Homotopy of Operads and Grothendieck–Teichmüller Groups
Part 2: The Applications of (Rational) Homotopy Theory Methods

Benoit Fresse
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Contents


References to chapters, sections, paragraphs, and statements of the book are given by §x.y.z when these cross references are done within a part (I, II, and III), and by §P.x.y.z where P = I, II, III otherwise. The cross references to the sections, paragraphs, and statements of the appendices are given by §P.x.y all along the book, where §P = §A, §B, §C. The preliminary part of the first volume of this book also includes a “foundations and conventions” section, whose paragraphs, numbered §§0.1-0.16, give a summary of the main conventions used in this work.

Preliminaries ix
Preface xi
Reminders xv
Reading Guide and Overview of the Volume xxiii

Part II. Homotopy Theory and its Applications to Operads 1

Part II(a). General Methods of Homotopy Theory 3

Chapter 1. Model Categories and Homotopy Theory 5
  1.0. Introduction: The problem of defining homotopy categories 6
  1.1. The notion of a model category 8
  1.2. The homotopy category of a model category 14
  1.3. The example of topological spaces and of simplicial sets 25
  1.4. The model category of operads and of algebras over operads 37

Chapter 2. Mapping Spaces and Simplicial Model Categories 45
### CONTENTS

Chapter 2. Applications of the Cotriple Cohomology of Operads

- 2.0. Multi-complexes
- 2.1. Modules of derivations associated to operads
- 2.2. The definition and the applications of the cotriple cohomology
- 2.3. Appendix: Hom-objects on the category of Λ-sequences

Chapter 3. Applications of the Koszul Duality of Operads

- 3.1. The applications of the cobar-bar and Koszul resolutions
- 3.2. The applications of the Koszul derivation complex

Part III(b). The Case of $E_n$-operads

Chapter 4. The Applications of the Koszul Duality for $E_n$-operads

- 4.1. The Koszul dual of the Gerstenhaber operads
- 4.2. The cotriple cohomology of the Gerstenhaber operads

Chapter 5. The Interpretation of the Result of the Spectral Sequence in the Case of $E_2$-operads

- 5.0. Reminders on the Grothendieck–Teichmüller group
- 5.1. The degree zero homotopy of the homotopy automorphism space
- 5.2. The action of the classifying space of the additive group and the concluding result
- 5.3. Appendix: Rationalization and homotopy spectral sequences

Conclusion: A Survey of Further Research on Operadic Mapping Spaces and their Applications

Chapter 6. Graph Complexes and $E_n$-operads

- 6.1. The operads of graphs
- 6.2. Mapping spaces of $E_n$-operads and graph complexes
- 6.3. Possible generalizations of the computations in positive characteristic and in pro-finite homotopy theory

Chapter 7. From $E_n$-operads to Embedding Spaces

Appendices

Appendix C. Cofree Cooperads and the Bar Duality of Operads

- C.0. Reminders on the language of trees
- C.1. The construction of cofree cooperads
- C.2. The bar duality of operads
- C.3. The Koszul duality of operads

Glossary of Notation

Bibliography

Index
Preliminaries
Preface

This volume is a follow-up of the study initiated in the first volume of this monograph, where we gave an introduction to operads, we provided a survey on the definition of the notion of an $E_n$-operad, and we explained the definition of the (pro-unipotent) Grothendieck–Teichmüller group from the viewpoint of the theory of algebraic operads.

Recall briefly that the class of $E_n$-operads consists of the topological operads which are weakly-equivalent to a reference model, namely the operad of little $n$-discs $D_n$ (an equivalent choice is given by the operad of little $n$-cubes). Recall also that the fundamental groupoid of the little 2-discs operad $D_2$ is equivalent to an operad in groupoids, the operad of parenthesized braids $Pa\hat{B}$, which governs operations acting on braided monoidal categories. In our approach, we precisely define the (pro-unipotent) Grothendieck–Teichmüller group $GT(\mathbb{k})$, where $\mathbb{k}$ is any characteristic zero ground field, as a group of automorphisms of a Malcev completion $Pa\hat{B}^\tau$ of this operad in groupoids $PaB$. The Malcev completion construction considered in this definition refers to a rationalization process for (possibly non-abelian) groups which we extend to groupoids and to operads in groupoids. These topics form the matter of the first part of this monograph, “From Operads to Grothendieck–Teichmüller Groups”.

In this second volume, we set up general methods for the study of the (rational) homotopy of operads in topological spaces and we give the proof of the following statement which is the ultimate goal of this book: The pro-unipotent Grothendieck–Teichmüller group is isomorphic to the group of homotopy automorphism classes of the rationalization of the little 2-disc operad. These topics form the matter of the second and third parts of this monograph, entitled “Homotopy Theory and its Applications to Operads” and “The Computation of Homotopy Automorphism Spaces of Operads”, which are both contained in this volume.

Most of this volume is independent from the results of the first volume. The reader interested in applications of homotopy theory methods and who is familiar enough with the general definition of an operad can tackle the study of this volume straight away by skipping the algebraic study of the first volume. (We also give a short reminder of our conventions in the next chapter of this preliminary section.)

The rational homotopy theory is the study of spaces modulo torsion. The idea of working modulo a class of groups in homotopy was introduced by Serre in [142]. The computation of the homotopy groups of spheres modulo the class of finite groups (which capture all the torsion in this case) was also achieved by Serre in [140] by relying on spectral sequence constructions (see also the article [31] by Cartan–Serre for an account of this method). The theory was revisited by Quillen in [128], who proved that the rational homotopy of simply connected spaces is captured by a model in the category of differential graded Lie algebras (Lie dg-algebras for
short) and by a dual model in the category of differential graded cocommutative coalgebras (cocommutative dg-coalgebras).

In Quillen’s formulation, the rational homotopy category of (simply connected) spaces is defined by formally inverting the maps which induce an isomorphism on the rationalized homotopy groups (the tensor products $\pi_*(-) \otimes \mathbb{Q}$ equivalent to the quotients of the groups $\pi_*(-)$ by the Serre class of torsion subgroups) and the rational homotopy type of a (simply connected) space is defined as the isomorphism class of a space in this localized category. We use the name rational weak-equivalence and we adopt the distinguishing mark $\sim_\mathbb{Q}$ for this class of maps which define the isomorphisms of the rational homotopy category of spaces.

To define the morphism sets of the localization of a category properly, Quillen axiomatized the usual construction of the homotopy category of spaces, where we essentially have to take a quotient with respect to the homotopy relation on maps to define the morphism sets of the localization. He coined the name model category for this general notion of category where the localization with respect to a class of weak-equivalences is defined by an analogue of the classical homotopy category of topological spaces. The category of Lie dg-algebras and the category of cocommutative dg-coalgebras inherit a natural model structure (we neglect some mild connectedness conditions) and Quillen precisely proved that the homotopy categories of both model categories are equivalent to the localization of the category of spaces with respect to the class of rational weak-equivalences.

The theory was again revisited by Sullivan in [151], who used a differential cochain graded algebra of piecewise linear differential forms $\Omega^*(X)$ (the Sullivan cochain dg-algebra for short), which is defined for any simplicial set $X$. This cochain dg-algebra is equivalent to the dual unitary commutative dg-algebra of the cocommutative dg-coalgebra defined by Quillen when $X$ is a simply connected space such that the rational homology $H_*(X) = H_*(X, \mathbb{Q})$ forms a finitely generated $\mathbb{Q}$-module degreewise. We elaborate on this model, introduced by Sullivan, to build our rational homotopy theory of operads.

In fact, both Quillen and Sullivan deal with simplicial sets, regarded as combinatorial models of spaces, rather than with actual topological spaces. In this context, we also use the name space to refer to the objects of the category of simplicial sets. In Sullivan’s approach, the rational homotopy type of a simplicial set $X$ is captured by a simplicial set $X^\wedge$ associated to $X$ and equipped with a map $\eta^* : X \to X^\wedge$ which induces the rationalization on homotopy groups. To be explicit, we get that the fundamental group of this simplicial set $X^\wedge$ is identified with the Malcev completion of the fundamental group of our original simplicial set $\pi_1(X^\wedge) = \pi_1(X)^\wedge$, while we have $\pi_n(X^\wedge) = \pi_n(X) \otimes \mathbb{Q}$ for $n \geq 2$. The map $\eta_* : \pi_n(X) \to \pi_n(X^\wedge)$ is identified with the universal morphism associated to this algebraic rationalization construction in each case. We precisely prove that this construction lifts to operads.

To be explicit, to any operad in simplicial sets $R$, we associate another operad $R^\wedge$ whose components $R^\wedge(r)$ are (under mild finiteness assumptions) weakly-equivalent to the Sullivan rationalization $R(r)^\wedge$ of the individual spaces $R(r)$. In good cases (when the classical Sullivan model works properly), this rationalized operad $R^\wedge$ captures the rational homotopy type of the object $R$ in the category of operads in spaces, where, to be precise, we define the rational homotopy type of an operad $R$ as the isomorphism class of our object $R$ in the localization of the category of
operads with respect to the operad morphisms $\phi : P \to Q$ which define a rational weak-equivalence of spaces aritywise $\phi : P(r) \xrightarrow{\sim} Q(r)$.

We also establish that the Sullivan cochain dg-algebra admits an operadic enhancement which associates a cooperad in the category of unitary commutative cochain dg-algebras $\Omega^*(R)$ to any operad in simplicial set $R$. We call cooperad the structure, dual to an operad in the categorical sense, which we naturally get in this context, because we rely on contravariant functors to build our model. We also use the short name Hopf cochain dg-cooperad for the objects of the category of cooperads in unitary commutative cochain dg-algebras. We precisely prove that (in good cases again) the Hopf cochain dg-cooperad $\Omega^*(R)$ captures the rational homotopy type of the operad in simplicial sets $R$ just like the Sullivan cochain dg-algebra $\Omega^*(X)$ captures the rational homotopy type of any (good) space $X$.

The latter result is the main goal of the second part of this book, “Homotopy Theory and its Applications to Operads”. We comprehensively review the homotopical background of our constructions, the theory of model categories, and the rational homotopy theory of spaces, before tackling the applications to operads. We also define a new model structure for the study of the homotopy of unitary operads (the Reedy model category of $\Lambda$-operads) by relying on ideas introduced in the first volume of this work. We use this model structure to adapt the definition of our Hopf dg-cooperad model to the case of unitary operads.

To complete this study of the applications of homotopy theory to operads, we make explicit the definition of rational models of $E_n$-operads. In short, we explain a result of [66] which asserts that the rational homotopy of $E_n$-operads in simplicial sets (and in topological spaces) is determined by a model which we deduce from the cohomology of these operads (we say that $E_n$-operads are rationally formal as operads). This formality result follows from a counterpart, in the category of Hopf cochain dg-cooperads, of the formality of the chain operad of little $n$-discs, established by Tamarkin in the case $n = 2$ (see [152]) and by Kontsevich [97] for $n \geq 2$. In the case $n = 2$, we also check that a model for the rationalization $\hat{D}_2$ of the little 2-discs operad $D_2$ (and a model of a rational $E_2$-operad) is given by the classifying space of the Malcev completion of the operad of parenthesized braids $PaB^\mathbb{P}$ studied in the previous volume. We explicitly have $\hat{D}_2 = B(PaB^\mathbb{P})$. We use this observation and the functoriality of the classifying space construction to define our map from the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ to the group of homotopy automorphism classes of the rationalization of the little 2-discs operad $\hat{D}_2$.

We prove that this map defines an isomorphism in the third and concluding part of this book, “The Computation of Homotopy Automorphism Spaces of Operads”. We briefly explained in the preface of the first volume that the group of homotopy classes of homotopy automorphisms of a (unitary) operad $R$ is identified with the degree zero homotopy of a homotopy automorphism space $\text{Aut}_{\text{Top}}^h(R)$, which in short consists of invertible connected components of a simplicial endomapping space $\text{Map}_{\text{Top}}(R, R)$ associated to our object $R$.

We use homotopy spectral sequence methods to determine the homotopy of this homotopy automorphism space $\text{Aut}_{\text{Top}}^h(R)$ for our rationalization of the little 2-discs operad $R = \hat{D}_2$. We give a short survey of the general definition of such homotopy spectral sequences, which we borrow from Bousfield-Kan [25], before tackling the applications to operads. We use an operadic cotriple cohomology theory and the Koszul duality theory of operads [23] to compute this homotopy spectral sequence.
We give a detailed account of the applications of these methods to our problem. We also provide an account of the Koszul duality of operads in an appendix of this volume, “Cofree Cooperads and the Bar Duality of Operads”.

The writing of this volume was mostly carried out in parallel to the writing of the first volume of this book, and I am grateful for numerous supports, which I received from colleagues and from institutions, for the whole project. I especially thank Christine Vespa, Darij Grinberg, Emily Burgunder, Michaël Mienné, and Hadrien Espic for reading significant parts of the manuscript and for their helpful observations. I am also grateful to Damien Calaque and to Bill Dwyer for pointing out a mistake in an early version of this project.

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Reminders

We briefly recall our conventions before giving an overview of the contents of this volume.

The notation of categories. First recall that we use calligraphic letters, like $\mathcal{C}$, $\mathcal{D}$, ..., $\mathcal{M}$, $\mathcal{N}$, ... as a generic notation for the categories in which we define our objects. The most fundamental examples of categories which we consider in this book include the category of modules over a fixed ground ring $k$, which we denote by $\mathcal{C} = \text{Mod}$, the category of topological spaces $\mathcal{C} = \text{Top}$, and the category of simplicial sets $\mathcal{C} = \text{sSet}$ (where the plain notation $\text{Set}$ refers to the category of sets).

We use the notation $\text{dgMod}$ for the category of differential graded modules over $k$ (which we also call $\text{dg-modules}$ for short), the notation $\text{grMod}$ for the category of graded modules (which we regard as the subcategory of dg-modules equipped with a trivial differential) and the notation $\text{sMod}$ for the category of simplicial modules (which we define as the category of simplicial objects in $\text{Mod}$). We review the precise definition of these categories in the course of this volume. We also recall the definition of a chain (respectively, cochain) graded variant of the category of dg-modules $\text{dg}^*\text{Mod}$ (respectively, $\text{dg}^*\text{Mod}$) in our study of the rational homotopy of spaces. We still consider a category of cosimplicial modules $\text{cMod}$ which we define, dually to the category of simplicial modules, as the category of cosimplicial objects in $\text{Mod}$.

We use expressions with a calligraphic capital (like $\mathcal{P} = \text{Com}$, $\mathcal{O}p$, ...) for the categories of structured objects (commutative algebras, operads, ...) which we may form in any of these base categories. We just add the base category $\mathcal{M} = \text{Top}, \text{sSet}, \ldots, \text{Mod}, \ldots$ as a prefix to the notation $\mathcal{P} = \mathcal{M}\mathcal{P}$ of any such category $\mathcal{P} = \text{Com}$, $\mathcal{O}p$, ... when this precision is necessary. The notation $\text{Com} = \text{dgModCom}$, for instance, refers to the category of commutative algebras $\mathcal{P} = \text{Com}$ in the base category of dg-modules $\mathcal{M} = \text{dgMod}$.

We withdraw the expression $\text{Mod}$ and reduce our notation of the base category to the prefix $p = \text{dg}, \text{gr}, s, \ldots$ when we deal with a category of structured objects $\mathcal{P} = \text{Com}, \mathcal{O}p, \ldots$ in any of our variants $\mathcal{M} = \text{dgMod}, \text{grMod}, \ldots$ of the category of ordinary modules $\text{Mod}$. We accordingly use the notation $\text{dgCom}$ (respectively, $\text{grCom}$, $\text{sCom}$, ...) for the category of commutative algebras in dg-modules (respectively, in graded modules, in simplicial modules, ...), and we adopt similar conventions in the case of operads.

Recall that $\text{Com}$ is actually our notation for the category of commutative algebras without unit. In what follows, we mostly deal with unitary commutative algebras and we adopt the notation $\text{Com}_+$, with the extra postfix subscript $+$, to refer to this category. We also use the notation $\text{Com}^c_+$, with the extra postfix superscript $c$, for the category of counitary cocommutative algebras in any base category.
we take the usual tensor product of modules over the ground ring
fine the symmetric monoidal structure of our base category. In the case
\(\otimes\)X the associativity relations (\(X \otimes 1 \simeq X \otimes 1 \otimes X\),
the unitary relations (\(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)\) and the symmetry relations
\(X \otimes Y \simeq Y \otimes X\) in \(M\).

In the case \(M = Set, Top, sSet, \ldots\), we take the cartesian product \(\otimes = \times\) to define
the symmetric monoidal structure of our base category. In the case \(M = Mod\),
we take the usual tensor product of modules over the ground ring \(\otimes = \otimes_k\). We also use this standard tensor product to define a tensor product operation on the
category of dg-modules \(M = dgMod\) (and of graded modules \(M = grMod\)) but we modify the ordinary symmetry isomorphism in the dg-module setting in order
to implement the permutation rules of differential graded algebra in the symmetric
monoidal structure of our category (see §I.3.2). We explain the definition of this
symmetric monoidal category of dg-modules with full details in §II.5.2. We explain the definition of a symmetric monoidal structure on the category of simplicial
modules and on the category of cosimplicial modules in §II.5.2 too. We use these
symmetric monoidal structures in order to formalize the definition of unitary commutative algebras in dg-modules, in simplicial modules, and in cosimplicial modules
(see §II.6.1).

Recall that a tensor product of unitary commutative algebras inherits a natural
unitary commutative algebra structure so that the category of unitary commutative
algebras \(Com_+ = M Com_+\) in a base symmetric monoidal category \(M\) forms a symmetric
monoidal category itself. Recall also that we have a similar assertion for the
categories of counitary cocommutative coalgebras \(Com^\circ_+ = M Com^\circ_+\) (see §I.3.0.4).
We use these symmetric monoidal structures to formalize the definition of our notion of a Hopf operad and of a Hopf cooperad. To be explicit, we define a Hopf
operad as an operad in the symmetric monoidal category of counitary cocommutative
calgebras and we define a Hopf cooperad as a cooperad in the symmetric
monoidal category of unitary commutative algebras. We go back to this subject
in §II.9.

We often assume that the tensor product of our base category \(M\) distributes over
colimits in the sense that we have the relation (\(\colim_\alpha X_\alpha) \otimes Y \cong \colim_\alpha (X_\alpha \otimes Y)\)
for any diagram of objects \(X_\alpha, \alpha \in J\), in the category \(M\) and for any fixed object
\(Y \in M\). We symmetrically have \(X \otimes (\colim_\beta Y_\beta) \cong \colim_\beta (X \otimes Y_\beta)\) when we fix
\(X \in M\), and we take the colimit of a diagram of objects \(Y_\beta, \beta \in J\), on the right-hand side (see §I.1.9). The cartesian product satisfies these relations in the category
of sets \(M = Set\), topological spaces \(M = Top\), and simplicial sets \(M = sSet\), and so do the usual tensor product of the category of modules over our ground ring
\(M = Mod\), the tensor product of the category of dg-modules \(M = dgMod, \ldots\).

The tensor product of counitary cocommutative coalgebras distributes over
colimits as well because the colimits of counitary cocommutative coalgebras are
created in the base category (see §II.3.0.4). We see on the other hand that the tensor
product of unitary commutative algebras does not distribute over coproducts (and
hence, over colimits in general) since the tensor product also realizes coproducts in this category (see §I.3.1). In this case, we will rather assume that the tensor product distributes over finite limits. We go back to this idea in §II.5.1 when we explain the definition of our category of Hopf cooperads.

To define generalizations of the Postnikov decomposition for operads, we also consider the direct sum operation \( \oplus : M \times M \to M \) as the tensor product operation of an additive symmetric monoidal structure on our module categories \( M = \text{dg} \text{Mod}, \text{s Mod}, \ldots \) (see §III.1.2). This tensor product \( \otimes = \oplus \) obviously does not distribute over colimits either.

**Morphisms, homomorphisms, hom-objects and duals.** We adopt the notation \( \text{Mor}_\mathcal{C}(X, Y) \) for the set of morphisms associated to any pair of objects \( X, Y \in \mathcal{C} \) in a category \( \mathcal{C} \).

Many categories which we consider in this book are also endowed with the structure of an enriched category over a base category. To be explicit, we often assume that our category \( \mathcal{C} \) is equipped with a hom-object bifunctor, which we usually denote by \( \text{Hom}_\mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{M} \), together with a unit operation \( \eta : 1 \to \text{Hom}_\mathcal{C}(X, X) \), defined for any \( X \in \mathcal{C} \), and a composition operation \( \circ : \text{Hom}_\mathcal{C}(Z, Y) \otimes \text{Hom}_\mathcal{C}(X, Z) \to \text{Hom}_\mathcal{C}(X, Y) \) which mimics the unit and the composition operations of the ordinary morphisms. We use the name *homomorphism* to distinguish the elements of these hom-objects \( f \in \text{Hom}_\mathcal{C}(X, Y) \) from the actual morphisms of our category, which are the elements of the morphism sets \( \text{Mor}_\mathcal{C}(X, Y) \) (see §I.13 for more details explanations on this convention).

In our study, we also consider simplicial mapping spaces \( \text{Map}_\mathcal{C}(X, Y) \) whose homotopy determines the morphism sets of the homotopy category of model categories (see §§II.2.3). In the case of a simplicial model category (see §II.2), we assume that these mapping spaces define the hom-objects of an enriched category structure over values in the category of simplicial sets, but in general, we just define such mapping spaces by homotopy theory constructions (see §II.3) and we do not have strict composition operations (and hence, a strict enriched category structure) on these objects.

In the case \( M = \text{Top}, \text{s Set}, \ldots, \text{Mod}, \ldots \), we generally use that our category is equipped with a closed symmetric monoidal structure to get that \( M \) is enriched over itself. We explicitly have an internal hom-object bifunctor \( \text{Hom}_M(-, -) : M^{\text{op}} \times M \to M \) which we characterize by the adjunction relation \( \text{Mor}_M(X \otimes Y, Z) = \text{Mor}_M(X, \text{Hom}_M(Y, Z)) \) in our category (see §I.14). (We just make the choice of a convenient category of topological spaces in order to ensure that such an adjunction relation holds in the case \( M = \text{Top} \).) In the case \( M = \text{Mod} \), we actually have \( \text{Hom}_{\text{Mod}}(-, -) = \text{Mor}_{\text{Mod}}(-, -) \) since the morphism set \( \text{Mor}_{\text{Mod}}(X, Y) \) associated to any pair of modules \( X, Y \in \text{Mod} \) inherits a natural module structure and satisfies this adjunction relation \( \text{Mor}_{\text{Mod}}(X \otimes Y, Z) = \text{Mor}_{\text{Mod}}(X, \text{Hom}_{\text{Mod}}(Y, Z)) \).

We can also use the mapping \( D : M \mapsto \text{Hom}_M(M, 1) \), where we consider a hom-object with values in the unit object \( 1 \in M \) of our symmetric monoidal structure to define a natural duality functor \( D : M^{\text{op}} \to M \) on our base category \( M \). We just get the standard duality functor \( D(M) = \text{Hom}_{\text{Mod}}(M, k) \) when we work in the category of modules \( M = \text{Mod} \) over our ground ring \( k \). We examine the definition of duality functors on the category of dg-modules \( M = \text{dg} \text{Mod}, \) of simplicial modules \( M = \text{s Mod} \) and of cosimplicial modules \( M = c \text{Mod} \) with full details in §II.5. We will see that we can use this hom-object construction \( D(M) = \text{Hom}_{\text{dg} \text{Mod}}(M, k) \), where we
regard the ground ring $k$ as a unit object for the tensor product of dg-modules, to get an internal duality functor on the category of dg-modules $D : dgMod^{op} \to dgMod$, while we rely on the duality of plain modules $D : Mod^{op} \to Mod$ to define duality functors $D : sMod^{op} \to cMod$ and $D : cMod^{op} \to sMod$ which exchange the category of simplicial modules $M = sMod$ and the category of cosimplicial modules $M = cMod$.

In what follows, we generally use the notation $D : M \mapsto D(M)$ to refer to these duality functors on our base categories. We also use the notation $M^{\vee} = D(M)$ for the image of individual objects $M \in M$ under such duality functors and when we deal with objects equipped with extra structures (like product or coproduct operations).

**The notion of an operad.** We mostly use the definition of operads in terms of partial composition operations in this volume. Recall that we tacitly assume that our operads are symmetric in this work. We therefore define an operad $P$ in a category $M$ as a collection of objects $P = \{P(r) \in M, r \in \mathbb{N}\}$, where we have an action of the symmetric group $\Sigma_r$ on $P(r)$, for any $r \in \mathbb{N}$, together with a unit morphism $\eta : 1 \to P(1)$ and (partial) composition operations $\circ_i : P(m) \otimes P(n) \to P(m + n - 1)$, defined for each pair $m, n \in \mathbb{N}$, for any $i \in \{1, \ldots, m\}$, and which satisfy natural equivariance, unit, and associativity relations in our ambient category $M$. (We go back to the equivariance relations in the next paragraph.) We assume use that our category $M$ is equipped with a symmetric monoidal structure in this definition. We use the structure of this symmetric monoidal category when we define the unit morphism and the composition operations of our operad. In the case where $M$ is a concrete symmetric monoidal category (see §0.15), the unit morphism $\eta : 1 \to P(1)$ is defined by the choice of an element, which we usually denote by $1 \in P(1)$, in the component of arity one of our operad $P$. Recall that we also call symmetric sequence, the structure $M$, underlying an operad, which consists of a collection $M = \{M(r) \in M, r \in \mathbb{N}\}$, where the object $M(r)$ is endowed with an action of the symmetric group $\Sigma_r$, for any $r \in \mathbb{N}$. By convention, we generally assume that the symmetric group acts on the left when we use operads and symmetric sequences. We just need to convert this left action into a right action when we deal with $\Lambda$-operads (we recall the definition of this notion in the next paragraphs).

We adopt the notation $\mathcal{O}p$ for the category of operads which we may form in any such symmetric monoidal category $M$. We just add this ambient symmetric monoidal category as a prefix to this notation $\mathcal{O}p = M \mathcal{O}p$ when this information is necessary. We use the notation $\mathcal{S}eq$ for the category of symmetric sequences, and we similarly write $\mathcal{S}eq = M \mathcal{S}eq$ when we need to specify the ambient category of our objects.

We assume that the collection of objects underlying an operad $P(r)$ is indexed by all natural numbers $r \in \mathbb{N}$ in the above definition. Recall that we also refer to the index $r \in \mathbb{N}$ as the arity of the component $P(r)$ of our operad. In what follows however we often assume that our operads are not defined in arity zero. We say that $P$ is a *non-unitary operad* in this case and we also write $P(0) = \emptyset$ to specify that the object $P(0)$ is undefined. We use the notation $\mathcal{O}p_\emptyset$ for the category of non-unitary operads.

If the tensor product of our base category distributes over colimits, then we can identify the category of non-unitary operads with the full subcategory of the
category of all operads generated by the objects $P \in \mathcal{O}p$ which have the initial object of our base category as component of arity zero. In the generic case, we use the notation $\emptyset \in \mathcal{M}$ for this initial object. Hence, the expression $P(0) = \emptyset$ also reflects this category embedding $\mathcal{O}p_{\emptyset} \hookrightarrow \mathcal{O}p$ in the case where the tensor product of our ambient symmetric monoidal distributes over colimits.

We also say that a (non-unitary) operad $P$ is connected when we have the relation $P(1) = 1$ (in addition to $P(0) = \emptyset$) and the unit morphism of our operad $\eta : 1 \to P(1)$ reduces to the identity morphism of the unit object of our ambient symmetric monoidal category $1 \in \mathcal{M}$. We use the notation $\mathcal{O}p_{11}$ for the full subcategory of the category of non-unitary operads generated by the connected (non-unitary) operads.

In the context of non-unitary operads (respectively, of connected operads), we also consider the category $\mathcal{S}eq_{>0}$ (respectively, $\mathcal{S}eq_{>1}$) formed by the symmetric sequences which are only defined in arity $r > 0$ (respectively, in arity $r > 1$). We call non-unitary symmetric sequences (respectively, connected symmetric sequences) the objects of this category. We can still identify this category of non-unitary symmetric sequences (respectively, of connected symmetric sequences) with the full subcategory of the category of all symmetric sequences generated by the objects such that $M(0) = \emptyset$ (respectively, $M(0) = M(1) = \emptyset$) when necessary.

The first example of an operad which we consider in this book is the non-unitary operad of commutative algebras (the commutative operad). This operad, which we denote by $\text{Com}$, can be defined in any symmetric monoidal $\mathcal{M}$. We explicitly have $\text{Com}(r) = 1$, for every $r > 0$, where we consider the unit object of our category $1 \in \mathcal{M}$ (see §I.2.1.11).

Unitary operads. Besides the category of non-unitary operads, of which we recall the definition in the previous paragraph, we also consider a category of operads $P$ which satisfy $P(0) = 1$ in arity zero. We say that $P$ is a unitary operad when this condition holds (the terminology ‘unital operad’ is used for such objects in [118], but the name ‘unital’ is also used to refer to the unit of our operads in other references, and we therefore prefer to introduce another name to avoid confusion).

We use the notation $\mathcal{O}p_{*}$ for the subcategory which has the unitary operads as objects and the operad morphisms which reduce to the identity of the unit object in arity one as morphisms. We actually use the notation $*$ in any concrete symmetric monoidal in order to distinguish the arity zero element $* \in P(0)$, which we can associate to the unit object $P(0) = 1$ in the arity zero component of a unitary operad, from the operadic unit $1 \in P(1)$. In what follows, we also consider a category of connected unitary operads $\mathcal{O}p_{*1} \subset \mathcal{O}p_{*}$ formed by the objects $P \in \mathcal{O}p_{*}$ which satisfy the connectedness condition $P(1) = 1$ in addition to the relation $P(0) = 1$.

The notion of an (augmented) non-unitary $\Lambda$-operad. In subsequent constructions, we crucially use that the category of unitary operads $\mathcal{O}p_{*}$ is equivalent to a category formed by non-unitary operads $P \in \mathcal{O}p_{\emptyset}$ equipped with extra operations which model the operadic composition with an extra unit object $P_{\emptyset}(0) = 1$ in a unitary extension $P_{\emptyset} \in \mathcal{O}p_{*}$ of our operad $P$. (To be precise, we only use this correspondence in our study of unitary operads. We therefore suggest the reader who is not interesting by this setting to skip the reminders of this paragraph in a first reading.)
We consider the category, denoted by $\Lambda$, which has the finite ordinals $\mathfrak{r} = \{1 < \cdots < r\}$ as objects and all injective maps between such ordinals as morphisms. Recall that any map $f \in \text{Mor}_\Lambda(m, n)$ in this category $\Lambda$ has a unique decomposition $f = us$, where $u : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$ is a non-decreasing injection and $s : \{1 < \cdots < m\} \to \{1 < \cdots < m\}$ is a bijection of the set $m = \{1 < \cdots < m\}$ which is also equivalent to a permutation on $m$ letters $s = (s(1), \ldots, s(m)) \in \Sigma_m$.

To express this property, we also say that the category $\Lambda$ has a decomposition $\Lambda = \Lambda^+ \Sigma$, where $\Lambda^+$ denotes the subcategory of $\Lambda$ with the same objects as $\Lambda$ but the non-decreasing injections as morphisms, and $\Sigma$ refers to the disjoint union of the symmetric groups $\Sigma_n$ (regarded as categories with a single object) in the category of categories. For our purpose, we also consider the full subcategory $\Lambda_{>0}$ (respectively, $\Lambda_{>1}$) of the category $\Lambda$ generated by the ordinals $\mathfrak{r} = \{1 < \cdots < r\}$ of cardinal $r > 0$ (respectively, $r > 1$). We still have the relation $\Lambda_{>0} = \Lambda^+_{>0} \Sigma_{>0}$ in this case, where we set $\Lambda^+_{>0} = \Lambda^+ \cap \Lambda_{>0}$ and $\Sigma_{>0} = \Lambda^+ \cap \Lambda_{>0}$ (and we similarly have $\Lambda_{>1} = \Lambda^+_{>1} \Sigma_{>1}$ in the case of the category $\Lambda_{>1} \subset \Lambda$).

We precisely call augmented non-unitary $\Lambda$-operad the structure defined by a collection $P = \{P(r), r > 0\}$ equipped with the structure of a contravariant diagram over the category $\Lambda_{>0}$ together with an augmentation $\epsilon : P \to \text{Cst}$ over the constant diagram $\text{Cst}(r) \equiv 1$, a unit morphism $\eta : 1 \to P(1)$ and composition products $\circ_i : P(m) \otimes P(n) \to P(m + n - 1)$ such that a natural extension of the equivariance, unit and associativity relations of operads hold. We usually use the notation $u^* : P(n) \to P(m)$ for the action of the morphisms $u \in \text{Mor}_\Lambda(m, n)$ of the category $\Lambda_{>0} \subset \Lambda$ on our object $P$ and we refer to these morphisms $u^* : P(n) \to P(m)$ as the restriction operators (or the restriction operations) associated to our operad.

We observed in §I.2.2 that the non-unitary operad of commutative algebras $\text{Com}$ inherits the structure of an augmented non-unitary $\Lambda$-operad and the constant diagram $\text{Cst}(r) \equiv 1$ in our definition actually represents the contravariant $\Lambda$-diagram underlying this operad $\text{Com}$. We therefore use the notation of this operad $\text{Com}$ rather than the notation of the constant diagram $\text{Cst}$ in our subsequent applications of the definition of an augmented non-unitary $\Lambda$-operad.

We also observed in §I.2.2 that the augmentation morphism of an augmented non-unitary $\Lambda$-operad actually forms a morphism of augmented non-unitary $\Lambda$-operads $\epsilon : P \to \text{Com}$. The commutative operad $\text{Com}$ therefore represents the terminal object of the category of augmented $\Lambda$-operads.

In general, we use the notation $\Lambda \text{Op}_\sigma / \text{Com}$ for the category of augmented non-unitary $\Lambda$-operads in a base symmetric monoidal category $\mathcal{M}$. But, on the other hand, we can forget about the augmentation when we work in a symmetric monoidal category (such as $\mathcal{M} = \text{Top}, s\text{Set}, \ldots$) where the unit object $\mathbb{1}$ is identified with the terminal object $* \in \mathcal{M}$. In this case, we shorten our terminology to ‘non-unitary $\Lambda$-operad’ for the objects of our category of augmented non-unitary $\Lambda$-operad and we use the abridged notation $\Lambda \text{Op}_\sigma = \Lambda \text{Op}_\sigma / \text{Com}$ for this category of operads. We also use this convention in the case of (augmented) non-unitary Hopf $\Lambda$-operads which we define as (augmented) non-unitary $\Lambda$-operads in counitary cocommutative coalgebras.

The equivariance relations of the composition products of an augmented $\Lambda$-operad actually divide in two classes, which involve (partially defined) operadic composition operations on the morphisms of our category $\Lambda_{>0} \subset \Lambda$ in the first
case, and operadic composition operations with the empty maps \( o \in \text{Mor}_\Lambda(\emptyset, n) \), with the empty ordinal \( \emptyset \) as domain, in the second case. In the context of a concrete symmetric monoidal category, the first class of equivariance relations read \((u \circ_{u(i)} v)^*(p \circ_{u(i)} q) = u^*(p) \circ_i v^*(q)\), for any pair of operad elements \( p \in P(m), q \in P(n) \), for any pair of injective maps \( u \in \text{Mor}_\Lambda(k, m) \), \( v \in \text{Mor}_\Lambda(l, n) \), for any composition index \( i \in \{1 < \cdots < k\} \), and where \( u \circ_{u(i)} v \in \text{Mor}_\Lambda(k - 1, m + n - 1) \) denotes the alluded to operadic composite of our maps \( u \) and \( v \) in the category \( \Lambda \). The second class of equivariance relations read \((u \circ_{u(i)} o)^*(p \circ_{u(i)} q) = \epsilon(q) \cdot u^* \partial_i(p)\), for any pair of operad elements \( p \in P(m), q \in P(n) \), any injective maps \( u \in \text{Mor}_\Lambda(k, m) \), and any composition index \( i \in \{1 < \cdots < k\} \), where we consider the empty map \( o \in \text{Mor}_\Lambda(\emptyset, n) \), the restriction operator \( \partial_i : P(m) \to P(m - 1) \) associated to the increasing map \( \partial^i \in \text{Map}_\Lambda(m - 1, m) \) which skips the value \( i \) in \( \{1, \ldots, m\} \) in the ordinal \( m = \{1 < \cdots < m\} \), and the operadic composite \( u \circ_{u(i)} o \in \text{Mor}_\Lambda(k - 1, m + n - 1) \) of the map \( u \) with the empty map \( o \). We refer to §2.2.12 for the explicit definition of these operadic composition operations of the maps of the category \( \Lambda \). (We also review the definition of these operations with full details in §II.11.1.2 when we explain the definition of the structure of a coaugmented \( \Lambda \)-cooperad dual to our augmented \( \Lambda \)-cooperads.) We moreover have the relations \( \epsilon(p \circ_{o(i)} q) = \epsilon(p) \epsilon(q) \) when we apply the augmentation \( \epsilon : P(m + n - 1) \to 1 \) to a composite operation \( p \circ_{o(i)} q \in P(m + n - 1) \).

The correspondence between unitary operads and augmented non-unitary \( \Lambda \)-operads. Recall that we call unitary extension of a non-unitary operad \( P \) any unitary operad \( P_+ \) such that \( P_+(0) = 1 \) and \( P_+(r) = P(r) \) for \( r > 0 \). The restriction operators \( u^* : P(n) \to P(m) \) in the definition of an augmented \( \Lambda \)-operad actually model substitution operations \((u^*p)(x_1, \ldots, x_m) = p(y_1, \ldots, y_n)\) such that

\[
y_j = \begin{cases} x_{u(i)}, & \text{if } j \in \{u(1), \ldots, u(m)\}, \\ *, & \text{otherwise,} \end{cases}
\]

where * refers to the distinguished element of the unitary operad \( P_+ \) which we associate to the extra unit object \( P_+(0) = 1 \) in arity zero. The augmentations \( \epsilon : P(r) \to 1 \) represent the full substitution operation \( \epsilon(p) = p(*, \ldots, *) \). The equivariance relations recalled in the previous paragraph reflect the distribution of arity zero elements * which we get in the composition of operations of our operad. From this correspondence, we tautologically get that the mapping \( P \mapsto P_+ \) defines an isomorphism of categories \( \Lambda \text{Op}_\varnothing / \text{Com} \simeq \text{Op}_{\ast} \) from the category of augmented non-unitary \( \Lambda \)-operads \( \Lambda \text{Op}_{\varnothing} / \text{Com} \) to the category of unitary operads \( \text{Op}_{\ast} \).

We have an analogous isomorphism of categories \( \Lambda \text{Op}_{\varnothing 1} / \text{Com} \simeq \text{Op}_{\ast 1} \) when we consider the category of connected unitary operads \( \text{Op}_{\ast 1} \) and a category of augmented connected \( \Lambda \)-operads \( \Lambda \text{Op}_{\varnothing 1} / \text{Com} \) for which we assume the relation \( P(1) = 1 \) in addition to \( P(0) = \varnothing \). In what follows, we use the notation \( \Lambda \text{Seq}_{>0} \) for the category of contravariant \( \Lambda_{>0} \)-diagrams underlying the structure of an augmented non-unitary \( \Lambda \)-operad. We also call non-unitary \( \Lambda \)-sequence the objects of this category of diagrams. In the case where we deal with augmented connected \( \Lambda \)-operads, we also consider a category of contravariant \( \Lambda_{>1} \)-diagrams, which we denote \( \Lambda \text{Seq}_{>1} \). We call connected \( \Lambda \)-sequence the objects of this category of diagrams. We moreover consider the category \( \Lambda \text{Seq}_{>0} / \text{Com} \) formed by the non-unitary \( \Lambda \)-sequences equipped with an augmentation over the constant diagram \( \text{Com}(r) = 1 \) underlying the commutative operad \( \text{Com} \), and the category
\( \Lambda \text{Seq}_{>1}/\text{Com} \) formed by the connected \( \Lambda \)-sequences equipped with an augmentation over the augmentation ideal of the commutative operad \( \text{Com} \). Recall that this object \( \text{Com} \) is the connected symmetric sequence which we obtain by forgetting about the term of arity one \( \text{Com}(1) = 1 \) of the commutative operad \( \text{Com} \).

Symmetric collections and operads shaped on the category of finite sets. In what follows, we most usually assume that the components of an operad \( P(r) \) are indexed by natural numbers \( r \in \mathbb{N} \) (or by positive natural numbers \( r > 0 \) in the context of non-unitary operads). Intuitively, we assume that an operad \( P \) collects operations \( p = p(x_1, \ldots, x_r) \) on variables indexed by the finite ordinals \( \tau = \{1 < \cdots < r\} \) when we use this convention.

For certain constructions however it is more convenient to allow operads whose components are indexed by arbitrary finite sets \( \tau \). We then consider the category \( \text{Bij} \) formed by the finite sets as objects and the bijective maps of finite sets as morphisms. The category of symmetric collections is equivalent to the category of (covariant) functors over this category \( \text{Bij} \), and an operad in the ordinary sense is equivalent to an object of this category of (covariant) functors equipped with a unit morphism \( \eta : 1 \rightarrow P(1) \), where \( 1 \) denotes the one-point set in the category \( \text{Bij} \), and with composition morphisms \( \circ_i : P(m) \otimes P(n) \rightarrow P(m \circ_i n) \), defined for all finite sets \( m = \{i_1, \ldots, i_m\} \), \( n \in \{j_1, \ldots, j_n\} \), for each composition index \( i_k \in m \), and with values in the component of our operad associated to the finite set such that \( m \circ_i n = \{i_1, \ldots, \hat{i}_k, \ldots, i_m\} \amalg \{j_1, \ldots, j_n\} \). We still obviously assume that these composition morphisms shaped on the category of finite sets fulfill natural equivariance, unit, and associativity relations. We refer to §I.2.5 for a detailed survey of this definition of an operad. We also consider the full subcategory \( \text{Bij}_{>0} \subset \text{Bij} \) (respectively, \( \text{Bij}_{>1} \subset \text{Bij} \)) generated by the finite sets \( \tau \in \text{Ob} \text{Bij} \) of cardinal \( r > 0 \) (respectively, \( r > 1 \)).

We have an analogous extension of our definitions in the context of augmented non-unitary \( \Lambda \)-operads. We then consider the category \( \text{Inj} \) formed by the finite sets as objects and the injective maps between finite sets as morphisms. We also consider the full subcategory of this category \( \text{Inj}_{>0} \subset \text{Inj} \) (respectively, \( \text{Inj}_{>1} \subset \text{Inj} \)) generated by the finite sets \( \tau \in \text{Ob} \text{Inj} \) of cardinal \( r > 0 \) (respectively, \( r > 1 \)). We get that an augmented non-unitary \( \Lambda \)-operad is equivalent to a contravariant diagram over the category \( \text{Inj}_{>0} \) equipped with a composition structure shaped on this category. We also go back to this correspondence in the dual context of cooperads in §II.11.1.6.

We mainly use the indexing by arbitrary finite sets in our study of free operads (see §§A-B) and of cofree cooperads (see §C).
Reading Guide and Overview of the Volume

Recall that this monograph comprises three main parts: Part I, “From Operads to Grothendieck–Teichmüller Groups” (in the first volume), which is mainly devoted to the algebraic foundations of our subject; Part II, “Homotopy Theory and its Applications to Operads”, where we develop our rational homotopy theory of operads after a comprehensive review of the applications of methods of homotopy theory; and Part III, “The Computation of Homotopy Automorphism Spaces of Operads”, where we work out our problem of giving a homotopy interpretation of the Grothendieck–Teichmüller group (the ultimate goal of this work).

These parts are widely independent from each others (as we explained in volume one). Recall also that each part of this book is divided into subparts which, by themselves, form self-contained groupings of chapters, devoted to specific topics, and organized according to an internal progression of the level of the chapters each. There is a progression in the level of the parts of the book too, but the chapters are written so that a reader with a minimal background could tackle any of these subparts straight away in order to get a reference and a self-contained overview of the literature on each of the subjects addressed in this monograph.

This volume comprises the second named parts of the book, “Part II: Homotopy Theory and its Applications to Operads”, “Part III: The Computation of Homotopy Automorphism Spaces of Operads”, and “Appendix C: Cofree Cooperads and the Bar Duality of Operads”.

The following overview is not intended for a linear reading but should serve as a guide each time the reader tackles new parts of this volume.

**Part II. Homotopy Theory and its Applications to Operads.** The second part of this book includes: an introduction to the concepts of the theory of model categories and its applications in homotopy; a detailed review of the rational homotopy theory, from the algebraic background of the subject to the definition of models for the rational homotopy of spaces; a new definition of a model for the rational homotopy of operads; and a study of the applications of this model to \( E_n \)-operads.

**Part II(a). General Methods of Homotopy Theory.** We give an introduction to the general applications of the theory of model categories in this part. Most of the ideas explained in this part are not original. Nevertheless we will provide a detailed account of some particular results, which are certainly well known for the experts of the domain, but for which we can hardly get a reference. The first chapter of the part (§I) is an introductory survey of the axioms of model categories and of the construction of homotopy categories in the model category framework. The second and third chapters (§§2,3) are devoted to the applications of simplicial structures in model categories (the definition of general mapping spaces, together
with the definition of generalizations of the classical geometric realization of simplicial sets and of the totalization of cosimplicial spaces). By the way, we also explain the general definition of a homotopy automorphism space in this second chapter. The fourth chapter of the part (§4) is a survey (mostly without proofs) of the definition of the notion of a cofibrantly generated model category, an abstract setting where we have an analogue of the classical cell approximations of topology. We heavily use cofibrantly generated model structures to define the model categories which we consider in our study of the rational homotopy theory.

Chapter 1 Model Categories and Homotopy Theory. In a preliminary section of this chapter (§1.0), we explain the general problem of defining the localization of a category with respect to a class of weak-equivalences. We make explicit the axioms of model categories afterwards (in §1.1). In brief, the main idea of the theory of model categories is to use two auxiliary classes of morphisms, called cofibrations and fibrations, which are endowed with lifting properties similar to the properties of the classical cofibrations and fibrations of topology, in order to handle the definition of the morphism sets of the localization of our category. We will more precisely see that the localization of a model category is given by a homotopy category, which we construct by using the extra structure of the cofibrations and fibrations of our model category and by generalizing the definition of the classical homotopy category of spaces (§1.2).

We then review the classical definitions of fundamental model structures on topological spaces and simplicial sets (§1.3). We conclude this chapter by a brief account of the definition of model category structures on operads and on the categories of algebras associated to operads (§1.4).

The purpose of this chapter is only to give an introductory survey of the subject of model categories and to recall the most fundamental definitions of the theory which we use all through this volume. We therefore omit (or abridged) most proofs and we refer to the literature of the domain for details in general.

Chapter 2 Mapping Spaces and Simplicial Model Categories. The main purpose of this chapter is to explain the definition of the concept of a simplicial model category, where we have simplicial mapping spaces which give a generalization of the classical mapping spaces of the category of topological spaces.

We devote a short preliminary section of the chapter to the determination of functors on simplicial sets from cosimplicial objects (§2.0). We use this correspondence to relate the mapping spaces (the hom-objects) of the structure of an enriched category over simplicial sets to tensor product operations (and function objects) over the category of simplicial sets. We then explain the axioms of a simplicial model category, which ensure that the mapping spaces of such an enriched category structure satisfy appropriate homotopy properties. We check that, in the context of simplicial model categories, the homotopy of the simplicial mapping spaces determine the morphisms sets of the homotopy category associated to our model category (§2.1). This section is only a survey of classical ideas which we recall for the sake of reference.

We explain the definition of homotopy automorphism spaces in simplicial model categories afterwards (§2.2). We notably check that homotopy automorphism spaces define homotopy invariant simplicial monoids associated to the objects of our model category. This statement is known to experts, but we can hardly find a detailed proof of this observation in the literature.
We conclude this chapter by an overview of the definition of simplicial model structures for operads and for the categories of algebras associated to operads (§2.3).

Chapter 3: Simplicial Structures and Mapping Spaces in General Model Categories. This chapter is a continuation of the study initiated in the previous chapter. Our first purpose is to explain the construction of simplicial mapping spaces in general model categories. The general simplicial mapping spaces do not inherit strictly defined composition operations, in contrast to the hom-objects of a category enriched over simplicial sets, but we will explain that these objects can still be used to compute the morphism sets of the homotopy category in general model categories. To achieve this goal, we first review the definition of a certain model structure, the Reedy model structure, on the category of simplicial (respectively, cosimplicial) objects in a model category (§3.1). We tackle the construction of the mapping spaces afterwards (in §3.2).

To complete the account of this chapter, we explain the definition of a generalization of the classical geometric realization of simplicial complexes in the setting of model categories and we explain the definition of a generalization of the totalization of cosimplicial spaces (§3.3). This subject is classical and is addressed in reference books on model categories. We just put more emphasis on the possibility to make choices when we determine the geometric realization of a simplicial object in a model category (and when we determine the totalization of a cosimplicial object). To be explicit, we will see that the definition of the geometric realization of a simplicial object depends on the choice of a cosimplicial framing in our model category (and the definition of the totalization dually depends on the choice of a simplicial framing). We precisely check that different choices of cosimplicial frames return homotopy equivalent objects when we pass to the geometric realization (and different choices of simplicial frames return homotopy equivalent totalizations similarly). We rely on a study of homotopy coends (respectively, ends), which we carry out in an appendix section (§3.4), to establish this homotopy invariance result for the geometric realization of simplicial objects in model categories (respectively, for the totalization of cosimplicial objects). By the way, we also survey the definition of general tower decompositions of geometric realizations (and of totalizations).

The content of this chapter is crucial for the subsequent constructions of this monograph. We notably use the tower decompositions of the geometric realization and of the totalization in order to define homotopy spectral sequences which we use to compute the homotopy of homotopy automorphisms spaces of operads in Part III.

Chapter 4: Cofibrantly Generated Model Categories. We review the definition of the notion of a cofibrantly generated model category in this chapter. Briefly say for the moment that a cofibrantly generated model structure enables us to give an effective definition of the class of cofibrations in our model category by using a generalization of the classical notion of a relative cell complex. We explain the definition of this abstract notion of a relative cell complex first (§4.1). We make the axioms of a cofibrantly generated model category explicit afterwards (in §4.2). We also review the applications of the concept of a cofibrantly generated model category to the category of topological spaces and to the category of simplicial sets. We then give an account of the applications of cofibrantly generated model structures to the definition of model structures by adjunction from a base model category (§4.3). We give a brief introduction to the theory of combinatorial model categories, which are
cofibrantly generated model categories equipped with a presentation in the sense of the classical theory of categories, to complete the account of this chapter (§4.4). This chapter does not contain any original result and most proofs are abridged or omitted.

**Part II(b). Modules, Algebras, and the Rational Homotopy of Spaces.** We comprehensively revisit the rational homotopy theory of spaces in this part. We start with a review of the algebraic background of this subject, the homotopy theory of dg-modules, of simplicial modules, and of unitary commutative dg-algebras. We devote the first and second chapters of the part (§§5-6) to this survey. We rely on the concepts of the theory of model categories recalled in the previous chapters of this volume. We tackle the applications to the rational homotopy of spaces afterwards, in the third chapter of the part (§7). We focus on topics which we subsequently use when we address the definition of our models for the rational homotopy of operads.

**Chapter 5. Differential Graded Modules, Simplicial Modules, and Cosimplicial Modules.** We assume by convention that the objects of the standard category of dg-modules are equipped with a lower grading (which may run over \( \mathbb{Z} \)) and with a differential which lowers degree by one. We denote this category of dg-modules by \( \text{dg} \text{Mod} \). We consider, besides, a category of chain graded dg-modules, which are equivalent to dg-modules concentrated in non-negative degrees, and a category of cochain graded dg-modules, which are equivalent to dg-modules concentrated in non-positive degrees. We adopt the notation \( \text{dg}^* \text{Mod} \) for the category of chain graded dg-modules and the notation \( \text{dg}^- \text{Mod} \) for the category of cochain graded dg-modules.

We give a detailed account of the definition of these categories of dg-modules in the preliminary section of this chapter (§5.0). We also give a brief summary of the classical Dold–Kan correspondence, the equivalence of categories between the category of chain graded dg-modules \( \text{dg}^* \text{Mod} \) and the category of simplicial modules \( s \text{Mod} \). We formally define the category of simplicial modules \( s \text{Mod} \) as the category of simplicial objects in the category of ordinary modules \( \text{Mod} \). We also deal with the category of cosimplicial modules in our study. We formally define this category \( c \text{Mod} \) as the category of cosimplicial objects in the category of ordinary modules \( \text{Mod} \). We mostly use cochain graded dg-modules when we build our model for the rational homotopy of spaces. We therefore explain the definition of a model structure on the category of cochain graded dg-modules with full details (§5.1).

We review the definition of symmetric monoidal structures on the category of chain graded (respectively, cochain graded) dg-modules and on simplicial (respectively, cosimplicial) modules afterwards. We also recall the definition of (various forms of) the Eilenberg–Zilber equivalence which we use to relate these symmetric monoidal categories. We address these subjects in §5.2. We still recall the definition of internal hom-objects in our categories of dg-modules and in simplicial modules, and we check the homotopy invariance of these constructions (§5.3).

We devote an appendix section of the chapter (§5.4) to a short review of the definition of the notion of a contracting homotopy in the context of chain graded (respectively, cochain graded) dg-modules and of the notion of an extra-degeneracy (respectively, extra-codegeneracy) in the context of simplicial (respectively, cosimplicial) modules.
Most ideas developed in this chapter are not original (apart from the definition of a hom-object counterpart of the Eilenberg–Zilber equivalence in \(\text{§5.3}\)). Our main purpose is to make explicit the applications of standard constructions of homotopy theory to cochain graded dg-modules after a survey (mostly without proofs) of the homotopy theory of chain graded dg-modules and of simplicial modules.

**Chapter 6** Differential Graded Algebras, Simplicial Algebras, and Cosimplicial Algebras. In this chapter, we elaborate on the study of the previous chapter to define a model category structure for unitary commutative algebras. We first make explicit the definition of a unitary commutative algebra in chain graded (respectively, cochain graded) dg-modules by using the symmetric monoidal structure which we attach to this category. We use a similar approach to define the notion of a unitary commutative algebra in simplicial (respectively, cosimplicial) modules. We devote the first section of the chapter to these topics \(\text{§6.1}\).

We use the notation \(\text{dgCom}^{+}\) for the category of unitary commutative algebra in chain graded dg-modules. For short, we also call unitary commutative chain (respectively, cochain) dg-algebras the objects of this category of unitary commutative algebras.

We mostly deal with the category of unitary commutative cochain dg-algebras in what follows. We prove that this category inherits a model structure (in the characteristic zero setting) in the second section of the chapter \(\text{§6.2}\). We study cell attachments in the category of unitary commutative cochain dg-algebras in-depth in the course of our verifications. We also explain that the homotopy type of a cell attachment of generating cofibrations of unitary commutative cochain dg-algebras can be determined by using a version with coefficients of the bar construction. We devote the third section of the chapter to this subject \(\text{§§6.2-6.3}\).

Most statements explained in this chapter are not original (like the constructions of the previous chapter).

**Chapter 7** Models for the Rational Homotopy of Spaces. We explain the applications of our model categories of unitary commutative algebras to the definition of models for the rational homotopy of spaces in this chapter. We mainly deal with the Sullivan model which is formed in the category of unitary commutative cochain dg-algebras.

We use a Quillen adjunction to formalize the correspondence between the category of simplicial sets (which we consider instead of topological spaces) and the category of unitary commutative cochain dg-algebras. The Sullivan cochain dg-algebras of piecewise linear forms \(\Omega^{*}(X)\), which is a version of the de Rham cochain complex functor with rational coefficients, gives a (contravariant) functor from simplicial sets to unitary commutative cochain dg-algebras. We recall the definition of this functor \(\Omega^{*}: X \mapsto \Omega^{*}(X)\) and we make explicit the corresponding left adjoint functor \(G_{*}: A \mapsto G_{*}(A)\) from the category of unitary commutative cochain dg-algebras to the category of simplicial sets.

In our account, we mainly revisit the proof of the homotopy properties of the Sullivan model, and we give a new interpretation of results of the literature. We notably explain that the homotopy invariance properties of the Sullivan cochain dg-algebra, which we use in the definition of our Quillen adjunction, are related to the definition of a simplicial framing in the category of unitary commutative cochain dg-algebras. We devote the first and the second section of the chapter to these topics \(\text{§§7.1-7.2}\).
The rational homotopy category of spaces can naively be defined as the category which we obtain by formally inverting the maps of spaces that induce an isomorphism on the rationalization of homotopy groups. We devote the third section of the chapter (§7.3) to the study of the correspondence between the homotopy category of the model category of unitary commutative cochain dg-algebras and this rational homotopy category of spaces. We also recall the definition of a rationalization functor on spaces in this section.

These results are well covered by the literature. Therefore we just provide abridged proofs of the statements which we review in this concluding section of the chapter and we refer to the literature for further details.

**Part [II(c)]. The (Rational) Homotopy of Operads.** In this part, we explain the definition of our models for the rational homotopy of operads. This construction represents the main original theoretical contribution of this monograph.

We start with a detailed study of the definition of model structures on the category of operads in simplicial sets (§8). We then explain the definition of the notion of a (Hopf) cooperad and we check that the category of (Hopf) cooperads in cochain graded dg-modules forms a model category (§9). We define an operadic counterpart of the Sullivan model afterwards (§10) by relying on the definition of this model category of (Hopf) cochain dg-cooperads and by using an operadic upgrade of the Sullivan dg-algebra functor considered in our study of the rational homotopy of spaces.

We actually need to restrict ourselves to (connected) non-unitary operads in our correspondence (in order to handle convergence difficulties with cooperad structures). We can however use an analogue of our notion of a Λ-operad in the category of (Hopf) cooperads in order to extend our model to (connected) unitary operads. We use the same plan as in the case of plain non-unitary operads to carry out the definition of this model for the rational homotopy of unitary operads. We explain the definition of our notion of a (Hopf) Λ-cooperad first (§11) and we check that the category of Hopf Λ-cooperads in cochain graded dg-modules gives a suitable model for the rational homotopy of (connected) unitary operads afterwards (§12).

Chapter [§8]. The Model Category of Operads in Simplicial Sets. We give a thorough account of the definition of model structures for the category of operads in simplicial sets. We start with a brief inspection of the definition of an operad in simplicial sets (§8.0). We just check that an operad in simplicial sets is equivalent to a simplicial object in the category of operads in sets.

We actually consider two model structures for operads. The first model structure, the one usually given in the literature, will be used in the context of non-unitary operads (operads governing non-unitary algebra structures). The second one, which we introduce in this monograph and call the Reedy model structure, is more appropriate for unitary operads (operads governing algebras with a unit), and will be used in this context. We use our notion of Λ-operad, equivalent to the category of unitary operads, to formalize the definition of this Reedy model structure. We define the model structure of the category of non-unitary operads first (§8.1-8.2) and the Reedy model structure of the category of Λ-operads afterwards (§8.3-8.4).

In each case, we use a general adjunction process (recalled in §4.3) to deduce the definition of our model structure on operads from the definition of a model structure.
on the category of symmetric sequences (respectively, Λ-sequences) underlying our objects.

To complete the study of this chapter, we explain the applications of a general construction of simplicial resolutions, the cotriple resolution, for the definition of cofibrant replacements in the category of operads in simplicial sets (§8.5).

Chapter 9 The Homotopy Theory of (Hopf) Cooperads. We explain the general definition of the notion of a cooperad in the setting of a symmetric monoidal category in the first section of this chapter (§9.1). We then explain the definition of a model structure on the category of cooperads in cochain graded dg-modules (§9.2).

We study the category of Hopf cooperads afterwards (§9.3). We formally define a Hopf cooperad in a base category as a cooperad in the category of unitary commutative algebras in this given base category. We check that the category of Hopf cooperads in cochain graded dg-modules inherits a model structure.

We then study the totalization of cosimplicial objects in the category of cochain dg-cooperads and in the category of Hopf cochain dg-cooperads. We mainly prove that the totalization of a cosimplicial object in the category of cochain dg-cooperads can be determined by performing a conormalized cochain construction in the category of cochain graded dg-modules. We devote an appendix section to this subject (§9.4).

Chapter 10 Models for the Rational Homotopy of (Non-unitary) Operads. We define our models for the rational homotopy of non-unitary operads in this chapter. We elaborate on the construction of the Sullivan dg-cochain algebra models for the rational homotopy of spaces. We begin our study with a brief survey of the definition of our model structure for operads in simplicial sets (§10.0). We just check that this model structure admits a restriction to the category of connected (non-unitary) operads, because we have to restrict ourselves to this subcategory of operads in our constructions.

The Sullivan cochain dg-algebra functor $\Omega^* : X \mapsto \Omega^*(X)$ does not preserve multiplicative structures and, as a consequence, does not carry operads to cooperads. This functor preserves multiplicative structures up to homotopy only. The main purpose of this chapter is to explain the definition of an operadic upgrade of the Sullivan functor so that we do can associate a cooperad in unitary commutative cochain dg-algebras (a Hopf cochain dg-cooperad) $\Omega^*_\ast(P)$ to any operad in simplicial sets $P$. We prove that, under reasonable assumptions on the operad $P$, the components of this Hopf cochain dg-cooperad $\Omega^*_\ast(P)$ are weakly-equivalent to the Sullivan cochain dg-algebras $\Omega^*(P(r))$ associated to the individual spaces $P(r)$. We use this correspondence to ensure that our construction returns an appropriate result.

We explain these constructions in §10.1. We tackle the applications of our constructions for the definition of our rationalization functor on operads in simplicial sets afterwards, in §10.2.

Chapter 11 The Homotopy Theory of (Hopf) Λ-cooperads. In this chapter, we study a dual notion, in the category of cooperads, of the category of augmented (connected) non-unitary Λ-operads which we introduced to model (connected) unitary operads in the first volume of this book. We use the name ‘coaugmented Λ-cooperad’ for these objects. We also call ‘Hopf Λ-cooperads’ the objects of the
category of coaugmented Λ-cooperads in any category of unitary commutative algebras. We explain the general definition of a coaugmented Λ-cooperad in the setting of a symmetric monoidal category first (§11.1).

We have an obvious forgetful functor from the category of coaugmented Λ-cooperads to the category of plain cooperads. We check that this functor admits a left adjoint by relying on standard Kan extension constructions (§11.2). We use this correspondence to establish that the category of coaugmented Λ-cooperads in cochain graded dg-modules inherits a model structure (§11.3). We then establish that this model structure lifts to the category of Hopf Λ-cooperads (§11.4).

Chapter 12. Models for the Rational Homotopy of Unitary Operads. In §10, we focus on the study of the rational homotopy of (connected) non-unitary operads. The goal of this chapter is to extend our model to (connected) unitary operads. For this aim, we use the Reedy model structure of the category of augmented Λ-operads in simplicial sets which we defined in §8.

We prove that the functor $\Omega^\ast : P \mapsto \Omega^\ast(P)$ of §10, which assigns a Hopf cooperad in cochain graded dg-modules to any (connected) operad in simplicial sets $P$, admits a lifting to the category of (connected) Λ-operads in simplicial sets, and hence, induces a functor between the category of (connected) Λ-operads in simplicial sets and the category of Hopf Λ-cooperads in cochain graded dg-modules.

We begin our study with a brief survey of the definition of our Reedy model structure for Λ-operads in simplicial sets (§12.0). We then explain the definition of our functor $\Omega^\ast : P \mapsto \Omega^\ast(P)$ on the category of Λ-operads (§12.1) and we explain the applications of this construction to the definition of a rationalization functor on the category of (connected) unitary operads in simplicial sets (§12.2).

Part II(d). Applications of the Rational Homotopy to $E_n$-operads. We make explicit models of the little discs operads to complete our study of the rational homotopy of operads. We precisely check, as we briefly explain in the introduction of this volume, that the Chevalley–Eilenberg cochain complex of (graded analogues of) the Drinfeld–Kohno Lie algebras define such models of the little discs operads in the category of Hopf cochain dg-cooperads. (We have a similar result when we pass to Λ-operads.) We will also explain that this statement can be interpreted as a formality theorem for the little 2-discs operad. Throughout our study, we also use the notation $\hat{p}$ for the completion of the ordinary Drinfeld–Kohno Lie algebra operad. We recall the definition of the Chevalley–Eilenberg cochain complex of Lie algebras and we make explicit the simplicial sets which correspond to these cochain complexes in the first chapter of the part (§13). We tackle the applications to the little discs operads afterwards (§14).

Chapter 13. Complete Lie Algebras and Rational Models of Classifying Spaces. We recall the definition of a Lie algebra and of the enveloping algebra of a Lie algebra in a preliminary section of this chapter (§13.0). We mainly apply the ideas of the first volume, where we explain a general definition of a Lie algebra in the setting of symmetric monoidal categories, to the base category of dg-modules which we consider in this part. By the way, we review the definition of the notion of a complete filtered module and of a weight graded module in the dg-module context.
We then study the Chevalley–Eilenberg cochain complex of complete Lie algebras in graded modules (§13.1). We prove that the Chevalley–Eilenberg cochain complex of a complete Lie algebra $g$ corresponds, under our model, to a simplicial set of Maurer–Cartan forms $\mathcal{MC}_\bullet(g)$ naturally associated to $g$. We explain the definition of a natural decomposition of the Chevalley–Eilenberg cochain complex into a tower of cofibrations in the category of unitary commutative cochain dg-algebras and a parallel decomposition of our simplicial set of Maurer–Cartan forms in the course of our study. We will use a generalization of these tower decompositions to define our homotopy spectral sequence for the computation of the homotopy of the mapping spaces of operads in the next part.

We explained in the first volume that any complete Lie algebra in the ordinary (ungraded) sense $g$ is associated to a Malcev complete group $G$ which we define by taking the group of group-like elements $G = \mathcal{G} \hat{U}(g)$ in the complete enveloping algebra $\hat{U}(g)$ of our complete Lie algebra $g$. We also explained that this Malcev complete group $G = \mathcal{G} \hat{U}(g)$ can be depicted as a group of exponential elements $e^\xi$, $\xi \in g$, in the complete enveloping algebra $\hat{U}(g)$.

We actually have a weak-equivalence between the simplicial set of Maurer–Cartan forms $\mathcal{MC}_\bullet(g)$ which we associate to the Chevalley–Eilenberg cochain complex of the Lie algebra $g$ in this chapter and the classifying space $\mathcal{B}(G)$ of the Malcev complete group $G = \mathcal{G} \hat{U}(g)$. We explain this relationship in the concluding section of the chapter (§13.2). We also make explicit the definition of an analogue, for the classifying space $\mathcal{B}(G)$ of the group $G = \mathcal{G} \hat{U}(g)$, of the tower decomposition of the simplicial set of Maurer–Cartan forms $\mathcal{MC}_\bullet(g)$.

This chapter does not contain any original result. We mainly revisit classical constructions in our framework for the applications to operads of the next chapter.

Chapter 14. Formality and Rational Models of $E_n$-operads. In this chapter, we study the applications of the Chevalley–Eilenberg cochain complex to operads in Lie algebras. We aim to make explicit the models of $E_n$-operads in our category of Hopf dg-cooperads. We also explain the definition of a natural tower decomposition of these Hopf dg-cooperad models of $E_n$-operads. In fact, we deal with Hopf $\Lambda$-cooperads rather than ordinary Hopf cooperads in our constructions. We therefore study models of $E_n$-operads in the category of Hopf $\Lambda$-cooperads in cochain graded dg-modules. We first study an additive version of the notion of a Hopf $\Lambda$-cooperad which naturally occurs when we consider the fibers (actually, the cofiber) of these tower decompositions.

We explain this concept in a preliminary section of the chapter (§14.0). We study the Chevalley–Eilenberg cochain complex of (graded generalizations of) the Drinfeld–Kohno Lie algebra operad $\hat{p}_n$ afterwards (§14.1). We check that the Chevalley–Eilenberg cochain complex $C_{CE}^*(\hat{p}_n)$ of this operad in the category of complete chain graded Lie algebras $\hat{p}_n$ forms a cofibrant object in the category of Hopf cochain dg-$\Lambda$-cooperads, for any $n \geq 2$. We explain, by the way, that the correspondence between the Chevalley–Eilenberg cochain complex and the simplicial sets of Maurer–Cartan forms studied in the previous chapter extends to the category of operads. We also study the applications of the tower decompositions of the previous chapter to the Chevalley–Eilenberg cochain complex $C_{CE}^*(\hat{p}_n)$ of the graded Drinfeld–Kohno Lie algebra operad $\hat{p}_n$ and to the corresponding spaces of Maurer–Cartan forms. We then explain the statement of formality results for
$E_n$-operads which imply that the object $\mathcal{C}^*_{E}(\hat{p}_n)$ is weakly-equivalent to our model $\Omega^*_\Delta(D_n)$ of the little $n$-discs operad $D_n$.

We devote the next section of the chapter (§11.2) to the particular case $n = 2$ of the study of the little discs operads. We then consider the operad of chord diagrams $CD^\wedge$, which consists of the Malcev complete groups $CD(r)^\wedge = \mathcal{C}(U(\hat{p}(r)))$, associated to the standard (ungraded) complete Drinfeld–Kohno Lie algebras $\hat{p}(r) = \hat{p}_2(r)$, $r > 0$. We prove that the classifying space of these Malcev complete groups defines an operad in simplicial sets $\mathcal{B}(CD^\wedge)$ which is weakly-equivalent to the rationalization $D_2^\wedge$ (in our sense) of the little $n$-discs operad $D_2$. We rely on the existence of Drinfeld’s associators, of which we explained the definition with full details in the first volume of this monograph, to establish this result. We also examine the applications, to this operad $\mathcal{B}(CD^\wedge)$, of the tower decomposition of the classifying spaces of Malcev complete groups.

We give a short reminder on the definition of the Drinfeld–Kohno Lie algebra operad in an appendix section (§11.3) to complete the account of this chapter.

Part III. The Computation of Homotopy Automorphism Spaces of Operads. We complete the computation of the homotopy of the homotopy automorphism space of the rationalization of $E_2$-operads in this part. We explain a general method of computation of the homotopy of mapping spaces of operads in a first step. We tackle the applications of this method to $E_n$-operads and to $E_2$-operads afterwards.

Introduction to the Results of the Computations for $E_2$-operads. We first recall the statement of our main theorem, the identity between the (pro-unipotent) Grothendieck–Teichmüller group and the group of homotopy automorphism classes of the rationalization of $E_2$-operads, which represents the main objective of this part, and we explain the plan of our computation method.

Part III(a). The Applications of Homotopy Spectral Sequences. Recall that the homotopy automorphism space of an object $X$ in a model category $\mathcal{C}$ consists of the invertible connected components of the mapping space with $X$ as source and target object. We use homotopy spectral sequences to compute the homotopy of such mapping spaces in the context of operads. We explain the general definition of these homotopy spectral sequences first (§1). We prove that, in the context of operads, the second page of our homotopy spectral sequences has a conceptual description in terms of a natural cohomology theory, the cotriple cohomology, which we define on the category of operads in graded modules (§2). We explain a general computation method of this cohomology of operads, by using duality theories, namely the bar duality and the Koszul duality of operads, which give small resolutions of operads (§3).

Chapter 4. Homotopy Spectral Sequences and Mapping Spaces of Operads. The homotopy spectral sequences, which we use in our computations, have actually been defined by Bousfield-Kan, and we give a short survey of the general definition of these spectral sequences (mostly without proofs) before tackling the applications to mapping spaces of operads.

We first briefly explain our terminological conventions for spectral sequences (§1.0). Let us mention that the homotopy spectral sequences are generally formed in the category of sets and some care is necessary in this context. We then recall the definition of a homotopy spectral sequence associated to a simplicial set equipped
with a decomposition into the limit of a tower of fibrations and the definition of a homotopy spectral sequence associated to the totalization of a cosimplicial space \(\text{§1.1}\).

We use both constructions when we deal with mapping spaces of operads. We actually take objects which naturally occur as a limit of a natural tower of fibrations in the category of operads on the target of our mapping spaces. (We deduce such decompositions from the study of the previous part in the case of \(E_n\)-operads.) This tower decomposition in the category of operads gives a decomposition as the limit of a tower of fibrations at the mapping space level. We consider, on the other hand, the geometric realization of a simplicial resolution (the cotriple resolution) as a source object in our mapping spaces. This construction implies that our mapping spaces occur as the totalization of cosimplicial spaces and we apply the homotopy spectral sequence of cosimplicial spaces to such objects. We explain these ideas in the concluding section of the chapter \(\text{§1.2}\).

Let us observe that we actually get a double spectral sequence for our mapping spaces, with a horizontal spectral sequence direction which arises from the simplicial decomposition of the source object and a vertical spectral sequence direction which arises from the tower decomposition of our target object. We mostly give methods to compute the horizontal (cosimplicial) homotopy spectral sequence in the next chapters of this part.

**Chapter 2. Applications of the Cotriple Cohomology of Operads.** We first explain that the terms on the second page of this operadic cosimplicial homotopy spectral sequence reduces to the cotriple cohomology of the homology of our operads. This result gives the starting point of our subsequent computations.

We heavily use multi-graded structures in our study of spectral sequences and we devote a preliminary section to a detailed survey of this background \(\text{§2.0}\). We then explain the definition of a category of abelian bimodules over operads, which give the general notion of coefficients which we associate to the cotriple cohomology of operads \(\text{§2.1}\). We explain the definition of the cotriple cohomology itself afterwards. We explicitly define the cotriple cohomology \(H^*_{\text{gr} \Lambda \text{Op}_{\emptyset}}(R, \mathcal{N})\) of an operad in graded modules \(R\) with coefficients in an abelian bimodule \(\mathcal{N}\) as the cohomology of a cosimplicial module of operadic derivations \(\text{Der}_{\text{gr} \Lambda \text{Op}_{\emptyset}}(R_{\bullet}, \mathcal{N})\) defined on the cotriple resolution \(R_{\bullet} = \text{Res}_{\bullet}(R)\) of our operad \(P\) and with values in our abelian bimodule \(\mathcal{N}\) (see \(\text{§2.2}\)). For our purpose, we just focus on the applications to augmented \(\Lambda\)-operads rather than to ordinary operads (as our notation indicates). We quickly check that this cotriple cohomology theory does determine the second page of our operadic cosimplicial homotopy spectral sequence (as we expect).

We devote an appendix section of this chapter to a thorough study of the homotopy properties of hom-objects on the categories of symmetric sequences and \(\Lambda\)-sequences which underlie our operads \(\text{§2.3}\).

**Chapter 3. Applications of the Koszul Duality of Operads.** We then explain the applications of the bar duality and of the Koszul duality of operads to the definition of reductions of the cotriple cohomology complex which we introduced in the previous chapter. We carry out this reduction process itself in the first section of this chapter \(\text{§3.1}\). To complete this study, we just review the definition of our derivation complex in order to make explicit the structure of the reduced complexes which we associate to the bar construction and to the Koszul construction of operads. We address these topics in the second section of the chapter \(\text{§3.2}\).
Part III(b). The Case of $E_n$-operads. In this part, we examine the application of the Koszul construction, which represents the final outcome of the study of the previous chapter, to the homology of the little discs operads (equivalently, of $E_n$-operads). We eventually completely determine the second page of our operadic cosimplicial homotopy spectral sequence in the case of $E_2$-operads (§4). Then we check that the classes which we obtain in this homotopy spectral sequence computation correspond to a natural decomposition of the Grothendieck–Teichmüller group to complete the verification of the main result of this work, the homotopy interpretation of the pro-unipotent Grothendieck–Teichmüller group as the group of homotopy automorphisms of the rationalization of $E_2$-operads (§5).

Chapter 4. The Applications of the Koszul Duality for $E_n$-operads. We recalled in the first volume of this monograph that the homology of the little $n$-discs operad is identified with a graded version of the operad of Poisson algebras. In this book, we also use the notation $\text{Gerst}_n$ and the name ‘$n$-Gerstenhaber operad’ for this operad such that $\text{Gerst}_n = H_*(D_n)$, because the 2-Gerstenhaber operad, for which we also use the simplified notation $\text{Gerst} = \text{Gerst}_2$, is identified with the operad that governs the kind of algebra structures introduced by Gerstenhaber for the study of the deformation complex of algebras.

The $n$-Gerstenhaber operad $\text{Gerst}_n$ is an instance of a Koszul operad. We recall the statement of this Koszul duality result in the first section of this chapter (§4.1) and we determine the cohomology of the associated derivation complex afterwards (§4.2).

Chapter 5. The Interpretation of the Result of the Spectral Sequence in the Case of $E_2$-operads. The main consequence of the result of the previous chapter is that the cotriple cohomology of the 2-Gerstenhaber operad, and hence, the second page of the operadic cosimplicial homotopy spectral sequence for $E_2$-operads, is identified with the graded Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}$. In this chapter, we review the definition of our mapping from the pro-unipotent Grothendieck–Teichmüller group $\text{GT}(\mathbb{Q})$ to the space of homotopy automorphisms of the rationalization of $E_2$-operads $\text{Aut}_{\text{op}}^{h}(E_2)$ (§5.0). We check that our identity at the spectral sequence level reflects a natural tower decomposition of this group $\text{GT}(\mathbb{Q})$. We deduce from these observations that our double spectral sequence degenerates and that our mapping gives a bijection $\text{GT}(\mathbb{Q}) \cong \pi_0 \text{Aut}_{\text{op}}^{h}(E_2)$ when we pass to the degree zero homotopy of our homotopy automorphism space $\text{Aut}_{\text{op}}^{h}(E_2)$ (§5.1). We also rely on our spectral sequence computations to check that the homotopy of this homotopy automorphism space reduces to a module of rank one in degree one and vanishes in degrees larger than one (§5.2). This verification completes the proof of our main statement, such announced in the foreword of the first volume.

We just devote an appendix section of the chapter to the verification of a (partial) idempotence property of the rationalization of $E_2$-operads (§5.3). We mainly use the observation of this appendix section to give a simple interpretation of our result.

Conclusion: A Survey of Further Research on Operadic Mapping Spaces and their Applications. To conclude this study, we outline new developments of the homotopy theory of $E_n$-operads. Notably, we give a brief statement of the generalization of the computation of the previous chapter for the homotopy automorphism spaces of the rationalization of $E_n$-operads. These computations, carried out by the author in a joint work with Victor Turchin and Thomas
Willwacher, heavily rely on graph complexes similar to the graph complexes introduced by Kontsevich at the origin of the renewal of the theory of operads in the 1990’s. We also briefly explain the applications of mapping spaces of $E_n$-operads to the study of the embedding spaces of euclidean spaces mentioned in the introduction of this volume.

Appendix C. Cofree Cooperads and the Bar Duality of Operads. In this appendix, we first examine a dualization of the constructions of §§A-B with the aim of giving an explicit definition of cofree objects in the category of cooperads. We briefly recall our conventions on trees first (§C.0). We tackle the construction of cofree cooperads afterwards (§C.1). We then survey the ideas of the bar duality and of the Koszul duality of operads (§§C.2-C.3).

Most concepts which we explain in this appendix chapter are not original. We mainly survey definitions of the literature and we generally skip the proof of theorems. We just check that the standard constructions extend to our $\Lambda$-cooperad (and $\Lambda$-operad) setting. (We will see that this extension enables us to apply the bar duality and the Koszul duality theory to unitary operads.)
Glossary of Notation

Background

Fundamental objects
\( \mathbb{k} \): the ground ring
\( \mathbb{D}^n \): the unit \( n \)-disc, see §I.4.1.1
\( \Delta^n \): the topological \( n \)-simplex, see §0.3 §II.1.3.4
\( pt \): the one-point set (also denoted by * when regarded as a terminal object)
\( \Delta \): the simplicial category, see §0.3 §II.1.3.2
\( \Delta^n \): the \( n \)-simplex object of the category of simplicial sets, see §0.3 §II.1.3.4

Generic categorical notation
\( A, B, C, \ldots \): general categories
\( I, J, \ldots \): indexing categories, as well as the set of generating cofibrations and the set of generating acyclic cofibrations in a cofibrantly generated model category, see §II.4.1.3
\( F, G, \ldots \): some classes of morphisms in a category
\( M, N, \ldots \): (symmetric) monoidal categories, see §0.8
\( 1 \): the unit object of a (symmetric) monoidal category, see §0.8
eq: the equalizer of parallel arrows in a category
coeq: the coequalizer of parallel arrows in a category

Fundamental categories
\( \text{Mod} \): the category of modules over the ground ring
\( \text{Set} \): the category of sets
\( \text{Top} \): the category of topological spaces, see §II.1.3
\( \text{sSet} \): the category of simplicial sets, see §0.3 §II.1.3
\( \text{Gr} \): the category of groups
\( \text{Grd} \): the category of groupoids, see §I.5.2.1
\( \text{Cat} \): the category of small categories, see §I.5.2.1
\( \text{Ab} \): the category of abelian groups

Categories of algebras and of coalgebras
\( \text{Com} \): the category of non-unitary commutative algebras
\( \text{As} \): the category of non-unitary associative algebras
\( \text{Lie} \): the category of Lie algebras
\( \text{As}_+ \): the category of unitary associative algebras
\( \text{Comm}_+ \): the category of unitary commutative algebras, see §I.3.0.1
\( \text{Comm}_+^c \): the category of counitary cocommutative coalgebras, see §I.3.0.4
HopfAlg: the category of Hopf algebras (defined as the category of bialgebras equipped with an antipode operation), see §I.7.8
HopfGrd: the category of Hopf groupoids (defined as the category of small categories equipped with an antipode operation), see §I.9.0.2

Functors and constructions for filtered objects
Fₙ: the nth layer of a decreasing filtration
E₀: the nth subquotient of a filtered object, see §I.7.3.6 (also used to denote the nth fiber of a tower of set maps in the context of homotopy spectral sequences, see §III.1.1.7)
Eⁿ: the weight graded object associated to a filtered object in a category (e.g. the weight graded module associated to a filtered module, see §I.7.3.6 the weight graded Lie algebra associated to a Malcev complete group, see §I.8.2.2 ...)
(−)̂: the completion functor on a category of objects equipped with a decreasing filtration, as well as the Malcev completion for groups and groupoids, see §I.7.3.4 §II.8.3 (also the rationalization functor on spaces, see the section about the constructions of homotopy theory in this glossary)

Functors and constructions on algebras and coalgebras
S: the symmetric algebra functor (in any symmetric monoidal category), see §I.7.2.4
T: the tensor algebra functor (in any symmetric monoidal category), see §I.7.2.4
L: the free Lie algebra functor (in any Q-additive symmetric monoidal category and in abelian groups), see §I.7.2.3
U: the enveloping algebra functor (on the category of Lie algebras in any Q-additive symmetric monoidal category), see §I.7.2.7
Ŝ, Ŵ, ...: the complete variants of the symmetric algebra functor, of the tensor algebra functor, ... in the context of a category of complete filtered modules, see §I.7.3.22
G: the group-like element functor on coalgebras, see §I.7.1.14 and on complete Hopf coalgebras, see §I.8.1.2
P: the primitive element functor on Hopf algebras, see §I.7.2.11
I(−): the augmentation ideal of Hopf algebras, see §I.8.1.1

Categorical prefixes
dg: prefix for a category of differential graded objects in a category (e.g. the category of dg-modules dgMod, see §II.5.0.1)
dg⁺, dg⁻: prefix for the chain graded and cochain graded variants of the categories of differential graded objects (e.g. the category of chain graded dg-modules dg⁺Mod, see §II.5.0.1 the category of cochain graded dg-modules dg⁻Mod, see §II.5.0.1 and the category of unitary commutative cochain dg-algebras dg⁺Com⁺, see §II.5.1.1 ...)
gr: prefix for a category of graded objects in a category when the grading underlies a differential graded structure (e.g. the category of graded modules grMod, see §II.5.0.2 ...)
s: prefix for a category of simplicial objects in a category (e.g. the category of simplicial modules sMod, see §II.5.0.4 the category of simplicial sets sSet, see §II.5.1.3 ...
c: prefix for a category of cosimplicial objects in a category (e.g. the category of cosimplicial modules \(c Mod\), see §II.6.0.4, the category of cosimplicial unitary commutative algebras \(c \mathbb{C}om_+\), see §II.6.1.3, ...)

\(f\): prefix for a category of filtered objects in a category (e.g. the category of filtered modules \(f Mod\), see §II.7.3.1)

\(\hat{f}\): prefix for a category of complete filtered modules in a category (e.g. the category of complete filtered modules \(\hat{f} Mod\), see §II.7.3.4, the category of Malcev complete groups \(\hat{f} \mathbb{G}rp\), see §II.8.2). Note that the categories of complete Hopf algebras \(\hat{f} \mathcal{H}opf \mathcal{A}lg\) and of complete Lie algebras \(\hat{f} \mathcal{L}ie\) consist of Hopf algebras and Lie algebras in complete filtered modules that satisfy an extra connectedness requirement and a similar convention is made for the category of complete Hopf groupoids \(\hat{f} \mathcal{H}opf \mathcal{G}pd\), see §II.7.3.15, §II.7.3.20, §II.9.1.2

\(w\): prefix for a category of weight graded objects in a category (e.g. the category of weight graded modules \(w Mod\), see §II.7.3.5)

**Morphisms, hom-objects, duals, and analogous constructions**

\(\text{Mor}\): the notation for the morphism sets of any category (e.g. \(\text{Mor}_{\mathbb{M}od}(-,-)\) for the morphism sets of the category of modules over the ground ring \(\mathbb{M}od\))

\(\text{Aut}\): the notation for the automorphism group of an object in a category

\(\text{Hom}\): the notation for the hom-objects of an enriched category structure (not to be confused with the morphism sets), see §II.0.12

\(\text{D}\): the duality functor for ordinary modules, dg-modules, simplicial modules and cosimplicial modules, see §II.5.0.13

\((-)^{\vee}\): the dual of individual objects, or of objects equipped with extra structures (algebras, operads, ...), see §II.5.0.13

\(\text{Der}\): the modules of derivations (for algebras, operads, ...), see §III.2.1

\(\text{Map, Aut}^{h}\): see the section of this glossary about the constructions of homotopy theory

**Constructions of homotopy theory**

**Fundamental constructions in model categories**

\(\text{Ho}(-)\): the homotopy of a model category, see §II.1.2

\(\text{Aut}^{h}\): the notation for the homotopy automorphism space of an object in a model category, see §II.2.2

\(\text{Map}\): the notation for the mapping spaces of a pair of objects in simplicial model categories and in general model categories, see §II.2.1, §II.3.2.11

**Fundamental simplicial and cosimplicial constructions**

\(\text{B}\): the classifying space construction for groups, groupoids, categories, ..., see §II.5.2.8 (also the bar construction of algebras and of operads, see the relevant sections of this glossary)

\(\text{sk}\): the \(r\)th skeleton of a simplicial set, of a simplicial and of a cosimplicial object in a model category, see §III.1.3.8, §II.3.1.7, §II.3.1.17

\(\text{Tot}\): the totalization of cosimplicial spaces, of cosimplicial objects in a model category, see §II.5.3.1.9
$| - |$: the geometric realization of simplicial sets, of simplicial objects in a model category, see §0.3 §II.1.3.5 §II.3.3.5

**Diag**: the diagonal complex of a bisimplicial set, of a bisimplicial object and of a bicosimplicial object in a model category, see §II.3.3.19

$L_r(X)$: the $r$th latching object of a simplicial object in a category, see §II.3.1.14

$M_r(X)$: the $r$th matching object of a simplicial object in a category, see §II.3.1.15

(also the matching objects of $\Lambda$-sequences, see the section about operads and related structures of this glossary)

$L^r(X), M^r(X)$: the cosimplicial variants of the matching and matching object constructions, see §II.3.1.3 §II.3.1.5

**Differential graded constructions**

$b^m, e^m$: notation for particular homogeneous elements (of upper degree $m$) notably used to define the generating (acyclic) cofibrations of the category of cochain graded dg-modules, see §II.5.1.2

$b_m, e_m$: same as $b^m$ and $e^m$ but in the chain graded context

$B^m$: source objects of the generating cofibrations of the category of cochain graded dg-modules, see §II.5.1.2

$E^m$: target objects of the generating (acyclic) cofibrations of the category of cochain graded dg-modules, see §II.5.1.2

$B_m, E_m$: dual objects of the dg-modules $B^m$ and $E^m$

$\sigma$: notation for particular homogeneous elements used in the definition of suspension functors on dg-modules, see C.2.3

$\rho_r, \rho_s$: notation for particular homogeneous elements used in the definition of the operadic suspension functor for operads in dg-modules, see §II.4.1.1

**Cyl**: the standard cylinder object functor on the category of dg-modules, see §II.13.1.10

$B$: the bar construction for algebras, see §II.6.3 (also the classifying space of groups, categories, and the bar construction of operads, see the relevant sections of this glossary)

$\tau_*: \text{ the right adjoint } \tau_*: \text{dg-Mod} \rightarrow \text{dg}_* \text{Mod}$ of the embedding $\iota: \text{dg}_* \text{Mod} \hookrightarrow \text{dg-Mod}$ of the category of chain graded dg-modules $\text{dg}_* \text{Mod}$ into the category of all dg-modules $\text{dg-Mod}$, see §II.5.3.2

$\tau^*: \text{ the left adjoint } \tau^*: \text{dg-Mod} \rightarrow \text{dg}^* \text{Mod}$ of the embedding $\iota: \text{dg}^* \text{Mod} \hookrightarrow \text{dg-Mod}$ of the category of cochain graded dg-modules $\text{dg}^* \text{Mod}$ into the category of all dg-modules $\text{dg-Mod}$, see §II.5.0.1

$(-)_0$: the forgetful functor from dg-modules to graded modules, see §II.1

**The Dold–Kan correspondence**

$N_*$: the normalized chain complex functor on the category of simplicial modules, see §0.6 §II.5.0.5

$N^*$: the conormalized cochain complex functor on the category of cosimplicial modules, see §II.5.0.9

$\Gamma_*$: the Dold–Kan functor on the category of chain graded dg-modules, see §II.5.0.6

$\Gamma^*$: the cosimplicial version of the Dold–Kan functor on the category of cochain graded dg-modules, see §II.5.0.9
Constructions of rational homotopy theory

\(-\hat{}\): the rationalization functor on spaces, see §II\[7.2.3\] and on operads in simplicial sets, see §II\[10.2\], §II\[12.2\] (also the completion of filtered objects, see the section of this glossary about the background of our constructions)

\(\Omega^*\): the Sullivan cochain dg-algebra functor on simplicial sets, see §II\[7.1\]

\(\Omega^*\#:\) the operadic upgrade of the cochain dg-algebra functor on operads in simplicial sets, see §II\[10.1\], §II\[12.1\]

\(G\): the functor from cochain dg-algebras to simplicial sets, see §II\[7.2\]

\(MC\): the Maurer–Cartan spaces associated to (complete) Lie algebras, see §II\[13.1.8\]

Operads and related structures

Indexing of operads

\(\Sigma_r\): the symmetric group on \(r\) letters

\(\Sigma\): the category of finite ordinals and permutations, see §I\[2.2.3\]

\(\Lambda^+\): the category of finite ordinals and increasing injections, see §I\[2.2.2\]

\(\Sigma_{>0}, \Sigma_{>1}, \Lambda_{>0}, \Lambda_{>1}, \ldots\): the full subcategory of the category \(\Sigma, \Lambda, \ldots\) generated by the ordinals of cardinal \(r > 0, r > 1\), see §I\[2.2.2\], §II\[2.4.1\]

\(Bij, \text{Inj}\): the category of finite sets and bijections, see §II\[2.5.1\]

\(B_{ij}>0, B_{ij}>1, \text{Inj}_{>0}, \text{Inj}_{>1}, \ldots\): the full subcategory of the categories \(Bij, \text{Inj}\), \(\ldots\) generated by the finite sets of cardinal \(r > 0, r > 1\), see §II\[2.5.9\]

\(m, n, \ldots, r, \ldots\): generic notation for finite ordinals \(r = \{1 < \cdots < r\}\) or for finite sets \(r = \{i_1, \ldots, i_r\}\) used to index the terms of operads, symmetric sequences and \(\Lambda\)-sequences

\(0, 1, 2, \ldots\): the empty ordinal, the ordinal of cardinal one \(1 = \{1\}\), of cardinal two \(2 = \{1 < 2\}\), \ldots

Categories of operads and related

\(\Op\): the category of (symmetric) operads, see §I\[1.1.2\]

\(\Op_{\#}\): the category of non-unitary (symmetric) operads, see §I\[1.1.20\]

\(\Op_{\#1}\): the category of connected (symmetric) operads, see §I\[1.1.21\]

\(\Op_{\#1}\): the category of (symmetric) cooperads, see §II\[9.1.8\]

\(\Lambda\Op_{\#} / \text{Com}\): the category of augmented non-unitary \(\Lambda\)-operads (the postfix expression \(-/ \text{Com}\) can be discarded when the augmentation is trivial), see §II\[2.2.17\]

\(\Lambda\Op_{\#1} / \text{Com}\): the category of augmented connected \(\Lambda\)-operads (the postfix expression \(-/ \text{Com}\) can be discarded when the augmentation is trivial), see §II\[2.4.1\]

\(\Seq\): the category of symmetric sequences, see §I\[1.2\]

\(\Seq_{>0}\): the category of non-unitary symmetric sequences, see §II\[1.2.13\]

\(\Seq_1\): the category of connected symmetric sequences, see §II\[1.2.13\]

\(\Seq^c, \Seq_{>0}^c, \Seq_{>1}^c\): same as \(\Seq, \Seq_{>0}, \Seq_{>1}\) but used instead of this notation in the context of cooperads

\(\Lambda\Seq\): the category of \(\Lambda\)-sequences, see §I\[2.3\]

\(\Lambda\Seq_{>0}\): the category of non-unitary \(\Lambda\)-sequences, see §I\[2.3\]

\(\Lambda\Seq_{>1}\): the category of connected \(\Lambda\)-sequences, see §II\[2.4.1\]

\(\Lambda\Seq^c\): the category of covariant \(\Lambda\)-sequences
**Categories of Hopf operads and related**

\( \mathcal{H}opf \mathcal{O}p \): the category of Hopf operads (defined as the category of operads in counitary cocommutative coalgebras), see §I.3.2

\( \mathcal{H}opf \mathcal{O}p_{\varnothing}, \mathcal{H}opf \mathcal{O}p_{\varnothing 1} \): the non-unitary and connected variants of the category of Hopf operads

\( \mathcal{H}opf \Lambda \mathcal{O}p_{\varnothing}, \mathcal{H}opf \Lambda \mathcal{O}p_{\varnothing 1} \): the \( \Lambda \)-operad variants of the categories of non-unitary and connected Hopf operads

\( \mathcal{H}opf \mathcal{S}eq \): the category of Hopf symmetric sequences (defined as the category of symmetric sequences in counitary cocommutative coalgebras), see §I.3.2.6

\( \mathcal{H}opf \mathcal{S}eq_{\varnothing}, \mathcal{H}opf \mathcal{S}eq_{\varnothing 1} \): the non-unitary and connected variants of the category of Hopf symmetric sequences

\( \mathcal{H}opf \Lambda \mathcal{S}eq_{\varnothing}, \mathcal{H}opf \Lambda \mathcal{S}eq_{\varnothing 1} \): the \( \Lambda \)-sequence variants of the categories of non-unitary and connected Hopf symmetric sequences

\( \mathcal{H}opf \mathcal{O}p_{\varnothing 1} \): the category of Hopf cooperads (defined as the category of cooperads in unitary commutative algebras), see §II.9.3.1

\( \mathcal{H}opf \Lambda \mathcal{O}p_{\varnothing 1} \): the category of Hopf \( \Lambda \)-cooperads (defined as the category of cooperads in unitary commutative algebras), see §II.11.4.1

\( \mathcal{H}opf \mathcal{S}eq_{\varnothing 1} \): the category of connected Hopf sequences underlying Hopf cooperads (defined as the category of symmetric sequences in unitary commutative algebras), see §II.9.3.1

\( \mathcal{H}opf \Lambda \mathcal{S}eq_{\varnothing 1} \): the category of connected Hopf \( \Lambda \)-sequences underlying Hopf \( \Lambda \)-cooperads (defined as the category of \( \Lambda \)-sequences in unitary commutative algebras), see §II.11.4.1

**Notation of operads**

\( P, Q, \ldots \): generic notation for operads (of any kind)

\( M, N, \ldots \): generic notation for symmetric sequences, \( \Lambda \)-sequences, covariant \( \Lambda \)-sequences

\( C, D, \ldots \): generic notation for cooperads (of any kind)

\( C_n \): the operad of little \( n \)-cubes, see §I.4.1.3

\( D_n \): the operad of little \( n \)-discs, see §I.4.1.7

\( \text{As} \): the (non-unitary) associative operad, see §I.1.1.16, §I.1.2.6, §I.1.2.10

\( \text{Com} \): the (non-unitary) commutative operad, see §I.1.1.16, §I.1.2.6, §§II.11.2.10, 2.1.11

\( \text{Lie} \): the Lie operad, see §I.1.2.10

\( \text{Pois} \): the Poisson operad, see §I.1.2.12

\( \text{Gerst}_n \): the \( n \)-Gerstenhaber operad (defined as a graded variant of the Poisson operad), see §I.1.2.13

\( \text{Com}^c \): the commutative cooperad, see §§II.9.1.3

\( \text{CoS}, \text{PaS}, \text{CoB}, \text{PaB}, \ldots \): see the section about the applications of operads to the definition of Grothendieck–Teichmüller groups
Constructions on operads and on cooperads

\(\tau\): the truncation functors from non-unitary operads to connected operads and from augmented non-unitary \(\Lambda\)-operads to augmented connected \(\Lambda\)-operads, see §I.2.15 Proposition I.2.1.5

\(\Theta\): the free operad functor, see §A.3

\(\Theta\): the cofree cooperad functor, see §C.1

\(\Theta_T(M)\): the treewise tensor product of a symmetric sequence \(M\) over a tree \(T\) when regarded as a term of the free operad and of the cofree cooperad (same as the object denoted by \(\Theta(M(T))\) in the section about trees), see §A.2

\(\Sigma F^r\): the \(r\)th free symmetric sequence, see §II.8.1.2

\(\Lambda F^r\): the \(r\)th free \(\Lambda\)-sequence, see §II.8.3.6

\(\partial \Lambda F^r\): the boundary of the \(r\)th free \(\Lambda\)-sequence, see §II.8.3.7

\(\partial' \Lambda F^r\): the boundary of the \(r\)th free \(\Lambda\)-sequence in the context of connected \(\Lambda\)-sequences, see §II.12.0.1

\(\text{Res}^r\): the cotriple resolution functor on operads, see §B.1, §II.8.5

\(B\): the bar construction of operads, see §C.2 (also the classifying space of groups, categories, and the bar construction of algebras, see the relevant sections of this glossary)

\(B^r\): the cobar construction of cooperads, see §C.2

\(K\): the Koszul dual of operads, see §C.3

\(M(M)(r)\): the \(r\)th matching object of a \(\Lambda\)-sequence, see §II.8.3.1

\(\text{ar}^r\leq_s\): the \(s\)th layer of the arity filtration of a \(\Lambda\)-sequence, see Proof of Theorem II.8.3.20

\(\text{ar}^r\leq_s\): the operadic upgrade of the arity filtration, see Proof of Theorem II.8.4.12

\(\text{cosk}^\Lambda_r\): the \(r\)th \(\Lambda\)-coskeleton of a \(\Lambda\)-sequence, see §II.8.3.3 of an augmented non-unitary \(\Lambda\)-operad, see Proof of Theorem II.8.4.12

Trees

\(\text{Tree}(r)\): the category of \(r\)-trees (where \(r\) is the indexing set of the inputs of the trees), see §A.1

\(\text{Tree}\): the operad of trees, see §A.1

\(\text{Tree}(r)\): the category of reduced \(r\)-trees (where \(r\) is the indexing set of the inputs of the trees),

\(\text{Tree}\): the operad of reduced trees, see §A.1.12

\(\text{Tree}^o(r)\): the category of planar \(r\)-trees (where \(r\) is the indexing set of the inputs of the trees), see §A.3.16

\(\text{Tree}^o\): the operad of planar trees, see §A.3.16

\(\Sigma, \Delta, \ldots\): generic notation for trees

\(\downarrow\): the unit tree (the tree with no vertex), see §A.1.4

\(\Upsilon\): the notation of a corolla (a tree with a single vertex), see §A.1.4

\(\Gamma\): the notation of a tree with two vertices, see §A.2.3

\(V(T)\): the vertex set of a tree

\(E(T)\): the edge set of a tree

\(E(T)\): the set of inner edges of a tree

\(r_v\): the set of ingoing edges of a vertex in a tree

\(\text{Out}^r(T)\): the treewise tensor product of a symmetric sequence \(M\) over a tree \(T\) (same as the object denoted by \(\text{Out}^r(T)(M)\) in the section about constructions on operads and on cooperads), see §A.2
\( \lambda_T \): the treewise composition products associated to an operad, see §A.2.7
\( \rho_T \): the treewise composition coproducts associated to a cooperad, see §C.1.5

**From operads to Grothendieck–Teichmüller groups**

*Permutations, braids, and related objects*

\( \Sigma_r \): the symmetric group on \( r \) letters
\( B_r \): the Artin braid group on \( r \) strands, see §I.5.0
\( P_r \): the pure braid group on \( r \) strands, see §I.5.0
\( p(r) \): the \( r \)th Drinfeld–Kohno Lie algebra (the Lie algebra of infinitesimal braids on \( r \) strands), see §I.10.0.2
\( \hat{p}(r) \): the complete Drinfeld–Kohno Lie algebras, see §I.10.0.6
\( p_n(r) \): the graded variants of the Drinfeld–Kohno Lie algebras (with \( p(r) = p_2(r) \)), see §II.14.1.1
\( \hat{p} \): the Drinfeld–Kohno Lie algebra operad, see §I.10.1.1
\( \hat{p}_n \): the graded variants of the Drinfeld–Kohno Lie algebra operad (with \( p = p_2 \)), see §II.14.1.1
\( \hat{a} \): the complete Drinfeld–Kohno Lie algebra operad, see §II.10.2.2
\( \hat{a}_c \): the operad of colored symmetries, see §I.6.3
\( \hat{a}_{c} \): the operad of parenthesized symmetries, see §I.6.3
\( \hat{a}_b \): the operad of colored braids, see §§5.2.8, 5.2.11
\( \hat{a}_{b} \): the operad of parenthesized braids, see §I.6.2
\( \hat{a}_{b} \): the Malcev completion of the colored and parenthesized braid operads, see §I.10.1
\( \hat{a}_{c} \): the operad of chord diagrams, see §I.10.2.4
\( \hat{a}_{c} \): the operad of parenthesized chord diagrams, see §II.10.3.2

*Grothendieck–Teichmüller groups and related objects*

\( \text{Ass}(k) \): the set of Drinfeld’s associators, see §I.10.2.11
\( \text{GT}(k) \): the pro-unipotent Grothendieck–Teichmüller group, see §I.11.1
\( \text{GRT} \): the graded Grothendieck–Teichmüller group, see §I.11.1
\( \text{GT} \): the profinite Grothendieck–Teichmüller group
\( \text{grt} \): the graded Grothendieck–Teichmüller Lie algebra, see §I.10.4.6, §II.11.4


Index

$E_2$-operads, 411
and the classifying spaces of the chord diagram operad, 440
formality of the, 444
homotopy automorphism space of, see also homotopy automorphism space of $E_2$-operads
rationalization of, 446, 587
$E_n$-operads, 411
cohomology cooperad of the, 433
formality of the, 438, 594
homology of the, 432
homotopy automorphism space of, 504
intrinsic formality of the, 436–438, 602
mapping spaces on, 604, 612
rationalization of, 438
$\Lambda$-cooperads, see also coaugmented $\Lambda$-cooperads
additive, see also additive $\Lambda$-cooperads
$\Lambda$-operads
additive, see also additive $\Lambda$-operads
connected, see also augmented connected $\Lambda$-operads
non-unitary, see also augmented non-unitary $\Lambda$-operads
$\Lambda$-sequences, 533
cofree — over symmetric sequences, see also cofree $\Lambda$-sequences
connected, see also connected $\Lambda$-sequences
hom-objects of, 510
hom-objects of — in dg-modules, 513
and the Künneth isomorphism formula, 517
and weak-equivalences, 514
hom-objects of — in graded modules, 513
hom-objects of — in simplicial modules, 516
and the Eilenberg–Zilber equivalence, 517
homomorphisms of, 512
homomorphisms of — in dg-modules, 513
homomorphisms of — in graded modules, 513
non-unitary, see also non-unitary $\Lambda$-sequences
Q-nilpotent spaces, 199
of finite Q-type, see also rational spaces, 199, 204, 206
$\kappa$-combinatorial model categories, see also combinatorial model categories
$\kappa$-filtered colimits, 120
$\kappa$-presentable categories, see also locally presentable categories
$\kappa$-small, see also small object
$n$-Gerstenhaber operad, see also Gerstenhaber operads, 482
abelian bimodules over operads, 489
and abelian group objects over operads, 490
and additive operads, 491
and operadic derivations, 491
and semi-direct products over operads, 489
abelian groups of finite Q-type, 199
acyclic cofibrations, 11
and adjunctions, 12
and compositions, 12
and pushouts, 12
and relative cell complexes, 112
and the left lifting property, 11
class of — in a model category, 11
acyclic fibrations, 11
and adjunctions, 12
and compositions, 12
and pullbacks, 12
and the right lifting property, 11
class of — in a model category, 11
additive $\Lambda$-cooperads, 413
and semi-direct products over cooperads, 416
and symmetric algebras, 417, 418
additive $\Lambda$-operads, 414, 475
and the Dold–Kan equivalence, 414
in simplicial modules, 417, 418
additive cooperads, 415
and semi-direct products over cooperads, 416
and symmetric algebras, 417, 418

and semi-direct products over cooperads, additive operads, and abelian bimodules over operads, and the Dold–Kan equivalence, algebras over an operad cotensored category structure of — in simplicial sets, cotensored category structure of — in topological spaces, function objects on — in simplicial sets, function objects on — in topological spaces, model category of, simplicial model category of — in simplicial sets, simplicial model category of — in topological spaces, tensored category structure of — in simplicial sets, tensored category structure of — in topological spaces, Arnold relations, augmented connected Λ-operads, and the cobar-bar resolution, and the Koszul resolution of operads, augmented non-unitary Λ-operads, Λ-coskeletons of, cofibrations of — as cofibrations of non-unitary operads, connected truncation of, cotriple resolution of, see also cotriple resolution, of operads generating acyclic cofibrations of, generating cofibrations of, model category of, Reedy model category of, see also augmented non-unitary Λ-operads, model category of bar construction of operads, weight decomposition of the, bar construction of unitary commutative cochain dg-algebras, bar-cobar resolution of cooperads, binary trees, boundary of the simplices, Bousfield–Kan spectral sequences, see also homotopy spectral sequences Brown’s Lemma, categories cotensored — over simplicial sets, enriched, locally presentable, model, see also model categories simplicial, symmetric monoidal, tensored — over simplicial sets, category of finite ordinals and injections, operadic compositions in the, cell attachments in model categories of non-unitary operads in simplicial sets, of non-unitary symmetric sequences, of non-unitary symmetric sequences in simplicial sets, of unitary commutative cochain dg-algebras, chain complexes of complete filtered chain graded modules, chain complexes of dg-modules, and twisted dg-modules, total dg-module of the, chain graded dg-modules, augmented, complete, see also complete filtered chain graded dg-modules model category of, tensor products of, weight graded, chain graded truncation of dg-modules, chain-homotopies, see also contracting chain-homotopies Chevalley–Eilenberg chain complexes, see also complete Chevalley–Eilenberg chain complexes Chevalley–Eilenberg cochain complexes, and the Sullivan cochain dg-algebras of Maurer–Cartan spaces, as cofibrant unitary commutative cochain dg-algebras, geometric realization of the — and Maurer–Cartan spaces, of the Drinfeld–Kohno Lie algebra operad, see also Drinfeld–Kohno Lie algebra operad, Chevalley–Eilenberg cochain complexes of the, of the graded Drinfeld–Kohno Lie algebra operads, see also graded Drinfeld–Kohno Lie algebra operads, Chevalley–Eilenberg cochain complexes of the, tower decomposition of the, chord diagram operad, classifying spaces of the,
and the Maurer–Cartan spaces of the Drinfeld–Kohno Lie algebra operad, 443.

and the rationalization of \( E_2 \)-operads, 446.

tower decomposition of the, 440, 480.

classifying spaces, 402.

and abelian groups, 402.

and central extensions, 403.

of groups, see also classifying spaces of Malcev complete groups, 404.

and Maurer–Cartan spaces, 408.

of the chord diagram operad, 439.

of the parenthesized braid operad, 443.

of the Maurer–Cartan spaces of the Drinfeld–Kohno Lie algebra operad, 443.

of the parenthesized braid operad, 443.

tower decomposition of the — of Malcev complete groups, 404.

coaugmentation coideal of coaugmented \( \Lambda \)-cooperads, 339.

coaugmentation coideal of cooperads, 277, 625, 635.

coaugmentation morphisms of coaugmented \( \Lambda \)-cooperads, 334, 337.

on cooperads and treewise tensor products, 635, 637.

treewise representation of the — on cooperads, 635, 637.

coaugmented \( \Lambda \)-cooperads, 335, 338.

and cofree cooperads, 340, 343.

and colimits, 344.

and limits, 344.

and reflexive equalizers, 344.

and symmetric collections, 339.

associated to plain cooperads, 345–350.

coaugmentation coideal of, 339.

coaugmentation morphisms of, 334, 337.

cofree, 343.

corestriction operators of, 334, 337.

in cochain graded dg-modules, see also coaugmented cochain dg-\( \Lambda \)-cooperads.

symmetric algebras of, 400, 363.

coaugmented cochain dg-\( \Lambda \)-cooperads, 350.

cofibrations of — and free structures, 350.

generating acyclic cofibrations of, 351.

model category of, 351.

coaugmented cochain dg-\( \Lambda \)-cooperads generating acyclic cofibrations of, 351.

model category of, 351.

coaugmented connected covariant \( \Lambda \)-sequences, 351.

cobar construction of coaugmented \( \Lambda \)-cooperads, 301, 521.

cobar-bar resolution of operads, 521, 531.

and augmented connected \( \Lambda \)-operads, 550, 655.

and cosimplicial homotopy spectral sequences of operadic mapping spaces, 558, 559.

and the cotriple cohomology of operads, 523, 529, 537.

and the cotriple resolution, 522, 559, 561.

cochain complexes of cochain graded modules, 388.

cochain complexes of dg-modules, 388, 588.

and twisted dg-modules, 587.

total dg-module of the, 587.

cochain dg-cooperads, 284.

conormalized complex of cosimplicial, 312, 316.

fibrant quasi-cofree, 291.

generating acyclic cofibrations of, 284.

generating acyclic cofibrations of coaugmented, 351.

generating cofibrations of, 284.

generating cofibrations of coaugmented, 351.

model category of, 284, 285.

model category of coaugmented, 351.

simplicial frames of, 302–305.

totalization of cosimplicial, 306–316.

cochain graded dg-modules, 128.

coaugmented, 160.

cofibrations of — over a field, 141.

connected, 139.

connected generating cofibrations of, 139.

conormalized complex of cosimplicial, 500–508.

generating acyclic cofibrations of, 139, 141.

generating cofibrations of, 139, 141.

connected, 139.

model category of, 139, 141.

relative cell complexes of, 143.

tensor products of, 146.

totalization of cosimplicial, 307.

cochain-homotopies, see also contracting cochain-homotopies.

codegeneracies, see also codegeneracy morphisms, 26.

see also codegeneracy operators of the simplices, 28.

codegeneracy morphisms, 26.

codegeneracy operators in a cosimplicial object, 26.

coface morphisms, 26.

coface operators in a cosimplicial object, 26.
cofaces, see also coface morphisms, 26 see also coface operators
of the simplices, 28
cofibrant objects
class of — in a model category, 11

cofibrant resolutions, 13 19
and the homotopy category, 20
of connected Λ-operads in simplicial sets
and the cotriple resolution, 369
of connected operads in simplicial sets
and the cotriple resolution, 321
of non-unitary Λ-operads in simplicial
sets and the cotriple resolution,
complete filtered chain graded dg-modules,
380
continuous dual of, 384
local finiteness of, 384
tensor products of, 384
complete filtered modules, 380
continuous dual of, 383
complete Hopf algebras, 400

group-like elements in, 401
complete Lie algebras, 401
in chain graded modules, see also
complete chain graded Lie algebras
completed tensor products, 381
completeness axioms of model categories,
11
composition products of operads
treewise representation, 624
compositions
and the left lifting property, 10
and the right lifting property, 10
of acyclic cofibrations, 12
of acyclic fibrations, 12
of cofibrations, 12
of fibrations, 12
stability of a class of morphisms under,
transfinite, 9
configuration spaces, 432
cohomology of, 432 434
connected Λ-operads, see also augmented
connected Λ-operads
cofibrant resolutions of — in simplicial
sets and the cotriple resolution, 369
cotriple resolution of — in simplicial sets,
generating acyclic cofibrations of — in
simplicial sets, 367
generating cofibrations of — in simplicial
sets, 367
in simplicial sets, 375
model category of — in simplicial sets,
rationalization of — in simplicial sets,
see also rationalization, of connected
Λ-operads
tower decomposition of — in simplicial
sets, 476
connected A-sequences, 833
c connected covariant Λ-sequences, 833 839
c coaugmented, 839
connected graph complex, see also graph
complex
connected Hopf covariant Λ-sequences, 358
c connected operads, 878
cofibrant resolutions of — in simplicial
sets and the cotriple resolution, 321
cotriple resolution of, see also cotriple
resolution, of operads, 657 659
cotriple resolution of — in simplicial sets, 369

generating acyclic cofibrations of — in simplicial sets, 319

generating cofibrations of — in simplicial sets, 319

model category of — in simplicial sets, 319

connected symmetric sequences, xviii

connected truncation of augmented non-unitary Λ-operads, 369

connected truncation of non-unitary operads, 320

conormalized cochain complex, see also conormalized complex

conormalized complex, 133

and extra-codegeneracies, 161

as a totalization, 137

of cosimplicial cochain dg-cooperads, 312, 316

of cosimplicial cochain graded dg-modules, 316, 445

of cosimplicial modules, 133

of the homotopy of a cosimplicial space, 171

continuous dual

of complete filtered chain graded dg-modules, 354

of complete filtered modules, 383

contracting chain-homotopies, 160

contracting cochain-homotopies, 160

cooperads, 275

and symmetric collections, 279

aritywise tensor product of, 285

bar-cobar resolution of, 290, 301

cartesian products of, 583, 610

coaugmentation coideal of, 277

colimits of, 288

composition coproducts of, 274, 277

limits of, 288

quasi-cofree, 290, 291, 613

reflexive equalizers of, 288

symmetric algebras of, 298

treewise composition coproducts of, 281

029, 031, 034

twisting coderivations of, 290, 291, 643

coproducts and the left lifting property, 101

corestriction operators

of coaugmented Λ-cooperads, 333, 337

of cooperads and treewise tensor products, 634, 637

treewise representation of the — on cooperads, 635, 637

corollas, 624

cosimplicial frames, 51, 51

and geometric realizations, 93

and mapping realizations, 58, 88

of simplicial objects, 10

cosimplicial homology spectral sequences, 608

cosimplicial homotopy spectral sequences, 171, 174, 608

of operadic mapping spaces, 179, 199

and operadic derivations, 508, 512

and the cobar-bar resolution of operads, 583, 610

and the Koszul resolution of operads, 539, 541

terms of the, 172

cosimplicial modules, xv, 129

cohomology of, 133

conormalized complex of, 133

dual of, 137

model category of, 140

tensor products of, 147

cosimplicial objects, 27

and functors on simplicial sets, 46

coaugmented, 160

extra-codegeneracies of, 160

cosimplicial unitary commutative algebras, 153, 155

cohomology of, 161

cotensored categories over simplicial sets, 51

and mapping spaces, 51

cotriple cohomology of augmented connected Λ-operads, see also cotriple cohomology of operads

cotriple cohomology of operads, 483, 498

and Koszul operads, 537

and the cobar-bar resolution of operads, 520, 521

and the Koszul resolution of operads, 520, 521

cotriple resolution

and cofibrant resolutions of connected Λ-operads in simplicial sets, 369

and cofibrant resolutions of connected operads in simplicial sets, 391

and cofibrant resolutions of non-unitary Λ-operads in simplicial sets, 250, 274

and cofibrant resolutions of operads in simplicial sets, 250, 274

and operadic mapping spaces, 179

and the cobar-bar resolution of operads, 522, 559, 561

genealization of the — of operads in simplicial sets, 274

of augmented non-unitary Λ-operads, see also cotriple resolution, of operads, 260

of connected Λ-operads in simplicial sets, 519
of connected operads, see also cotriple resolution, of operads, 559
of connected operads in simplicial sets, 321
of operads, 266, 521, 655–659
counitary comonative coalgebras, xvi
in a symmetric monoidal category, xvii
tensor products of, xvii

covariant Λ-sequences, 333, 337
and symmetric collections, 438
connected, see also connected covariant Λ-sequences
free — over symmetric sequences, see also free covariant Λ-sequences
non-unitary, see also non-unitary covariant Λ-sequences
cylinder objects, 15
good, 127–129
degeneracies, see also degeneracy operators
degeneracy operators, 27
in a simplicial object, 27
degenerate simplices in simplicial sets, 29
derivations on augmented connected Λ-operads, see also operadic derivations
derivations on augmented non-unitary Λ-operads, see also operadic derivations
derived functors, see also Quillen adjunctions, and derived functors, 26
desuspension of dg-modules, 384
dg-algebras of piecewise linear forms, see also Sullivan cochain dg-algebras
dg-modules, xv
chain complexes of, 385
chain graded, see also chain graded dg-modules
chain graded truncation of, 383
cochain complexes of, 385
cochain graded, see also cochain graded dg-modules
desuspension of, 384
dual of, 387
hom-objects of, 383
hom-objects of — and weak-equivalences, 154
homomorphisms of, 159
model category of, 159
morphisms of — and homomorphisms, 154
Reedy model category of cosimplicial objects in, 154
Reedy model category of simplicial objects in, 149

suspension of, 383
tensor products of, 140
diagonalization of bisimplicial sets, 58

dimension grading, 463
Dold–Kan equivalence, 131
and additive Λ-operads, 116
and additive operads, 116
Dold–Kan functor, 131, 133
and the Sullivan cochain dg-algebras, 194
Drinfeld’s associators, 411

tower decomposition of the set of, 561
563
Drinfeld–Kohno Lie algebra operad, 396, 397
and the cohomology of the little 2-discs operad, 394
Chevalley–Eilenberg cochain complexes of the, 125, 129
tower decomposition of the, 125, 129
complete, 447
the cofree Λ-sequence structure of the,
complete enveloping algebras of the, 418
graded versions of the, see also graded Drinfeld–Kohno Lie algebra operads, 418
Maurer–Cartan spaces of the, 429
and the classifying spaces of the chord diagram operad, 413
the complete, 447
weight decomposition of the, 447
Drinfeld–Kohno Lie algebras, 414
and the cohomology of configuration spaces, 434
complete enveloping algebras of the, 418
graded versions of the, 418
the complete, 447
dual Λ-cooperad of a Λ-operad, 336
Λ-operad of a Λ-cooperad, 336
and hom-objects, xvii
continuous, see also continuous dual cooperad of an operad, 276
cooperad of the Gerstenhaber operads, 183
Hopf cooperad of a Hopf operad, 256
Hopf operad of a Hopf cooperad, 256
of cosimplicial modules, 137
of dg-modules, 157
of simplicial modules, 137
operad of a cooperad, 276
Dupont’s homotopy, 192, 194
edge contractions in a tree, 621
edge set of a tree, 619
Eilenberg–MacLane map, 148, 159
cosimplicial, 149, 151
Eilenberg–MacLane spaces, 198, 402
cohomology of, 199
Sullivan cochain dg-algebras of, 201
Eilenberg–Zilber equivalence, 148–150
and symmetric algebras, 168
and the Sullivan cochain dg-algebras, 160–192
and unitary commutative algebras, 166
cosimplicial, 150–151, 308–312
on hom-objects, 156
of \( A \)-sequences, 517
of symmetric sequences, 517
enriched categories, xviii
extension functor for algebras over operads, 111–112
extra-codegeneracies, 160
and cohomology, 162
extra-degeneracies, 159
and homology, 160
face operators, 27
in a simplicial object, 27
faces, see also face operators
factorization axioms
and cofibrantly generated model categories, 112
factorization axioms of model categories, 11
factorization homology, 612
fibrant objects
class of — in a model category, 11
fibrant resolutions, 13–19
and the homotopy category, 20
of operads in simplicial sets, 230
fibrations, 11
and adjunctions, 12
and compositions, 12
and pullbacks, 12
and the right lifting property, 11
class of — in a model category, 11
Kan, 83
filtered colimits, see also \( \kappa \)-filtered colimits
formality
intrinsic — of the \( E_n \)-operads, 436–438
of the \( E_2 \)-operads, 414–416
Kontsevich’s — result for \( E_n \)-operads, 436
of the \( E_2 \)-operads, 424
of the chain little 2-discs operad, 426
of the chain little discs operads, 438
of the little 2-discs operad, 434
of the little discs operads, 438–439
Tamarkin’s — result for \( E_2 \)-operads, 445
frames
cosimplicial, see also cosimplicial frames
simplicial, see also simplicial frames
free covariant \( A \)-sequences, 448–449
cofibrations of cosimplicated chain dg-A-cooperads, 550
and cooperads, 518
and — of a cofree \( \Lambda \)-sequence, 224
over symmetric sequences, see also free
of symmetric sequences, 630
function objects, see also cotensored
categories over simplicial sets, 50
and mapping spaces, 51
generating acyclic cofibrations, see also the
corresponding entry of each category for the definition of the set of
generating acyclic cofibrations associated to specific examples of model categories, 110–117
generating cofibrations, see also the
corresponding entry of each category for the definition of the set of
generating cofibrations associated to specific examples of model categories, 110–117
geometric realization
of Hopf cochain dg-\( \Lambda \)-cooperads, 370
of Hopf cochain dg-cooperads, 822–823
of simplicial sets, 25, 50
and operads, 10
of unitary commutative cochain
dg-algebras, see also Sullivan cochain
dg-algebras, adjoint functor of, 197
geometric realizations
and cosimplicial frames, 93
and cosimplicial-simplicial objects,
and singular complexes, 94
and the diagonalization of bisimplicial
sets, 90
and weak-equivalences, 93
in model categories, 92
in simplicial model categories, 93
deformation retraction of, 94
in model categories, 90
Koszul dual cooperad of the, 547
Koszul resolution of the, 519
monomial basis of the, 515
operadic cotriple cohomology of the, 550
presentation of the — by generators and
relations, 555
good \( \Lambda \)-operads, 397
good \( \Lambda \)-operads with respect to the
rationalization, see also good
\( \Lambda \)-operads
good operads, 823–824
and operads with respect to the
rationalization, see also good operads
good spaces, 207–208
INDEX

and Λ-operads, 374
and operads, 331

good spaces with respect to the rationalization, see also good spaces
Goodwillie–Weiss Taylor tower, 607
graded Drinfeld–Kohno Lie algebra operads, 418
and the cohomology of the little discs operads, 332
Chevalley–Eilenberg cochain complexes of the, 424–429
tower decomposition of the, 424–429
cofree Λ-sequence structure of the, 422–424
Maurer–Cartan spaces of the, 429
tower decomposition of the, 424–429
graded Drinfeld–Kohno Lie algebras, 418
and the cohomology of configuration spaces, 444
graded Grothendieck–Teichmüller Lie algebra, 562, 563, 605
graded modules, xv, 128

graph complex, 601
hairy, 600
homology of the — and the graded Grothendieck–Teichmüller Lie algebra, 605
graph cooperads, 593
graph dg-algebras, 591
graph operads, 594
and Gerstenhaber operads, 594
Grothendieck–Teichmüller group, 550, 560
and the homotopy automorphism space of $E_2$-operads, 565
filtration of the, 562
pro-finite, 605
group-like elements
and the exponential correspondence, 401
in a complete Hopf algebra, 401
hairy graph complex, 600
Harrison homology, 209
hom-objects, xvii
of Λ-sequences, 510
of Λ-sequences in dg-modules, 513
and the Künneth isomorphism formula, 517
and weak-equivalences, 514
of Λ-sequences in graded modules, 513
of symmetric sequences in graded modules, 513
of symmetric sequences in simplicial modules, 514
and the Eilenberg–Zilber equivalence, 517
of symmetric sequences in graded modules, 513
of symmetric sequences in simplicial modules, 514
of symmetric sequences in simplicial model categories, 555

homotopy
and weak-equivalences, 18
classes of morphisms, 17
left — relation, 16–18
relation, 17, 19
relation and compositions, 18
right — relation, 16–18
homotopy automorphism space of $E_2$-operads, 565
and the classifying space of the additive group, 582–586
and the Grothendieck–Teichmüller group, 565, 580
main result on the, 580
homotopy automorphism spaces of $E_n$-operads, 604
homotopy automorphism spaces, 57
and weak-equivalences, 59
homotopy automorphisms of operads, 13
homotopy category, 89
and cofibrant resolutions, 20
and fibrant resolutions, 20
and mapping spaces, 55
of a model category, 140, 222
of simplicial model categories, 55
homotopy equivalences in a model category, 18
homotopy exact sequences associated to a tower of fibrations, 404
derived — associated to a tower of fibrations, 165, 187
homotopy groups
and Harrison homology, 209
and unitary commutative cochain
dg-algebras, 209
homotopy spectral sequences
and operadic mapping spaces, 477
of a tower of fibrations, 467–471
of cosimplicial spaces, see also
cosimplicial homotopy spectral
sequences
terms of the — of a tower of fibrations,
467–468
Hopf Λ-cooperads, 358
and cofree cooperads, 359
cofree, 359
in cochain graded dg-modules, see also
Hopf cochain dg-Λ-cooperads
Hopf cochain dg-Λ-cooperads, 357
and augmented connected Λ-operads,
548, 549
and cosimplicial homotopy spectral
sequences of operadic mapping spaces,
538, 539
and operadic derivations, 531
and the cotriple cohomology of operads,
537
latching morphisms
cosimplicial, 67
double, 101
relative cosimplicial, 72
relative simplicial, 58
simplicial, 77
Hopf cochain dg-cooperads
cofibrations of, 301
conormalized complex of cosimplicial,
317
generating acyclic cofibrations of, 298
generating cofibrations of, 298
geometric realization of, 370
model category of, 368, 369, 370
Hopf cochain dg-cooperads
cofibrations of, 291
cofree, 296
colimits of, 297
limits of, 297
Hopf covariant Λ-sequences, 358
connected, 358
Hopf symmetric sequences, 294
colimits of, 297
horizontal degree, 463
horizontal grading, 463
of the bar construction of operads, 665
horizontal weak-equivalences of bisimplicial
sets, 88, 100
horns of the simplices, 29, 54
ingoing edges
of a tree, 619
of a vertex in a tree, 619
inner edges
of a tree, 619
intrinsic formality of the $E_n$-operads,
436, 438, 502
iterated loop spaces, 611
Künneth isomorphism formula, 451
and symmetric algebras, 165
on hom-objects, 458
and Λ-sequences, 517
and symmetric sequences, 517
Künneth morphisms, 451
on hom-objects, 158
Kan complexes, 34
Kan fibrations, 34
Koszul construction, 666
Koszul dual cooperad of an operad, 539
and augmented connected Λ-operads,
548, 549
and cosimplicial homotopy spectral
sequences of operadic mapping spaces,
538, 539
and operadic derivations, 531
and the cotriple cohomology of operads,
537
left derived functor of a Quillen adjunction,
23
left homotopies, 16, 18
and compositions, 18
left lifting property, 9
and compositions, 10
and coproducts, 10
and pushouts, 10
and retracts, 10
left proper model categories, 35
lifting axioms of model categories, 11
lifting property
left, see also left lifting property
right, see also right lifting property
little 2-discs operad
formality of the, 445
little $n$-discs operad, see also little discs
operads, 111
little discs operads
and configuration spaces, 432
cohomology cooperads of the, 432
cohomology of the, 435
formality of the, 435
homology of the, 432
local coefficient system operads, 475
locally κ-presentable categories, see also
locally presentable categories
"
locally presentable categories, 121

Malcev complete groups, 400
   and Maurer–Cartan spaces, 408
   filtration of, 400
   tower decomposition of, 400

mapping spaces
   and cosimplicial frames, 85–88
   and function objects, 51
   and homotopy classes, 55, 87
   and Quillen adjunctions, 89
   and simplicial frames, 85–88
   and tensor products over simplicial sets, 54
   in model categories, 58, 87
   in simplicial categories, 48
   of operads in simplicial sets, 62, 270
   of simplicial modules, 156

mapping spaces of $E_n$-operads, 604, 612
mapping spaces of abelian bimodules over $E_n$-operads, 610

matching morphisms
   cosimplicial, 69
   double, 102
   relative cosimplicial, 72, 80
   simplicial, 77

matching objects
   cosimplicial, 69
   double, 102
   simplicial, 77

Maurer–Cartan equation, 393
Maurer–Cartan spaces, 393, 506, 399
   and the geometric realization of the
   Chevalley–Eilenberg cochain complexes, 394
   of complete Lie algebras and classifying spaces, 408
   of the Drinfeld–Kohno Lie algebra operad
   and the classifying spaces of the chord
diagram operad, 413
   tower decomposition of, 394

minimal models, 178

Mittag–Leffler convergence of spectral sequences, 170

model categories, see also the corresponding entry of each category for specific examples
   axioms of, 111 111
   cofibrantly generated, 112
   combinatorial, 121 123
   defined by transfer from cofibrantly generated model categories, 117 120
   homotopy category of, 119 20 240
   left proper, 35
   right proper, 55

   simplicial, see also simplicial model categories, 52
   morphisms, XVII

   nilpotent cell complexes of unitary commutative cochain dg-algebras of finite type, 204 207
   nilpotent spaces of finite Q-type, 199 207
   rationalization of, 204
   Sullivan cochain dg-algebras of, 200
   non-degenerate simplices in a simplicial set, 32

   non-unitary $\Lambda$-operads, see also augmented non-unitary $\Lambda$-operads, 213
   cofibrant resolutions of — in simplicial sets and the cotriple resolution, 249 271
   generating acyclic cofibrations of — in simplicial sets, see also augmented non-unitary $\Lambda$-operads, generating acyclic cofibrations of generating cofibrations of — in simplicial sets, see also augmented non-unitary $\Lambda$-operads, model category of non-unitary $\Lambda$-sequences, 531
   $\Lambda$-co skeleta of, 248 201
   boundary of the free, 249 247 307
   cofibrations of — as cofibrations of non-unitary symmetric sequences, 292
   cofibrations of — in dg-modules, 509
   cofibrations of — in simplicial modules, 509
   free, 249 271 509
   generating acyclic cofibrations of, 248
   generating cofibrations of, 248
   matching morphisms of, 249
   matching objects of, 241 212 507
   model category of, 243 251
   model category of — in dg-modules, 507
   model category of — in simplicial modules, 507
   model category of — in simplicial sets, see also non-unitary $\Lambda$-sequences, model category of
   Reedy model category of, see also non-unitary $\Lambda$-sequences, model category of
   non-unitary covariant $\Lambda$-sequences, 335 337
   non-unitary operads, XVIII
   cell attachments of — in simplicial sets, 209
   cofibrant — in simplicial sets, 230
   cofibrant resolutions of — and the cotriple resolution, 209 211
INDEX

connected truncation of, 320

cotriple resolution of, see also cotriple resolution, of operads, 266

fibrant resolutions of — in simplicial sets, 220

generating acyclic cofibrations of, 224

generating cofibrations of, 227

model category of, 227, 228, 234

path objects of — in simplicial sets, 221

quasi-free — in simplicial sets, 255, 257

relative cell complexes of — in simplicial sets, 230

non-unitary symmetric sequences, xviii

cell attachments of, 219

cell attachments of — in simplicial sets, 220

cofibrant — in simplicial sets, 226

free, 216, 218

generating acyclic cofibrations of, 216

generating cofibrations of, 216

model category of, 216, 217, 231

quasi-free — in simplicial sets, 222, 220

relative cell complexes of, 219, 221

relative cell complexes of — in simplicial sets, 220

normalized chain complex, see also

normalized complex

and extra-degeneracies, 169

as a geometric realization, 136

of simplicial dg-modules, 187

of simplicial groups, 131

of simplicial modules, 130

of simplicial unitary commutative algebras, 166

operadic biderivations, 508

bicomplex of, 508

bicomplex of — on the dual cooperad of the n-Gerstenhaber operad, 602

operadic compositions of trees, 622

operadic cotriple cohomology, 608, 609

of the Gerstenhaber operads, 550

with coefficients in the Drinfeld–Kohno Lie algebra operad, 551, 557

with coefficients in the graded Drinfeld–Kohno Lie algebra operads, 551, 557

operadic derivations, 491

and abelian bimodules over operads, 491

and abelian group objects over operads, 492

and cosimplicial homotopy spectral sequences of operadic mapping spaces, 504

of operads, 504

and mapping spaces of operads, 504

and the Koszul resolution of operads, 531

dg-modules of, 191

on free operads, 193, 198

on the cobar-bar resolution of operads, 535, 537

on the cotriple resolution of augmented connected A-operads, see also operadic derivations, on the cotriple resolution of operads

on the cotriple resolution of operads, 533

on the Koszul resolution of operads, 533, 537

on the Koszul resolution of the Gerstenhaber operads, 550

simplicial modules of, 491

operadic Hochschild cohomology, 499

operadic mapping spaces

cosimplicial homotopy spectral sequences of, 179, 499, 505, 600

and operadic derivations, 503, 504

on $E_n$-operads, see also mapping spaces on $E_n$-operads

operadic suspension, 544

operads, xviii

additive, see also additive operads and symmetric collections, xxxii

cobar-bar resolution of, 650, 652

connected, see also connected operads
totensor category structure of — in simplicial sets, 60

totensor category structure of — in topological spaces, 60

totensor category structure of — in simplicial sets, 60

function objects on — in simplicial sets, 60

function objects on — in topological spaces, 60

homotopy automorphisms of, 143, 144

in simplicial sets as simplicial operads in sets, 215, 217

mapping spaces of — in simplicial sets, 217

model category of, see also

non-unitary operads, model category of, 224

model category of — in simplicial sets, 50

model category of — in topological spaces, 50

non-unitary, see also non-unitary operads

quasi-free, 547

rationalization of — in simplicial sets, see also rationalization, of operads

simplicial model category of — in simplicial sets, 62, 240

simplicial model category of — in topological spaces, 62
tensored category structure of — in simplicial sets, 61

tensored category structure of — in topological spaces, 61

twisting derivations of, 647

outgoing edge

of a tree, 619
of a vertex in a tree, 619

parenthesized braid operad, 411, 443, 560

classifying spaces of the, 443
Malcev completion of the, 560
universal property of the, 564

path objects, 15

of operads in simplicial sets, 231

Poisson operad

graded versions of the, see also Gerstenhaber operads

Postnikov towers

Sullivan cochain dg-algebras of, 203

presentable categories, see also locally presentable categories

principal fibrations, 105

and classifying spaces, 406
in the tower decomposition of Maurer–Cartan spaces, 597
in the tower decomposition of the classifying spaces of Malcev complete groups, 406

in the tower decomposition of the classifying spaces of the chord diagram operad, 449

in the tower decomposition of the Maurer–Cartan spaces of the Drinfeld–Kohno Lie algebra operad, 451

in the tower decomposition of the Maurer–Cartan spaces of the graded Drinfeld–Kohno Lie algebra operads, 451

Sullivan cochain dg-algebras of, 201

pro-finite Grothendieck–Teichmüller group, 606

pro-unipotent Grothendieck–Teichmüller group, see also Grothendieck–Teichmüller group

products and the right lifting property, 10
projective model category of operads, see also non-unitary operads, model category of, 215

projective model category of symmetric sequences, see also non-unitary symmetric sequences, model category of, 216

pullback-corner morphisms, 59, 51
pullbacks, 9

and acyclic fibrations, 12

and fibrations, 12

and the right lifting property, 10

stability of a class of morphisms under, 10

pure braid groups, 581

centers of the, 581

rationalization of the classifying spaces of the, 586

pushout-corner morphisms, 59, 51

pushouts, 9

and acyclic cofibrations, 12

and cofibrations, 12

and the left lifting property, 10

stability of a class of morphisms under, 10

quasi-cofree cooperads, 290

quasi-free operads, 617

operads in simplicial sets, 235

symmetric sequences in simplicial sets, 222

unitary commutative cochain dg-algebras, 177

quasi-free extensions of operads in simplicial sets, 235

of symmetric sequences in simplicial sets, 222

Quillen adjoint functors, see also Quillen adjunctions

Quillen adjunctions, 13

and cofibrantly generated model categories, 116

and derived functors, 23

and mapping spaces, 89

and model categories of algebras over operads, 42

Quillen derived adjoint functors, see also Quillen adjunctions, and derived functors

Quillen equivalences, 12

and model categories of algebras over operads, 12

Quillen model, 592

rational A-operads, 374

rational operads, 331

rational spaces, 207

and A-operads, 331

and operads, 331

rationalization and mapping spaces, 209

of E2-operads, 146

and the chord diagram operad, 146

of En-operads, 138

and the graded Drinfeld–Kohno Lie algebra operads, 138

and connected A-operads, 331
and mapping spaces, 374
of good spaces, 208
and $\Lambda$-operads, 374
of operads, 208
of nilpotent spaces, 208
and mapping spaces, 331
of spaces, 197
of the little 2-discs operad, 446
and the chord diagram operad, 446
reduced trees, 610
Reedy acyclic cofibrations
cosimplicial, 73
Reedy acyclic fibrations
cosimplicial, 73
Reedy cofibrations, see also Reedy model categories
cosimplicial, 72 74
simplicial, 80
Reedy fibrations, see also Reedy model categories
cosimplicial, 72 74
simplicial, 80
Reedy indexing categories, 66
Reedy model categories
of cosimplicial objects, 72 74
of simplicial objects, 80
Reedy’s patching lemma, 73
reflexive equalizers, 274
relative cell complexes
and acyclic cofibrations, 112
and cofibrations, 112
in model categories, 110
of cochain graded dg-modules, 143
of generating acyclic cofibrations in model categories, 110
of generating cofibrations in model categories, 110
of non-unitary operads in simplicial sets, 210 230
of non-unitary symmetric sequences, 210 221
of non-unitary symmetric sequences in simplicial sets, 223 226
of simplicial sets, 115 116
of topological spaces, 113
of unitary commutative cochain dg-algebras, 177 177
relative tensor products of unitary commutative algebras, 169
resolutions, see also the name of the resolution for specific resolution constructions
cofibrant, 140 140
fibrant, 10 10
restriction functor for algebras over operads, 41 42
restriction operators on trees, 628
retract axiom of model categories, 11
retracts, 0
and the left lifting property, 10
and the right lifting property, 10
stability of a class of morphisms under, 10
right derived functor of a Quillen adjunction, 23
right homotopies, 16 18
and compositions, 18
right lifting property, 10
and compositions, 10
and products, 10
and pullbacks, 10
and retracts, 10
right proper model categories, 55
semi-alternate two-colored trees, 638
treewise tensor products over, 638
semi-direct products over cooperads, 116
semi-direct products over operads, 489
semi-model categories, 58
simplices, 28 39
boundary of the, 29 34 35 68
horns of the, 29 34
topological, 28
simplicial categories, 48
simplicial category (the indexing category), 26 86
simplicial frames, 53 81 84 85
and mapping spaces, 85 88
and totalizations, 97
of cochain dg-cooperads, 302 305
of cosimplicial objects, 91
simplicial model categories, see also the corresponding entries at simplicial sets, topological spaces, operads, and algebras over operads for specific examples, 52
axioms of, 52
homotopy category of, 55
simplicial modules, 5x 129
dual of, 137
hom-objects of, 155
and the Eilenberg–Zilber equivalence, 156
homology of, 130
mapping spaces of, 156
normalized complex of, 130
tensor products of, 147
simplicial monoids, 52
simplicial objects, 24
and functors on simplicial sets, 46
augmented, 169
extra-degeneracies of, 159
simplicial sets
cotensored category structure of, see also simplicial sets, function objects on, degenerate simplices in, function objects on, functors on, generating acyclic cofibrations of, generating cofibrations of, geometric realization of, and operads, homotopy category of — and topological spaces, model category of, relative cell complexes of, simplicial model category of, skeletons of, tensored category structure of, vertices in, simplicial unitary commutative algebras, homology of, singular complex of topological spaces, Sinha’s cosimplicial space, skeletons cosimplicial, cosimplicial — of the simplices, of operads in simplicial sets, of simplicial sets, of the simplices, simplicial, small object, argument, see also small object argument with respect to $\kappa$-filtered colimits, with respect to a class of relative cell complexes, with respect to a class of transfinite composites, small object argument, and generating acyclic cofibrations, and generating cofibrations, spaces of embeddings with compact support, spaces of embeddings with compact support modulo immersions, spaces of long knots, spaces of long knots modulo immersions, stability of a class of morphisms under compositions, under coproducts, under products, under pullbacks, under pushouts, subtrees, Sullivan cochain dg-algebras, acyclicity of the, adjoint functor of, see also geometric realization, of unitary commutative cochain dg-algebras, and $\Lambda$-operads, see also Sullivan cochain dg-algebras, operadic upgrade of the, and $\Lambda$-operads and operads, see also Sullivan cochain dg-algebras, operadic upgrade of the and simplicial frames, the Dold–Kan functor, and the Dupont homotopy, and the Eilenberg–Zilber equivalence, and the rationalization of spaces, and totalizations, codiagonal map on the, integration map on the, of Eilenberg–MacLane spaces, of nilpotent spaces, of Postnikov towers, of principal fibrations, operadic upgrade of the, and $\Lambda$-operads, Sullivan model, see also Sullivan cochain dg-algebras suspension of an operad, see also operadic suspension suspension of dg-modules, symmetric algebras, and coaugmented $\Lambda$-cooperads, and cooperads, and the Eilenberg–Zilber equivalence, and the K"unneth isomorphism formula, of additive $\Lambda$-cooperads, symmetric collections, symmetric monoidal categories, symmetric monoidal category of chain graded dg-modules, of cochain graded dg-modules, of cosimplicial modules, of dg-modules, of simplicial modules, symmetric sequences, aritywise tensor product of, cofibrations of — in dg-modules, cofibrations of — in simplicial modules, connected, generating acyclic cofibrations of, generating cofibrations of, hom-objects of, hom-objects of — in dg-modules,
and the Künneth isomorphism
formula, 617
and weak-equivalences, 514
hom-objects of — in graded modules, 513
hom-objects of — in simplicial modules, 516
and the Eilenberg–Zilber equivalence, 517
homomorphisms of, 512
homomorphisms of — in dg-modules, 513
homomorphisms of — in graded modules, 513
model category of, see also non-unitary symmetric sequences, model category of
model category of — in dg-modules, 505
model category of — in simplicial modules, 505
non-unitary, xviii
tensor products
and terminal objects, 274
aritywise — of cooperads, 295
aritywise — of symmetric sequences, 295
completed, 381
distribution of — over colimits, xvi
distribution of — over limits, 274
distribution of — over reflexive equalizers, 274
of chain graded dg-modules, 146
of cochain graded dg-modules, 146
of complete filtered chain graded dg-modules, 383
of cosimplicial modules, 147
of counitary cocommutative coalgebras, xvi
of dg-modules, 146
of simplicial modules, 147
of unitary commutative algebras, xvi
treewise, 620
tensor products over simplicial sets, see also tensored categories over simplicial sets, 50
and mapping spaces, 54
tensored categories over simplicial sets, 50
and mapping spaces, 51
topological chiral homology, 612
topological spaces
cotensored category structure of, see also topological spaces, function objects on,
function objects on, 34
generating acyclic cofibrations of, 113
generating cofibrations of, 113
homotopy category of — and simplicial sets, 57
model category of, 26
relative cell complexes of, 118
simplicial model category of, 49
singular complex of, 36
tensored category structure of, 51
total degree, 463
of a chain complex of dg-modules, 486
of a cochain complex of dg-modules, 487
of a cochain complex of dg-modules, 187
total grading, 463
of the bar construction of operads, 605
totalizations
and cosimplicial-simplicial objects, 106
and simplicial frames, 94
and weak-equivalences, 97, 106
in model categories, 96
in simplicial model categories, 98
of cosimplicial cochain dg-cooperads, 305, 316
of cosimplicial cochain graded dg-modules, 307
of cosimplicial Hopf cochain dg-cooperads, 307
of cosimplicial Hopf cochain dg-cooperads, 117
tower decomposition of, 96, 98
tower decomposition of $E_2$-operads, 480
of $E_n$-operads, 481
of connected $\Lambda$-operads in simplicial sets, 175
of Malcev complete groups, 100
of Maurer–Cartan spaces, 395
of the Chevalley–Eilenberg cochain complexes, 389, 393
of the Chevalley–Eilenberg complexes of the Drinfeld–Kohno Lie algebra operad, 425, 429
of the Chevalley–Eilenberg complexes of the graded Drinfeld–Kohno Lie algebra operands, 125, 129
of the classifying spaces of Malcev complete groups, 100
of the classifying spaces of the chord diagram operad, 140, 148
of the Maurer–Cartan spaces of the Drinfeld–Kohno Lie algebra operad, 129
of the Maurer–Cartan spaces of the graded Drinfeld–Kohno Lie algebra operads, 129
of the set of Drinfeld’s associators, 501
of the set of Drinfeld’s associators, 501
trees, 619
trees, 619
tree morphisms, 623
binary, 664
edge set of, 619
ingoing edges of, 619
inner edge set of, 619
isomorphisms of, 619
operadic compositions of, 622
outgoing edge of, 619
reduced, 619
restriction operators on, 623
semi-alternate two-colored, 638
unit, 621
vertex set of, 619
with one vertex, see also corollas, 621
with two vertices, 621
treewise composition coproducts of cofree cooperads, 631–633
treewise composition coproducts of cooperads, 629–631, 635
treewise tensor products, 620
and cofree cooperads, 626, 633
and composition products of operads, 621
coaugmentation morphisms on, 635–637
corestriction operators on, 635–637
over semi-alternate two-colored trees, 638
truncation
chain graded — of dg-modules, 153
connected — of augmented non-unitary operads Ly-operads, 369
connected — of non-unitary operads, 320
twisted dg-modules
and chain complexes of dg-modules, 486
and cochain complexes of dg-modules, 487
twisting coderivations of cooperads, 290, 293, 543
twisting derivations of operads, 647
twisting derivations of unitary commutative cochain dg-algebras, 172
twisting differential
of the bar construction of operads, 644
of the cobar construction of cooperads, 648
twisting homomorphisms, 143
of covariant A-sequences, 834
two-out-of-three axiom of model categories, 11
unit tree, 621
unital operads, see also unitary operads, xvi
unitary commutative algebras, xvi
colimits of, 160
coproducts of, 160
free, see also symmetric algebras
in a symmetric monoidal category, xvi
in chain graded dg-modules, see also
unitary commutative chain dg-algebras
in cochain graded dg-modules, see also
unitary commutative cochain dg-algebras
in cosimplicial modules, see also
cosimplicial unitary commutative algebras
in simplicial modules, see also simplicial unitary commutative algebras
limits of, 160
pushouts of, see also unitary commutative algebras, relative tensor products of
relative tensor products of, 169
tensor products of, xvi
unitary commutative chain dg-algebras, 163, 165
homology of, 167
unitary commutative cochain dg-algebras, 163, 165
cell attachments of, 173, 175
cofibrations of, 175
cohomology of, 167
connected, 167, 172, 176
generating acyclic cofibrations of,
172, 176
quasi-free, 177
relative cell complexes of, 176, 177
relative tensor products of, 176
simplicial frames of, 187
totalization of, 188
twisting homomorphisms, 143
unitary connected operads, xix, xxi
Vassiliev’s homology spectral sequence, 607
vertex set of a tree, 619
vertical degree, 463
vertical grading, 463
vertical weak-equivalences of bisimplicial sets, 88
vertices in simplicial sets, 240
weak-equivalences, see also the entry of each example of model category for the
definition of the class of weak-equivalences in particular categories, 11
class of — in a model category, 11
weight graded chain graded dg-modules, 580
weight grading, 463
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<thead>
<tr>
<th>Title</th>
<th>Authors</th>
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The ultimate goal of this book is to explain that the Grothendieck–Teichmüller group, as defined by Drinfeld in quantum group theory, has a topological interpretation as a group of homotopy automorphisms associated to the little 2-disc operad. To establish this result, the applications of methods of algebraic topology to operads must be developed. This volume is devoted primarily to this subject, with the main objective of developing a rational homotopy theory for operads.

The book starts with a comprehensive review of the general theory of model categories and of general methods of homotopy theory. The definition of the Sullivan model for the rational homotopy of spaces is revisited, and the definition of models for the rational homotopy of operads is then explained. The applications of spectral sequence methods to compute homotopy automorphism spaces associated to operads are also explained. This approach is used to get a topological interpretation of the Grothendieck–Teichmüller group in the case of the little 2-disc operad.

This volume is intended for graduate students and researchers interested in the applications of homotopy theory methods in operad theory. It is accessible to readers with a minimal background in classical algebraic topology and operad theory.