Toric Topology
Toric Topology

Victor M. Buchstaber
Taras E. Panov
## Contents

**Introduction**  
Chapter guide  
Acknowledgements

### Chapter 1. Geometry and Combinatorics of Polytopes
1. **Convex polytopes**  
2. **Gale duality and Gale diagrams**  
3. **Face vectors and Dehn–Sommerville relations**  
4. **Characterising the face vectors of polytopes**  
Polytopes: Additional Topics  
5. **Nestohedra and graph-associahedra**  
6. **Flagtopes and truncated cubes**  
7. **Differential algebra of combinatorial polytopes**  
8. **Families of polytopes and differential equations**

### Chapter 2. Combinatorial Structures
1. **Polyhedral fans**  
2. **Simplicial complexes**  
3. **Barycentric subdivision and flag complexes**  
4. **Alexander duality**  
5. **Classes of triangulated spheres**  
6. **Triangulated manifolds**  
7. **Stellar subdivisions and bistellar moves**  
8. **Simplicial posets and simplicial cell complexes**  
9. **Cubical complexes**

### Chapter 3. Combinatorial Algebra of Face Rings
1. **Face rings of simplicial complexes**  
2. **Tor-algebras and Betti numbers**  
3. **Cohen–Macaulay complexes**  
4. **Gorenstein complexes and Dehn–Sommerville relations**  
5. **Face rings of simplicial posets**  
Face Rings: Additional Topics  
6. **Cohen–Macaulay simplicial posets**  
7. **Gorenstein simplicial posets**  
8. **Generalised Dehn–Sommerville relations**

### Chapter 4. Moment-Angle Complexes
1. **Basic definitions**  
2. **Polyhedral products**
4.3. Homotopical properties 139  
4.4. Cell decomposition 143  
4.5. Cohomology ring 144  
4.6. Bigraded Betti numbers 150  
4.7. Coordinate subspace arrangements 157  
4.8. Free and almost free torus actions on moment-angle complexes 162  
4.9. Massey products in the cohomology of moment-angle complexes 168  
4.10. Moment-angle complexes from simplicial posets 171  

Chapter 5. Toric Varieties and Manifolds 179  
5.1. Classical construction from rational fans 179  
5.2. Projective toric varieties and polytopes 183  
5.3. Cohomology of toric manifolds 185  
5.4. Algebraic quotient construction 188  
5.5. Hamiltonian actions and symplectic reduction 195  

Chapter 6. Geometric Structures on Moment-Angle Manifolds 201  
6.1. Intersections of quadrics 201  
6.2. Moment-angle manifolds from polytopes 205  
6.3. Symplectic reduction and moment maps revisited 209  
6.4. Complex structures on intersections of quadrics 212  
6.5. Moment-angle manifolds from simplicial fans 215  
6.6. Complex structures on moment-angle manifolds 219  
6.7. Holomorphic principal bundles and Dolbeault cohomology 224  
6.8. Hamiltonian-minimal Lagrangian submanifolds 231  

Chapter 7. Half-Dimensional Torus Actions 239  
7.1. Locally standard actions and manifolds with corners 240  
7.2. Toric manifolds and their quotients 242  
7.3. Quasitoric manifolds 243  
7.4. Locally standard $T$-manifolds and torus manifolds 257  
7.5. Topological toric manifolds 278  
7.6. Relationship between different classes of $T$-manifolds 282  
7.7. Bounded flag manifolds 284  
7.8. Bott towers 287  
7.9. Weight graphs 302  

Chapter 8. Homotopy Theory of Polyhedral Products 313  
8.1. Rational homotopy theory of polyhedral products 314  
8.2. Wedges of spheres and connected sums of sphere products 321  
8.3. Stable decompositions of polyhedral products 326  
8.4. Loop spaces, Whitehead and Samelson products 331  
8.5. The case of flag complexes 340  

Chapter 9. Torus Actions and Complex Cobordism 347  
9.1. Toric and quasitoric representatives in complex bordism classes 347  
9.2. The universal toric genus 357  
9.3. Equivariant genera, rigidity and fibre multiplicativity 364  
9.4. Isolated fixed points: localisation formulae 367
CONTENTS

9.5. Quasitoric manifolds and genera 376
9.6. Genera for homogeneous spaces of compact Lie groups 380
9.7. Rigid genera and functional equations 385

Appendix A. Commutative and Homological Algebra 395
A.1. Algebras and modules 395
A.2. Homological theory of graded rings and modules 398
A.3. Regular sequences and Cohen–Macaulay algebras 405
A.4. Formality and Massey products 409

Appendix B. Algebraic Topology 413
B.1. Homotopy and homology 413
B.2. Elements of rational homotopy theory 424
B.3. Eilenberg–Moore spectral sequences 426
B.4. Group actions and equivariant topology 428
B.5. Stably complex structures 433
B.6. Weights and signs of torus actions 434

Appendix C. Categorical Constructions 439
C.1. Diagrams and model categories 439
C.2. Algebraic model categories 444
C.3. Homotopy limits and colimits 450

Appendix D. Bordism and Cobordism 453
D.1. Bordism of manifolds 453
D.2. Thom spaces and cobordism functors 454
D.3. Oriented and complex bordism 457
D.4. Characteristic classes and numbers 463
D.5. Structure results 467
D.6. Ring generators 468

Appendix E. Formal Group Laws and Hirzebruch Genera 473
E.1. Elements of the theory of formal group laws 473
E.2. Formal group law of geometric cobordisms 477
E.3. Hirzebruch genera (complex case) 479
E.4. Hirzebruch genera (oriented case) 486
E.5. Krichever genus 488

Bibliography 495

Index 511
Introduction

Traditionally, the study of torus actions on topological spaces has been considered as a classical field of algebraic topology. Specific problems connected with torus actions arise in different areas of mathematics and mathematical physics, which results in permanent interest in the theory, new applications and penetration of new ideas into topology.

Since the 1970s, algebraic and symplectic viewpoints on torus actions have enriched the subject with new combinatorial ideas and methods, largely based on the convex-geometric concept of polytopes.

The study of algebraic torus actions on algebraic varieties has quickly developed into a highly successful branch of algebraic geometry, known as toric geometry. It gives a bijection between, on the one hand, toric varieties, which are complex algebraic varieties equipped with an action of an algebraic torus with a dense orbit, and on the other hand, fans, which are combinatorial objects. The fan allows one to completely translate various algebraic-geometric notions into combinatorics. Projective toric varieties correspond to fans which arise from convex polytopes. A valuable aspect of this theory is that it provides many explicit examples of algebraic varieties, leading to applications in deep subjects such as singularity theory and mirror symmetry.

In symplectic geometry, since the early 1980s there has been much activity in the field of Hamiltonian group actions on symplectic manifolds. Such an action defines the moment map from the manifold to a Euclidean space (more precisely, the dual Lie algebra of the torus) whose image is a convex polytope. If the torus has half the dimension of the manifold, the image of the moment map determines the manifold up to equivariant symplectomorphism. The class of polytopes which arise as the images of moment maps can be described explicitly, together with an effective procedure for recovering a symplectic manifold from such a polytope. In symplectic geometry, as in algebraic geometry, one translates various geometric constructions into the language of convex polytopes and combinatorics.

There is a tight relationship between the algebraic and the symplectic pictures: a projective embedding of a toric manifold determines a symplectic form and a moment map. The image of the moment map is a convex polytope that is dual to the fan. In both the smooth algebraic-geometric and the symplectic situations, the compact torus action is locally isomorphic to the standard action of $(S^1)^n$ on $\mathbb{C}^n$ by rotation of the coordinates. Thus the quotient of the manifold by this action is naturally a manifold with corners, stratified according to the dimension of the stabilisers, and each stratum can be equipped with data that encodes the isotropy torus action along that stratum. Not only does this structure of the quotient provide a powerful means of investigating the action, but some of its subtler combinatorial properties may also be illuminated by a careful study of the equivariant topology.
of the manifold. Thus, it should come as no surprise that since the beginning of the 1990s, the ideas and methodology of toric varieties and Hamiltonian torus actions have started penetrating back into algebraic topology.

By 2000, several constructions of topological analogues of toric varieties and symplectic toric manifolds had appeared in the literature, together with different seemingly unrelated realisations of what later has become known as moment-angle manifolds. We tried to systematise both known and emerging links between torus actions and combinatorics in our 2000 paper [67] in Russian Mathematical Surveys, where the terms ‘moment-angle manifold’ and ‘moment-angle complex’ first appeared. Two years later it grew into a book Torus Actions and Their Applications in Topology and Combinatorics [68] published by the AMS in 2002 (the extended Russian edition [69] appeared in 2004). The title ‘Toric Topology’ coined by our colleague Nigel Ray became official after the 2006 Osaka conference under the same name. Its proceedings volume [177] contained many important contributions to the subject, as well as the introductory survey An Invitation to Toric Topology: Vertex Four of a Remarkable Tetrahedron by Buchstaber and Ray. The vertices of the ‘toric tetrahedron’ are topology, combinatorics, algebraic and symplectic geometry, and they have symbolised many strong links between these subjects. With many young researchers entering the subject and conferences held around the world every year, toric topology has definitely grown into a mature area. Its various aspects are presented in this monograph, with an intention to consolidate the foundations and stimulate further applications.

Chapter guide

1. Polytopes
2. Combinatorial structures
3. Face rings
4. Moment-angle complexes
5. Toric varieties
6. Moment-angle manifolds
7. Half-dim torus actions
8. Homotopy theory
9. Cobordism

Each chapter and most sections have their own introductions with more specific information about the contents. ‘Additional topics’ of Chapters 1, 3 and 4 contain more specific material which is not used in an essential way in the following chapters. The appendices at the end of the book contain material of more general nature,
not exclusively related to toric topology. A more experienced reader may refer to them only for notation and terminology.

At the heart of toric topology lies a class of torus actions whose orbit spaces are highly structured in combinatorial terms, that is, have lots of orbit types tied together in a nice combinatorial way. We use the generic terms toric space and toric object to refer to a topological space with a nice torus action, or to a space produced from a torus action via different standard topological or categorical constructions. Examples of toric spaces include toric varieties, toric and quasitoric manifolds and their generalisations, moment-angle manifolds, moment-angle complexes and their Borel constructions, polyhedral products, complements of coordinate subspace arrangements, intersections of real or Hermitian quadrics, etc.

In Chapter 1 we collect background material related to convex polytopes, including basic convex-geometric constructions and the combinatorial theory of face vectors. The famous $g$-theorem describing integer sequences that can be the face vectors of simple (or simplicial) polytopes is one of the most striking applications of toric geometry to combinatorics. The concepts of Gale duality and Gale diagrams are important tools for the study of moment-angle manifolds via intersections of quadrics. In the additional sections we describe several combinatorial constructions providing families of simple polytopes, including nestohedra, graph associahedra, flagtopes and truncated cubes. The classical series of permutahedra and associahedra (Stasheff polytopes) are particular examples. The construction of nestohedra takes its origin in singularity and representation theory. We develop a differential algebraic formalism which links the generating series of nestohedra to classical partial differential equations. The potential of truncated cubes in toric topology is yet to be fully exploited, as they provide an immense source of explicitly constructed toric spaces.

In Chapter 2 we describe systematically combinatorial structures that appear in the orbit spaces of toric objects. Besides convex polytopes, these include fans, simplicial and cubical complexes, and simplicial posets. All these structures are objects of independent interest for combinatorialists, and we emphasised the aspects of their combinatorial theory most relevant to subsequent topological applications.

The subject of Chapter 3 is the algebraic theory of face rings (also known as Stanley–Reisner rings) of simplicial complexes, and their generalisations to simplicial posets. With the appearance of face rings at the beginning of the 1970s in the work of Reisner and Stanley, many combinatorial problems were translated into the language of commutative algebra, which paved the way for their solution using the extensive machinery of algebraic and homological methods. Algebraic tools used for attacking combinatorial problems include regular sequences, Cohen–Macaulay and Gorenstein rings, Tor-algebras, local cohomology, etc. A whole new thriving field appeared on the borders of combinatorics and algebra, which has since become known as combinatorial commutative algebra.

Chapter 4 is the first ‘toric’ chapter of the book; it links the combinatorial and algebraic constructions of the previous chapters to the world of toric spaces. The concept of the moment-angle complex $\mathcal{Z}_K$ is introduced as a functor from the category of simplicial complexes $\mathcal{K}$ to the category of topological spaces with torus actions and equivariant maps. When $\mathcal{K}$ is a triangulated manifold, the moment-angle complex $\mathcal{Z}_K$ contains a free orbit $\mathcal{Z}_\emptyset$ consisting of singular points. Removing this orbit we obtain an open manifold $\mathcal{Z}_K \setminus \mathcal{Z}_\emptyset$, which satisfies the relative version of
Poincaré duality. Combinatorial invariants of simplicial complexes $\mathcal{K}$ therefore can be described in terms of topological characteristics of the corresponding moment-angle complexes $\mathcal{Z}_\mathcal{K}$. In particular, the face numbers of $\mathcal{K}$, as well as the more subtle bigraded Betti numbers of the face ring $\mathbb{Z}[\mathcal{K}]$ can be expressed in terms of the cellular cohomology groups of $\mathcal{Z}_\mathcal{K}$. The integral cohomology ring $H^*(\mathcal{Z}_\mathcal{K})$ is shown to be isomorphic to the Tor-algebra $\text{Tor}_{\mathbb{Z}}[v_1,\ldots,v_m](\mathbb{Z}[\mathcal{K}],\mathbb{Z})$. The proof builds upon a construction of a ring model for cellular cochains of $\mathcal{Z}_\mathcal{K}$ and the corresponding cellular diagonal approximation, which is functorial with respect to maps of moment-angle complexes induced by simplicial maps of $\mathcal{K}$. This functorial property of the cellular diagonal approximation for $\mathcal{Z}_\mathcal{K}$ is quite special, due to the lack of such a construction for general cell complexes. Another result of Chapter 4 is a homotopy equivalence (an equivariant deformation retraction) from the complement $U(\mathcal{K})$ of the arrangement of coordinate subspaces in $\mathbb{C}^m$ determined by $\mathcal{K}$ to the moment-angle complex $\mathcal{Z}_\mathcal{K}$. Particular cases of this result are known in toric geometry and geometric invariant theory. It opens a new perspective on moment-angle complexes, linking them to the theory of configuration spaces and arrangements.

Toric varieties are the subject of Chapter 5. This is an extensive area with a vast literature. We outline the influence of toric geometry on the emergence of toric topology and emphasise combinatorial, topological and symplectic aspects of toric varieties. The construction of moment-angle manifolds via nondegenerate intersections of Hermitian quadrics in $\mathbb{C}^m$, motivated by symplectic geometry, is also discussed here. Some basic knowledge of algebraic geometry may be required in Chapter 5. Appropriate references are given in the introduction to the chapter.

The material of the first five chapters of the book should be accessible for a graduate student, or a reader with a very basic knowledge of algebra and topology. These five chapters may be also used for advanced courses on the relevant aspects of topology, algebraic geometry and combinatorial algebra. The general algebraic and topological constructions required here are collected in Appendices A and B respectively. The last four chapters are more research-oriented.

Geometry of moment-angle manifolds is studied in Chapter 6. The construction of moment-angle manifolds as the level sets of toric moment maps is taken as the starting point for the systematic study of intersections of Hermitian quadrics via Gale duality. Following a remarkable discovery by Bosio and Meersseman of complex-analytic structures on moment-angle manifolds corresponding to simple polytopes, we proceed by showing that moment-angle manifolds corresponding to a more general class of complete simplicial fans can also be endowed with complex-analytic structures. The resulting family of non-Kähler complex manifolds includes the classical series of Hopf and Calabi–Eckmann manifolds. We also describe important invariants of these complex structures, such as the Hodge numbers and Dolbeault cohomology rings, study holomorphic torus principal bundles over toric varieties, and establish collapse results for the relevant spectral sequences. We conclude by exploring the construction of A.E. Mironov providing a vast family of Lagrangian submanifolds with special minimality properties in complex space, complex projective space and other toric varieties. Like many other geometric constructions in this chapter, it builds upon the realisation of the moment-angle manifold as an intersection of quadrics.
In Chapter 7 we discuss several topological constructions of even-dimensional manifolds with an effective action of a torus of half the dimension of the manifold. They can be viewed as topological analogues and generalisations of compact nonsingular toric varieties (or toric manifolds). These include quasitoric manifolds of Davis and Januszkiewicz, torus manifolds of Hattori and Masuda, and topological toric manifolds of Ishida, Fukukawa and Masuda. For all these classes of toric objects, the equivariant topology of the action and the combinatorics of the orbit spaces interact in a harmonious way, leading to a host of results linking topology with combinatorics. We also discuss the relationship with GKM-manifolds (named after Goresky, Kottwitz and MacPherson), another class of toric objects having its origin in symplectic topology.

Homotopy-theoretical aspects of toric topology are the subject of Chapter 8. This is now a very active area. Homotopy techniques brought to bear on the study of polyhedral products and other toric spaces include model categories, homotopy limits and colimits, higher Whitehead and Samelson products. The required information about categorical methods in topology is collected in Appendix C.

In Chapter 9 we review applications of toric methods in a classical field of algebraic topology, complex cobordism. It is a generalised cohomology theory that combines both geometric intuition and elaborate algebraic techniques. The toric viewpoint brings an entirely new perspective on complex cobordism theory in both its nonequivariant and equivariant versions.

The later chapters require more specific knowledge of algebraic topology, such as characteristic classes and spectral sequences, for which we recommend respectively the classical book of Milnor and Stasheff [273] and the excellent guide by McCleary [260]. Basic facts and constructions from bordism and cobordism theory are given in Appendix D while the related techniques of formal group laws and multiplicative genera are reviewed in Appendix E.

Acknowledgements

We wish to express our deepest thanks to

- our teacher Sergei Petrovich Novikov for encouragement and support of our research on toric topology;
- Mikiya Masuda and Nigel Ray for long and much fruitful collaboration;
- our coauthors in toric topology;
- Peter Landweber for most helpful suggestions on improving the text;
- all our colleagues who participated in conferences on toric topology for the insight gained from discussions following talks and presentations.

This work was supported by the Russian Science Foundation (grant no. 14-11-00414). We also thank the Russian Foundation for Basic Research, the President of the Russian Federation Grants Council and Dmitri Zimin’s ‘Dynasty’ Foundation for their support of our research related to this monograph.
Bibliography


## Index

<table>
<thead>
<tr>
<th>Term</th>
<th>Page(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action (of group)</td>
<td>428</td>
</tr>
<tr>
<td>almost free</td>
<td>162, 429</td>
</tr>
<tr>
<td>effective</td>
<td>429</td>
</tr>
<tr>
<td>free</td>
<td>129</td>
</tr>
<tr>
<td>proper</td>
<td>195, 196, 216, 224</td>
</tr>
<tr>
<td>semifree</td>
<td>293, 129</td>
</tr>
<tr>
<td>transitive</td>
<td>129</td>
</tr>
<tr>
<td>Acyclic (space)</td>
<td>263, 419</td>
</tr>
<tr>
<td>Adjoint functor</td>
<td>439, 441</td>
</tr>
<tr>
<td>Affine equivalence</td>
<td>2</td>
</tr>
<tr>
<td>$A$-genus</td>
<td>394, 486</td>
</tr>
<tr>
<td>Alexander duality</td>
<td>67, 114</td>
</tr>
<tr>
<td>Algebra</td>
<td>395</td>
</tr>
<tr>
<td>bigraded</td>
<td>396</td>
</tr>
<tr>
<td>connected</td>
<td>396</td>
</tr>
<tr>
<td>exterior</td>
<td>396</td>
</tr>
<tr>
<td>finitely generated</td>
<td>395</td>
</tr>
<tr>
<td>graded</td>
<td>395</td>
</tr>
<tr>
<td>graded-commutative</td>
<td>395</td>
</tr>
<tr>
<td>free</td>
<td>396</td>
</tr>
<tr>
<td>polynomial</td>
<td>396</td>
</tr>
<tr>
<td>multigraded</td>
<td>396</td>
</tr>
<tr>
<td>with straightening law (ASL)</td>
<td>116</td>
</tr>
<tr>
<td>Almost complex structure</td>
<td>433</td>
</tr>
<tr>
<td>associated with omniorientation</td>
<td>247</td>
</tr>
<tr>
<td>integrable</td>
<td>484</td>
</tr>
<tr>
<td>$G$-invariant</td>
<td>351</td>
</tr>
<tr>
<td>$T$-invariant</td>
<td>214, 251, 434</td>
</tr>
<tr>
<td>Ample divisor</td>
<td>157</td>
</tr>
<tr>
<td>Annihilator (of module)</td>
<td>408</td>
</tr>
<tr>
<td>Arrangement (of subspaces)</td>
<td>65</td>
</tr>
<tr>
<td>coordinate</td>
<td>157</td>
</tr>
<tr>
<td>Aspherical (space)</td>
<td>415, 247</td>
</tr>
<tr>
<td>Asociahedron</td>
<td>53</td>
</tr>
<tr>
<td>$f$-vector of</td>
<td>52</td>
</tr>
<tr>
<td>generalised</td>
<td>23</td>
</tr>
<tr>
<td>Associated polyhedron (of intersection of quadrics)</td>
<td>204</td>
</tr>
<tr>
<td>Atiyah–Hirzebruch formula</td>
<td>474</td>
</tr>
<tr>
<td>Augmentation</td>
<td>413</td>
</tr>
<tr>
<td>Augmentation genus</td>
<td>485</td>
</tr>
<tr>
<td>Axial function</td>
<td>267, 304</td>
</tr>
<tr>
<td>$n$-independent</td>
<td>304</td>
</tr>
<tr>
<td>2-independent</td>
<td>304</td>
</tr>
<tr>
<td>Baker–Akhiezer function</td>
<td>488</td>
</tr>
<tr>
<td>Bar construction</td>
<td>443</td>
</tr>
<tr>
<td>Barnette sphere</td>
<td>74, 172, 180, 282</td>
</tr>
<tr>
<td>Barycentre (of simplex)</td>
<td>63</td>
</tr>
<tr>
<td>Barycentric subdivision</td>
<td>65</td>
</tr>
<tr>
<td>of polytope</td>
<td>65</td>
</tr>
<tr>
<td>of simplicial complex</td>
<td>65</td>
</tr>
<tr>
<td>of simplicial poset</td>
<td>83, 122</td>
</tr>
<tr>
<td>Base (of Schlegel diagram)</td>
<td>70</td>
</tr>
<tr>
<td>Based map</td>
<td>113</td>
</tr>
<tr>
<td>Based space</td>
<td>113</td>
</tr>
<tr>
<td>Basepoint</td>
<td>113</td>
</tr>
<tr>
<td>Basis (of free module)</td>
<td>396</td>
</tr>
<tr>
<td>Bernoulli number</td>
<td>483</td>
</tr>
<tr>
<td>Betti numbers (algebraic)</td>
<td>97, 120, 126</td>
</tr>
<tr>
<td>Betti numbers (topological)</td>
<td>120</td>
</tr>
<tr>
<td>bigraded</td>
<td>143</td>
</tr>
<tr>
<td>Bidegree</td>
<td>396</td>
</tr>
<tr>
<td>Bistellar equivalence</td>
<td>70</td>
</tr>
<tr>
<td>Bistellar move</td>
<td>78</td>
</tr>
<tr>
<td>Blow-down</td>
<td>310</td>
</tr>
<tr>
<td>Blow-up</td>
<td>310</td>
</tr>
<tr>
<td>of $T$-graph</td>
<td>310</td>
</tr>
<tr>
<td>of $T$-manifold</td>
<td>275</td>
</tr>
<tr>
<td>Boolean lattice</td>
<td>81</td>
</tr>
<tr>
<td>Bordism</td>
<td>453</td>
</tr>
<tr>
<td>complex</td>
<td>457</td>
</tr>
<tr>
<td>oriented</td>
<td>457</td>
</tr>
<tr>
<td>unoriented</td>
<td>453</td>
</tr>
<tr>
<td>Borel construction</td>
<td>429</td>
</tr>
<tr>
<td>Borel spectral sequence</td>
<td>228</td>
</tr>
<tr>
<td>Bott manifold</td>
<td>288</td>
</tr>
<tr>
<td>generalised</td>
<td>288</td>
</tr>
<tr>
<td>Bott–Taubes polytope</td>
<td>37</td>
</tr>
<tr>
<td>Bott tower</td>
<td>288</td>
</tr>
<tr>
<td>generalised</td>
<td>300</td>
</tr>
<tr>
<td>real</td>
<td>301</td>
</tr>
<tr>
<td>topologically trivial</td>
<td>228</td>
</tr>
<tr>
<td>Boundary</td>
<td>397</td>
</tr>
<tr>
<td>Bounded flag</td>
<td>294</td>
</tr>
<tr>
<td>Bounded flag manifold</td>
<td>283, 289, 361</td>
</tr>
<tr>
<td>Bouquet (of spaces)</td>
<td>138, 413</td>
</tr>
<tr>
<td>Brückner sphere</td>
<td>75</td>
</tr>
<tr>
<td>Buchsbaum complex</td>
<td>112</td>
</tr>
<tr>
<td>Term</td>
<td>Page</td>
</tr>
<tr>
<td>------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>Buchstaber invariant</td>
<td>165</td>
</tr>
<tr>
<td>Building set, graphical</td>
<td>34</td>
</tr>
<tr>
<td>Calabi–Eckmann manifold</td>
<td>230</td>
</tr>
<tr>
<td>Catalan number</td>
<td>34</td>
</tr>
<tr>
<td>Categorical quotient</td>
<td>188</td>
</tr>
<tr>
<td>CAT(0) inequality</td>
<td>74</td>
</tr>
<tr>
<td>Cell complex</td>
<td>414</td>
</tr>
<tr>
<td>Cellular approximation</td>
<td>114</td>
</tr>
<tr>
<td>Cellular chain</td>
<td>221</td>
</tr>
<tr>
<td>Cellular map</td>
<td>114</td>
</tr>
<tr>
<td>Chain complex, augmented</td>
<td>397</td>
</tr>
<tr>
<td>Chain homotopy</td>
<td>398</td>
</tr>
<tr>
<td>Characteristic function, directed</td>
<td>244</td>
</tr>
<tr>
<td>Characteristic matrix, refined</td>
<td>243</td>
</tr>
<tr>
<td>Characteristic number, refined</td>
<td>241</td>
</tr>
<tr>
<td>Characteristic pair, combinatorial</td>
<td>245</td>
</tr>
<tr>
<td>Characteristic pair, equivalence of</td>
<td>247</td>
</tr>
<tr>
<td>Characteristic submanifold</td>
<td>244</td>
</tr>
<tr>
<td>Charney–Davis Conjecture</td>
<td>98, 74</td>
</tr>
<tr>
<td>Chern class, 163, 170</td>
<td>243, 147</td>
</tr>
<tr>
<td>in complex cobordism, 164</td>
<td>147</td>
</tr>
<tr>
<td>in equivariant cohomology, 143</td>
<td></td>
</tr>
<tr>
<td>in generalised cohomology theory, 184</td>
<td></td>
</tr>
<tr>
<td>of stably complex manifold, 164</td>
<td></td>
</tr>
<tr>
<td>Chern–Dold character, 305</td>
<td></td>
</tr>
<tr>
<td>Chern number, 264, 179</td>
<td>204, 220</td>
</tr>
<tr>
<td>Chow ring</td>
<td>185</td>
</tr>
<tr>
<td>CHP (covering homotopy property), 415</td>
<td></td>
</tr>
<tr>
<td>Classifying functor (algebraic), 148</td>
<td></td>
</tr>
<tr>
<td>Classifying space, 129</td>
<td></td>
</tr>
<tr>
<td>Clique, 30, 244</td>
<td></td>
</tr>
<tr>
<td>Cobar construction, 335</td>
<td>148</td>
</tr>
<tr>
<td>Cobordism, 155, 156</td>
<td>158</td>
</tr>
<tr>
<td>complex, 158</td>
<td></td>
</tr>
<tr>
<td>equivariant</td>
<td></td>
</tr>
<tr>
<td>geometric, 357</td>
<td>359</td>
</tr>
<tr>
<td>homotopic, 357</td>
<td>358</td>
</tr>
<tr>
<td>unoriented, 357</td>
<td>357</td>
</tr>
<tr>
<td>Coboundary, 397</td>
<td></td>
</tr>
<tr>
<td>Cochain complex, 397, 152</td>
<td>415</td>
</tr>
<tr>
<td>Cochain homotopy, 397</td>
<td></td>
</tr>
<tr>
<td>Cocycle, 397</td>
<td></td>
</tr>
<tr>
<td>Cofibrant (object, replacement, approximation), 440</td>
<td></td>
</tr>
<tr>
<td>Cofibration, 440</td>
<td></td>
</tr>
<tr>
<td>in model category, 140</td>
<td></td>
</tr>
<tr>
<td>Cofibre, 416</td>
<td></td>
</tr>
<tr>
<td>Coformality, 416</td>
<td>418</td>
</tr>
<tr>
<td>Cohen–Macaulay</td>
<td></td>
</tr>
<tr>
<td>algebra, 408</td>
<td>418</td>
</tr>
<tr>
<td>module, 408</td>
<td></td>
</tr>
<tr>
<td>simplicial complex, 106</td>
<td></td>
</tr>
<tr>
<td>simplicial poset, 121</td>
<td></td>
</tr>
<tr>
<td>Cohomological rigidity, 301</td>
<td></td>
</tr>
<tr>
<td>Cohomology</td>
<td></td>
</tr>
<tr>
<td>cellular, 422</td>
<td></td>
</tr>
<tr>
<td>of cochain complex, 397</td>
<td>422</td>
</tr>
<tr>
<td>simplicial, 415</td>
<td>422</td>
</tr>
<tr>
<td>reduced, 418</td>
<td>422</td>
</tr>
<tr>
<td>singular, 420</td>
<td>422</td>
</tr>
<tr>
<td>Cohomology product, 422</td>
<td></td>
</tr>
<tr>
<td>Cohomology product length, 150</td>
<td></td>
</tr>
<tr>
<td>Colimit, 172, 181, 313, 130</td>
<td></td>
</tr>
<tr>
<td>Combinatorial equivalence</td>
<td></td>
</tr>
<tr>
<td>of polyhedral complexes, 70</td>
<td></td>
</tr>
<tr>
<td>of polytopes, 2</td>
<td></td>
</tr>
<tr>
<td>Combinatorial equivalence of polytopes, 2</td>
<td></td>
</tr>
<tr>
<td>Complex orientable map, 458</td>
<td></td>
</tr>
<tr>
<td>Complete intersection algebra, 111</td>
<td></td>
</tr>
<tr>
<td>Cone</td>
<td></td>
</tr>
<tr>
<td>over simplicial complex, 61</td>
<td>147</td>
</tr>
<tr>
<td>over space, 147</td>
<td>147</td>
</tr>
<tr>
<td>Cone (convex polyhedral), 56</td>
<td></td>
</tr>
<tr>
<td>dual, 56</td>
<td></td>
</tr>
<tr>
<td>rational, 56</td>
<td></td>
</tr>
<tr>
<td>regular, 56</td>
<td></td>
</tr>
<tr>
<td>simplicial, 56</td>
<td></td>
</tr>
<tr>
<td>strongly convex, 56</td>
<td></td>
</tr>
<tr>
<td>Connected sum</td>
<td></td>
</tr>
<tr>
<td>of manifolds, 344, 361, 161</td>
<td></td>
</tr>
<tr>
<td>equivariant, 351</td>
<td></td>
</tr>
<tr>
<td>of simple polytopes, 6, 17</td>
<td></td>
</tr>
<tr>
<td>of simplicial complexes, 92</td>
<td></td>
</tr>
<tr>
<td>of stably complex manifolds, 162</td>
<td></td>
</tr>
<tr>
<td>Connection (on graph), 304</td>
<td></td>
</tr>
<tr>
<td>Contractible (space), 113</td>
<td></td>
</tr>
<tr>
<td>Contraction (of building set), 30</td>
<td></td>
</tr>
<tr>
<td>Coordinate subspace, 157</td>
<td></td>
</tr>
<tr>
<td>Coproduct, 139</td>
<td></td>
</tr>
<tr>
<td>Core (of simplicial complex), 63</td>
<td></td>
</tr>
<tr>
<td>Cross-polytope, 2</td>
<td></td>
</tr>
<tr>
<td>Cube, 2</td>
<td></td>
</tr>
<tr>
<td>standard, 2</td>
<td></td>
</tr>
<tr>
<td>topological, 83</td>
<td></td>
</tr>
<tr>
<td>Cubical complex</td>
<td></td>
</tr>
<tr>
<td>abstract, 83</td>
<td></td>
</tr>
<tr>
<td>polyhedral, 83</td>
<td></td>
</tr>
<tr>
<td>topological, 83</td>
<td></td>
</tr>
<tr>
<td>Cubical subdivision, 58</td>
<td></td>
</tr>
<tr>
<td>Cup product, 422</td>
<td></td>
</tr>
<tr>
<td>CW complex, 414</td>
<td></td>
</tr>
<tr>
<td>Cycle, 397</td>
<td></td>
</tr>
<tr>
<td>Cyclohedron, 37</td>
<td></td>
</tr>
<tr>
<td>Davis–Januszkiewicz space, 141, 316</td>
<td></td>
</tr>
<tr>
<td>Deformation retraction, 158</td>
<td></td>
</tr>
<tr>
<td>Degree (grading), 395</td>
<td></td>
</tr>
<tr>
<td>external, 395</td>
<td></td>
</tr>
<tr>
<td>internal, 395</td>
<td></td>
</tr>
<tr>
<td>total, 395</td>
<td></td>
</tr>
</tbody>
</table>
Dehn–Sommerville relations
for polytopes, 13 15
for simplicial complexes, 113
for simplicial posets, 127
for triangulated manifolds, 128
Depth (of module), 406
Derived functor, 141
Diagonal approximation, 146
Diagonal map, 122
Diagram (functor), 139
Diagram category, 139
Differential graded algebra (dga), 398 409 442 445
formal, 111
homologically connected, 109
simply connected, 110
Differential graded coalgebra, 442 445
Differential graded Lie algebra, 442 447
Dimension
of module, 408
of polytope, 1
of simplicial complex, 59
of simplicial poset, 81
Discriminant (of elliptic curve), 175
Dolbeault cohomology, 225
Dolbeault complex, 225 483
Double (simplicial), 165
Edge, 2
Eilenberg–Mac Lane space, 113
Eilenberg–Moore spectral sequence, 127
Elliptic cohomology, 488
Elliptic curve
Jacobi model, 175
Weierstrass model, 187
Elliptic formal group law, 275
universal, 175
Elliptic genus, 364 392 386 389
universal, 386
Elliptic sine, 364 392 386 389
Equivariant bundle, 131
Equivariant characteristic class, 133
Equivariant cohomology, 483
of T-graph, 305
Equivariant map, 128
Euler class
in complex cobordism, 401 417
in equivariant cobordism, 399
in equivariant cohomology, 261 432
in generalised cohomology theory, 483
Euler formula, 15
Euler characteristic, 254 377 118 119 483
Eulerian poset, 127
Exact sequence, 197
of fibration, 116
of pair, 130
Excision, 120
Exponential (of formal group law), 174
Ext (functor), 107
Face
of cubical complex, 53
of cone, 56
of manifold with corners, 241 283
of manifold with faces, 241
of polytope, 11
of simplicial complex, 59
of simplicial poset, 82
of T-graph, 305
Face category (of simplicial complex), 82
Face coalgebra, 335
Face poset, 2
Face ring (Stanley–Reisner ring)
of manifold with corners, 265
of simple polytope, 92
of simplicial complex, 92
exterior, 142
of simplicial poset, 115
Facet, 2 241 305
Face truncation, 5
Fan, 56 72
complete, 56
normal, 57
rational, 56
regular, 56
simplicial, 56
Fat wedge, 138
Fibrant (object, replacement, approximation), 440
Fibration, 115
in model category, 140
locally trivial, 113
Fibre bundle, 113
associated (with G-space), 429
Fixed point, 129
Fixed point data, 368
Flag complex (simplicial), 465 510
Flag manifold, 382 465
Flagtope (flag polytope), 38 66
Folding map, 82
Formal group law, 173
Abel, 476
elliptic, 475
linearisable, 473
of geometric cobordisms, 477
universal, 475
Formality
in model category, 444
integral, 316
of dg-algebra, 111
of space, 165 313 320 125
F-polynomial (of polytope), 13
Frölicher spectral sequence, 228
Full subcomplex (of simplicial complex), 69
Fundamental group, 413
Fundamental homology class, 156
f-vector (face vector)
of cubical complex, 84
of polytope, 14
of simplicial complex, 60
of simplicial poset, 117

Gal Conjecture, 39, 74
Gale diagram, 11, 208, 236
combinatorial, 13, 207
Gale duality, 10, 204, 210, 212, 227
Gale transform, 10
Ganea’s Theorem, 143
g-conjecture, 73, 113
Generalised (co)homology theory, 454,
complex oriented, 483
multiplicative, 457
Generalised Lower Bound Conjecture
(GLBC), 22, 138
Generating series
of face polynomials, 51
of polytopes, 49
Genus, 164, 429
equivariant, 466
fibre multiplicative, 367
oriented, 486
rigid, 466
universal, 482
Geometric cobordism, 461
Geometric quotient, 189
Geometric realisation
of cubical complex, 84
of simplicial complex, 60
of simplicial set, 142
Ghost vertex, 59
GKM-graph, 303
GKM-manifold, 303
Gold (ring, simplicial complex), 171, 346
Gorenstein, Gorenstein*
simplicial, 112, 157
simplicial poset, 125
Goresky–MacPherson formula, 101
Graded Lie algebra, 417
Graph
chordal, 345
of polytope, 9
simple, 9, 34
Graph-associahedron, 341
Graph product, 341
Grassmannian, 188
H-polynomial (of polytope), 14
H-polynomial (of polytope), 14
h-vector
of polytope, 14
of simplicial complex, 60
Gysin homomorphism
in equivariant cohomology, 261
in cobordism, 361
Gysin–Thom isomorphism, 369
Half-smash product (left, right), 322
Hamiltonian action, 195
Hamiltonian-minimal submanifold, 231
Hamiltonian vector field, 231
Hard Lefschetz Theorem, 231
Hauptvermutung, 76
Heisenberg group, 126
HEP (homotopy extension property), 116
Hermitian quadric, 197
Hilton–Milnor Theorem, 138
Hirzebruch genus, 364
Hirzebruch surface, 152
Hirzebruch–Thom isomorphism, 369
Hirzebruch surface, 152
H-minimal submanifold, 231
Hodge algebra, 116
Hodge number, 225
Homological dimension (of module), 399
Homology
cellular, 421
of chain complex, 507
simplicial, 485
reduced, 418
singular, 419
of pair, 120
reduced, 419
Homology polytope, 264
Homology sphere, 76
Homotopy, 113
Homotopy category (of model category),
111
Homotopy cofibre, 417
Homotopy colimit, 314
Homotopy equivalence, 113
Homotopy fibre, 416
Homotopy group, 413
Homotopy Lie algebra, 314
Homotopy limit, 314
Homotopy quotient, 429
Homotopy type, 113
Hopf Conjecture, 74
Hopf equation, 50
Hopf line bundle, 330
Hopf manifold, 222
H-polynomial (of polytope), 14
hsop (homogeneous system of parameters), 408
in face rings, 105
Hurwitz series, 171
Hurwitz series, 171
Hurwitz series, 171
Hurwitz series, 171
h-vector
of polytope, 14
INDEX 515

of simplicial complex, 60
of simplicial poset, 117
Hyperplane cut, 6

Intersection homology, 24 187
Intersection of quadrics, 197 202
nondegenerate (transverse), 202
Intersection poset (of arrangement), 161
Isotropy representation, 389 430

Join (least common upper bound), 82 114
Join (operation)
of simplicial complexes, 61
of simplicial posets, 172
of spaces, 322

Klein bottle, 234
Koszul algebra, 403
Koszul complex, 403
Koszul resolution, 400
Krichever genus, 389 489
universal, 389 192
K-theory, 515

Lagrangian immersion, 231
Lagrangian submanifold, 231
Landweber Exact Functor Theorem, 515
Latching functor, 114
Lattice, 180 210
Lefschetz pair, 134
Left lifting property, 416
Leibniz identity, 398
L-genus (signature), 483
Limit (of diagram), 439
Link (in simplicial complex), 62 121
Link (of intersection of quadrics), 207
Linkage, 130
Localisation formula, 368
Locally standard (torus action), 240 257
Logarithm (of formal group law), 148
Loop functor (algebraic), 407
Loop space, 331 115
Lower Bound Theorem (LBT), 22
lsop (linear system of parameters), 408
in face rings, 190 117
integral, 108
LVM-manifold, 213 223

Manifold with corners, 133 211
face-acyclic, 261
nice, 241
Manifold with faces, 241
Mapping cone, 417
Mapping cylinder, 417
Massy product, 105 111
indecomposable, 174
indeterminacy of, 412
trivial (vanishing), 412
Matching functor, 444

Mayer–Vietoris sequence, 120
Maximal action (of torus), 228
Meet (greatest common lower bound), 82

Milnor hypersurface, 484 469
Minimal basis (of graded module), 400
Minimal model
of dg-algebra, 410 443
of space, 425
Minimal submanifold, 231
Minkowski sum, 26
Missing face, 62 92 182 151
Model (of dg-algebra), 410
Model category, 440
Module, 395
finitely generated, 395
free, 396
graded, 396
projective, 396
Moment-angle complex, 131 172
real, 134 139
Moment-angle manifold, 134 197
polytopal, 134 196
non-formal, 169
Moment map, 195 209
proper, 196 210
Monoid, 136
Monomial ideal, 92
Moore loops, 131
Morava K-theory, 473
Multi-fan, 239 480
Multidegree, 515
Multigrading, 100 148 178
M-vector, 23

Nerve complex, 60 101
Nested set, 51
Nestohedron, 41
Nilpotent space, 424
Non-PL sphere, 424 479
Normal complex structure, 433

Octahedron, 4
Omniorientation, 247 261 279 300
Opposite category, 439
Orbifold, 128
Orbit (of group action), 228
Orbit space, 224
Order complex (of poset), 65
Overcategory, 440

Partition, 469
Path space, 415
Pair (of spaces), 418
Perfect elimination order, 545
Permuthahedron, 25
Picard group, 159
PL map, 61
homeomorphism, 61
PL manifold, 76
PL sphere, 69
Poincaré algebra, 112
Poincaré–Atiyah duality, 157 460
Poincaré duality, 424
Poincaré duality space, 164
Poincaré series, 311 399
of face ring, 94
Poincaré sphere, 76
Pointed map, 414
Pointed space, 413
Polar set, 3
Polyhedral complex, 70
Polyhedral product, 155 172 418
Polyhedral smash product, 326
Polyhedron (convex), 1
Delzant, 210
Polyhedron (simplicial complex), 59
Polytope
  combinatorial, 2
  convex, 11
  cyclic, 77
  Delzant, 377 184 190
  dual, 4
  generic, 5
  lattice, 183
  neighbourly, 6 174
  nonrational, 185
  polar, 3
  regular, 4
  self-dual, 4
  simple, 3 46
  simplicial, 3 46
  stacked, 22
  triangle-free, 13
Polytope algebra, 23
Pontryagin algebra, 331
Pontryagin class, 486
Pontryagin number, 164
Pontryagin product, 123
Pontryagin–Thom map, 142 154
Poset (partially ordered set), 2
Poset category, 139
Positive orthant, 3 9 131
Presentation (of a polytope by inequalities), 2
  generic, 5
  irredundant, 2
  rational, 240
Primitive (lattice vector), 56
Principal bundle, 420
Principal minor (of matrix), 291
Product
  of building sets, 47
  of polytopes, 6
Product (categorical), 439
Products (in cobordism), 450
Projective dimension (of module), 399
Projectivisation (of vector module), 450
Pseudomanifold, 152 307
  orientable, 153
Pullback, 439
Pushout, 459
Quadratic algebra, 92 340
Quasi-isomorphism, 398 110 142
Quasitoric manifold, 244 518 560
  almost complex structure on, 254
  canonical complex structure on, 249
  complex structure on, 254
  equivalence of, 246 247
Quillen pair (of functors), 411
Quotient space, 129
Rank (of free module), 396
Rank function, 51
Rational equivalence, 423
Rational homotopy type, 424
Ray’s basis, 363
Redundant inequality, 2
Reedy category, 444 450
Regular sequence, 105
Regular subdivision, 121
Regular value (of moment map), 195 210
Reisner Theorem, 107
Resolution (of module)
  free, 399
  minimal, 400
  projective, 399
Restriction (of a building set), 26
Restriction map
  algebraic, 94 105 110 266
  in equivariant cohomology, 259
Right-angled Artin group, 341
Right-angled Coxeter group, 341
Right lifting property, 416
Ring of polytopes, 44
Root (of representation), 851
  complementary, 851
  of almost complex structure, 851
Samelson product, 332 423 443
  higher, 533
Schlegel diagram, 70
Segre embedding, 469
Seifert fibration, 226
Sign (of fixed point), 251 377 334
Signature, 377 183
Simplex, 2 82
  abstract, 59
  regular, 2
  standard, 2
Simplicial cell complex, 82
Simplicial chain, 417
Simplicial cochain, 418
Weak equivalence
  of dg-algebras, 410
  in model category, 440
Wedge (of spaces), 138, 314
  of spheres, 171, 321, 345
Weierstrass $\wp$-function, 387, 487
Weierstrass $\sigma$-function, 487, 488
Weight (of torus representation), 240, 347
  graph, 302
Weight lattice (of torus), 210, 434
Weyl group, 380
Whitehead product, 422
  higher, 332
Witten genus, 487

Yoneda algebra, 448
Zigzag (of maps), 410
Zonotope, 29
$\gamma$-polynomial, 18
$\gamma$-vector
  of polytope, 18
  of triangulated sphere, 24
$\chi_{a,b}$-genus, 373, 483, 484
$\chi_y$-genus, 373, 483

2-truncated cube, 40
24-cell, 1
5-lemma, 398
This book is about toric topology, a new area of mathematics that emerged at the end of the 1990s on the border of equivariant topology, algebraic and symplectic geometry, combinatorics, and commutative algebra. It has quickly grown into a very active area with many links to other areas of mathematics, and continues to attract experts from different fields.

The key players in toric topology are moment-angle manifolds, a class of manifolds with torus actions defined in combinatorial terms. Construction of moment-angle manifolds relates to combinatorial geometry and algebraic geometry of toric varieties via the notion of a quasitoric manifold. Discovery of remarkable geometric structures on moment-angle manifolds led to important connections with classical and modern areas of symplectic, Lagrangian, and non-Kaehler complex geometry. A related categorical construction of moment-angle complexes and polyhedral products provides for a universal framework for many fundamental constructions of homotopical topology. The study of polyhedral products is now evolving into a separate subject of homotopy theory. A new perspective on torus actions has also contributed to the development of classical areas of algebraic topology, such as complex cobordism.

This book includes many open problems and is addressed to experts interested in new ideas linking all the subjects involved, as well as to graduate students and young researchers ready to enter this beautiful new area.

For additional information and updates on this book, visit
www.ams.org/bookpages/surv-204