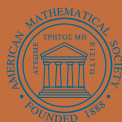


Mathematical
Surveys
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Volume 163

The Ricci Flow: Techniques and Applications

Part III: Geometric-Analytic Aspects

Bennett Chow
Sun-Chin Chu
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American Mathematical Society

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Preface

I didn't have time to write you a short letter, so I wrote you a long one instead.

– Samuel Clemens

What Part III is about

I'm taking the time for a number of things

That weren't important yesterday.

– From “Fixing a Hole” by The Beatles

This is Part III (a.k.a. ΔR_{ijkl}), a sequel to Part I ([40]; a.k.a. R_{ijkl}) and Part II ([41]; a.k.a. $\frac{\partial}{\partial t} R_{ijkl}$) of this volume (Volume Two) on techniques and applications of the Ricci flow (we shall refer to Volume One ([42]; a.k.a. g_{ij}) as Volume One).

In Part I we discussed various *geometric* topics in Ricci flow such as Ricci solitons, an introduction to the Kähler–Ricci flow, Hamilton's Cheeger–Gromov-type compactness theorem, Perelman's energy and entropy monotonicity, the foundations of Perelman's reduced distance function, the reduced volume, applications to the analysis of ancient solutions, and a primer on 3-manifold topology.

In Part II we discussed mostly *analytic* topics in Ricci flow including weak and strong maximum principles for scalar heat-type equations and systems on compact and noncompact manifolds, Böhm and Wilking's classification of closed manifolds with 2-positive curvature operator, Shi's local derivative estimates, Hamilton's matrix estimate, and Perelman's estimate for fundamental solutions of the adjoint heat equation.

Here, in Part III, we discuss mostly *geometric-analytic* topics in Ricci flow. In particular, we discuss properties of Perelman's entropy functional, point picking methods, aspects of Perelman's theory of κ -solutions including the κ -gap theorem, compactness theorem, and derivative estimates, Perelman's pseudolocality theorem, and aspects of the heat equation with respect to static and evolving metrics related to Ricci flow. In the appendices we review metric and Riemannian geometry including the space of points at infinity and Sharafutdinov retraction for complete noncompact manifolds with nonnegative sectional curvature. As in previous volumes, we have endeavored, as much as possible, to make the chapters independent of each other.

In Part IV we shall discuss some topics originally slated for Part III such as Hamilton's classification of nonsingular solutions, the linearized Ricci flow, stability of the Ricci flow, the space-time formulation of the Ricci flow, and Type II singularities from the numerical perspective.

Caveat: Many of the chapter numbers of references in Part II to Part III have changed and some of the referred chapters are in Part IV.

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Now that your rose is in bloom, a light hits the gloom on the grey.

– From "Kiss from a Rose" by Seal

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Contents of Part III of Volume Two

Well, you know, we're doing what we can.

– From “Revolution” by The Beatles

Chapter 17. Perelman’s entropy \mathcal{W} leads to the μ -invariant. We discuss qualitative properties of the μ -invariant such as lower and upper bounds and we give a proof of the fact that $\lim_{\tau \rightarrow 0^+} \mu(g, \tau) = 0$. We also discuss applications of the μ -invariant monotonicity formula. This includes the recent classification by Z.-L. Zhang of compact finite time singularity models as shrinking gradient Ricci solitons. We revisit the proof of the existence of a smooth minimizer for \mathcal{W} , providing more details than in Part I, and we also show that when the isometry group acts transitively, the minimizer is not unique for sufficiently small τ . Related to renormalization group considerations, some low-loop calculations are presented.

Chapter 18. We discuss some tools used in the study of the Ricci flow including the changing distances estimate for solutions of Ricci flow, point picking methods, rough monotonicity of the size of necks in complete noncompact manifolds with positive sectional curvature, and a local form of the weakened no local collapsing theorem.

Chapter 19. With the goal of understanding compactness in higher dimensions, we introduce the notion of ‘ κ -solution with Harnack’, which is a variant of Perelman’s notion of κ -solution. In dimensions 2 and 3 we show that κ -solutions with Harnack must have bounded curvature. We also discuss the construction of Perelman’s rotationally symmetric ancient solution on \mathcal{S}^n , the result that κ -solutions with Harnack must have bounded curvature, the existence of an asymptotic shrinker in a κ -solution (correcting a gap (no pun intended) in Part I), and the κ -gap theorem.

Chapter 20. We show that noncompact κ -solutions have asymptotic scalar curvature ratio $\text{ASCR} = \infty$ and asymptotic volume ratio $\text{AVR} = 0$; the latter result does not require the κ -noncollapsed at all scales assumption. We show that solutions which are almost ancient and have bounded nonnegative curvature operator are collapsed at large scales and we obtain a curvature estimate in noncollapsed balls. We prove that the collection of κ -solutions with Harnack is compact modulo scaling. In dimension 3 this is equivalent to Perelman’s compactness theorem and implies scaled derivative of curvature estimates.

Chapter 21. We discuss Perelman’s pseudolocality theorem. Assuming an initial ball with scalar curvature bounded from below and which is almost Euclidean isoperimetrically, we obtain a curvature estimate in a smaller ball; this estimate gets worse as time approaches the initial time. One may consider this as sort of a pseudolocalization of the curvature doubling time estimate. One of the ideas in the proof is that one can localize the entropy monotonicity formula by multiplying the integrand by a suitable time-dependent cutoff function. In the setting of a proof by contradiction, a main idea is to use point picking methods to locate an infinite sequence of ‘good’ high curvature points and to study a local entropy in their neighborhoods via Perelman’s Harnack-type estimate for fundamental solutions of the adjoint heat equation coupled to the Ricci flow.

Chapter 22. We discuss tools used in the proof of the pseudolocality theorem such as the point picking ‘Claims 1 and 2’, convergence of heat kernels under Cheeger–Gromov convergence, a uniform negative upper bound for the local entropies centered at the well-chosen bad points at time zero, and a sharp form of the logarithmic Sobolev inequality related to the isoperimetric inequality.

Chapter 23. We discuss existence and asymptotics for heat kernels with respect to *static* metrics. We follow the parametrix method of Levi and its Riemannian adaptation by Minakshisundaram and Pleijel. Starting with a good approximation to the heat kernel, we prove the existence of the heat kernel by establishing the convergence of the ‘convolution series’. With this construction we compute some low-order asymptotics for the heat kernel.

Chapter 24. We adapt the methods of the previous chapter to study the existence and asymptotics for heat kernels with respect to *evolving* metrics. We consider aspects of the adjoint heat kernel for evolving metrics related to §9.6 of Perelman’s paper [152]. We also discuss the existence of Dirichlet heat kernels on compact manifolds with boundary and heat kernels on noncompact manifolds with respect to evolving metrics.

Chapter 25. We discuss estimates for solutions to the heat equation with respect to evolving metrics including the parabolic mean value property for solutions to heat equations and the Li–Yau differential Harnack estimate for positive solutions to heat equations.

Chapter 26. Applying the estimates of the previous chapter, we discuss estimates for heat kernels with respect to evolving metrics including upper and lower bounds and the space-time mean value property. We also discuss the existence of distance-type functions on complete noncompact Riemannian manifolds with bounded gradient and Laplacian.

Appendix G. With Perelman’s work, the space-time of a solution of the Ricci flow is given a quasi-length space structure. This geometric structure is foundational in the understanding of singularity formation under the

Ricci flow. We discuss notions of (quasi-)metric and (quasi-)length spaces, Gromov–Hausdorff convergence, and Aleksandrov spaces.

Appendix H. We discuss convex analysis on Euclidean spaces and on locally convex subsets in Riemannian manifolds.

Appendix I. We discuss the points at infinity for nonnegatively curved manifolds, the Sharafutdinov retraction theorem, and some consequences.

Appendix J. We provide solutions to some of the exercises in the book.

Notation and Symbols

Confusion never stops, closing walls and ticking clocks.

– From “Clocks” by Coldplay

The following is a list of some of the notation and symbols which we use in this book.

\times	multiplication, when a formula does not fit on one line
∇	covariant derivative
$\nabla_{\star} r$	‘set gradient’ of the distance function to a point
\square	heat operator
\square^*	adjoint heat operator
\doteq	defined to be equal to
\cdot	dot product or multiplication
$\nabla\nabla f$	Hessian of f
α^{\sharp}	dual vector field to the 1-form α
ALE	asymptotically locally Euclidean
Area	area of a surface or volume of a hypersurface
ASCR	asymptotic scalar curvature ratio
AVR	asymptotic volume ratio
$B(p, r)$	ball of radius r centered at p
bounded curvature	bounded <i>sectional</i> curvature (for time-dependent metrics, the bound may depend on time)
$C_V \mathcal{J}$	tangent cone at V of a convex set $\mathcal{J} \subset \mathbb{R}^k$
const	constant
$\frac{d^+}{dt}, \frac{d^-}{dt}, \frac{d_+}{dt}, \frac{d_-}{dt}$	various Dini time derivatives
d	distance
d_{GH}	Gromov–Hausdorff distance
$d\mu$	volume form
$d\mu_{\mathbb{E}}$	Euclidean volume form
$d\sigma$ or dA	volume form on boundary or hypersurface
$\Delta, \Delta_L, \Delta_d$	Laplacian, Lichnerowicz Laplacian, Hodge Laplacian

diam	diameter
div	divergence
\mathbb{E}^n	\mathbb{R}^n with the flat Euclidean metric
$E_r(x, t)$	heat ball of radius r based at (x, t)
exp	exponential map
\mathcal{F}	Perelman's energy functional
Γ_{ij}^k	Christoffel symbols
$g(X, Y) = \langle X, Y \rangle$	metric or inner product
$g(t)$	time-dependent metric, e.g., solution of the Ricci flow
g_∞ or $g_\infty(t)$	limit Riemannian metric or solution of Ricci flow
h or Π	second fundamental form
H	mean curvature
$H_V \mathcal{J}$ for $V \in \partial \mathcal{J}$	set of closed half-spaces H containing $\mathcal{J} \subset \mathbb{R}^k$ with $V \in \partial H$
Hess f	Hessian of f (same as $\nabla \nabla f$)
\mathcal{I}	a time interval for the Ricci flow
id	identity
int	interior
inj	injectivity radius
Isom	group of isometries of a Riemannian manifold
IVP	initial-value problem
J	Jacobian of the exponential map
λ	λ -invariant
L	length
LHS	left-hand side
log	natural logarithm
L	L -distance
ℓ	reduced distance or ℓ -function
\mathcal{L}	Lie derivative or \mathcal{L} -length
$\mathcal{L} \text{Cut}$	\mathcal{L} -cut locus
$\mathcal{L} \text{exp}$	\mathcal{L} -exponential map
$\mathcal{L} \text{I}$	\mathcal{L} -index form
$\mathcal{L} \text{J}_V$	\mathcal{L} -Jacobian
$L(v, X)$	linear trace Harnack quadratic
μ	μ -invariant
(\mathcal{M}, \hat{g})	static Riemannian manifold
\mathfrak{Met}	space of Riemannian metrics on a manifold
MVP	mean value property

$\mathfrak{M}_{n,\kappa}$	collection of n -dimensional κ -solutions
$\mathfrak{M}_{n,\kappa}^{\text{Harn}}$	n -dimensional κ -solutions with Harnack
ν	ν -invariant or unit outward normal
$n\omega_n$	volume of the unit Euclidean $(n - 1)$ -sphere
ω_n	volume of the unit Euclidean n -ball
ODE	ordinary differential equation
P_{ijk}	the symmetric 3-tensor $\nabla_i R_{jk} - \nabla_j R_{ik}$
PCO	positive curvature operator
PDE	partial differential equation
R_{ijkl}	$\sum_m R_{ijk}^m g_{ml}$ (opposite of Hamilton's convention)
R_{jk}	$\sum_i R_{ijk}^i = \sum_{i,\ell} g^{i\ell} R_{ijkl}$ (components of Ricci)
\mathcal{R}_{jk}	a symmetric 2-tensor ($\mathcal{R}_{jk} = R_{jk}$ is a special case)
RF	Ricci flow
RG flow	renormalization group flow
RHS	right-hand side
R, Rc, Rm	scalar, Ricci, and Riemann curvature tensors
$\text{Rm}^\#$	the quadratic $\text{Rm} \# \text{Rm}$
\mathbb{R}^n	n -dimensional Euclidean space
$\text{SO}(n, \mathbb{R})$	real orthogonal group
$S_V \mathcal{J}$ for $V \in \partial \mathcal{J}$	set of support functions of $\mathcal{J} \subset \mathbb{R}^k$ at V
sect	sectional curvature
S^n	unit radius n -dimensional sphere
supp	support of a function
$\tau(t)$	function satisfying $\frac{d\tau}{dt} = -1$
$T_x \mathcal{M}$	tangent space of \mathcal{M} at x
$T_x^* \mathcal{M}$	cotangent space of \mathcal{M} at x
tr or trace	trace
\tilde{V}	reduced volume
\hat{V}_∞	mock reduced volume
\mathcal{V}	vector bundle
Vol	volume of a manifold
\mathcal{W}	Perelman's entropy functional
\mathcal{W}_{lin}	linear entropy functional
$W^{k,p}$	Sobolev space of functions with $\leq k$ weak derivatives in L^p
$W_{\text{loc}}^{k,p}$	space of functions locally in $W^{k,p}$
W_\top	tangential component of the vector W
W_\perp	normal component of the vector W
WMP	weak maximum principle

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