\frac{d_+}{dt}$, $\frac{d_-}{dt}$	various Dini time derivatives
d	distance
d_{GH}	Gromov–Hausdorff distance
$d\mu$	volume form
$d\mu_{\mathbb{E}}$	Euclidean volume form
$d\sigma$ or dA	volume form on boundary or hypersurface
Δ, Δ_L, Δ_d	Laplacian, Lichnerowicz Laplacian, Hodge Laplacian

diam	diameter
div	divergence
$\mathbb{E}^n$	$\mathbb{R}^n$ with the flat Euclidean metric
$E_r(x,t)$	heat ball of radius r based at (x,t)
exp	exponential map
$\mathcal{F}$	Perelman's energy functional
Γ_{ij}^k	Christoffel symbols
$g(X,Y) = \langle X,Y \rangle$	metric or inner product
$g(t)$	time-dependent metric, e.g., solution of the Ricci flow
g_∞ or $g_\infty(t)$	limit Riemannian metric or solution of Ricci flow
h or II	second fundamental form
H	mean curvature
$H_V\mathcal{J}$ for $V \in \partial\mathcal{J}$	set of closed half-spaces H containing $\mathcal{J} \subset \mathbb{R}^k$ with $V \in \partial H$
$\operatorname{Hess} f$	Hessian of f (same as $\nabla\nabla f$)
$\mathcal{I}$	a time interval for the Ricci flow
id	identity
int	interior
inj	injectivity radius
Isom	group of isometries of a Riemannian manifold
IVP	initial-value problem
J	Jacobian of the exponential map
λ	λ-invariant
L	length
LHS	left-hand side
log	natural logarithm
L	L-distance
ℓ	reduced distance or ℓ-function
$\mathcal{L}$	Lie derivative or $\mathcal{L}$-length
$\mathcal{L}\,\mathrm{Cut}$	$\mathcal{L}$-cut locus
$\mathcal{L}\exp$	$\mathcal{L}$-exponential map
$\mathcal{L}\,\mathrm{I}$	$\mathcal{L}$-index form
$\mathcal{L}\,\mathrm{J}_V$	$\mathcal{L}$-Jacobian
$L(v,X)$	linear trace Harnack quadratic
μ	μ-invariant
$(\mathcal{M},\hat{g})$	static Riemannian manifold
$\mathfrak{Met}$	space of Riemannian metrics on a manifold
MVP	mean value property

$\mathfrak{M}_{n,\kappa}$	collection of n-dimensional κ-solutions
$\mathfrak{M}_{n,\kappa}^{\mathrm{Harn}}$	n-dimensional κ-solutions with Harnack
ν	ν-invariant or unit outward normal
$n\omega_n$	volume of the unit Euclidean $(n-1)$-sphere
ω_n	volume of the unit Euclidean n-ball
ODE	ordinary differential equation
P_{ijk}	the symmetric 3-tensor $\nabla_i R_{jk} - \nabla_j R_{ik}$
PCO	positive curvature operator
PDE	partial differential equation
$R_{ijk\ell}$	$\sum_m R_{ijk}^m g_{m\ell}$ (opposite of Hamilton's convention)
R_{jk}	$\sum_i R_{ijk}^i = \sum_{i,\ell} g^{i\ell} R_{ijk\ell}$ (components of Ricci)
$\mathcal{R}_{jk}$	a symmetric 2-tensor ($\mathcal{R}_{jk} = R_{jk}$ is a special case)
RF	Ricci flow
RG flow	renormalization group flow
RHS	right-hand side
R, Rc, Rm	scalar, Ricci, and Riemann curvature tensors
$\mathrm{Rm}^{\#}$	the quadratic $\mathrm{Rm} \# \mathrm{Rm}$
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathrm{SO}(n, \mathbb{R})$	real orthogonal group
$S_V \mathcal{J}$ for $V \in \partial \mathcal{J}$	set of support functions of $\mathcal{J} \subset \mathbb{R}^k$ at V
sect	sectional curvature
$\mathcal{S}^n$	unit radius n-dimensional sphere
supp	support of a function
$\tau(t)$	function satisfying $\frac{d\tau}{dt} = -1$
$T_x \mathcal{M}$	tangent space of $\mathcal{M}$ at x
$T_x^* \mathcal{M}$	cotangent space of $\mathcal{M}$ at x
tr or trace	trace
$\tilde{V}$	reduced volume
$\hat{V}_\infty$	mock reduced volume
$\mathcal{V}$	vector bundle
Vol	volume of a manifold
$\mathcal{W}$	Perelman's entropy functional
$\mathcal{W}_{\mathrm{lin}}$	linear entropy functional
$W^{k,p}$	Sobolev space of functions with $\leq k$ weak derivatives in L^p
$W_{\mathrm{loc}}^{k,p}$	space of functions locally in $W^{k,p}$
$W_\top$	tangential component of the vector W
$W_\perp$	normal component of the vector W
WMP	weak maximum principle

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