

Mathematical  
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Volume 279

# Automorphic Forms Beyond $GL_2$

Lectures from the 2022  
Arizona Winter School

Ellen Elizabeth Eischen  
Wee Teck Gan  
Aaron Pollack  
Zhiwei Yun  
Hang Xue, Editor



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Providence, Rhode Island

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## Preface

Hang Xue, editor

The Southwest Center for Arithmetic Geometry (SWC) was founded in 1997 by a group of seven mathematicians working in the southwest United States, and has been continuously supported by the National Science Foundation since that time. In the beginning, the SWC served as a true *center* for Arithmetic Geometry in the Southwest, but survives today in name only, having been subsumed by its principal activity, the *Arizona Winter School* (AWS).

The AWS was started with the ambitious goal of creating an intense and immersive workshop in which graduate students – especially those who may not be studying at traditional centers of number theory – would work under the guidance of leading experts to solve research-level problems at the forefront of the field. The very first school was held in the Spring of 1998 under the title “Workshop on Diophantine Geometry Related to the ABC Conjecture.” In the twenty-one years that have followed, the AWS has been held annually each March on a different topic in arithmetic geometry and related areas, and has become a pillar of graduate education and training in these subjects throughout the country and abroad.

Over the years, the Arizona Winter School model has been adjusted and refined to meet the needs of an ever-growing and increasingly diverse audience: the five-day meeting, organized around a different central topic each year, now features a set of four lecture series by leading and emerging experts. A month before the school begins, each speaker proposes one or more research projects related to their lectures, and is assigned 10-15 graduate students who will work on these projects. At that time, speakers also provide detailed lecture notes for their courses. During the school, students attend lectures daily from 9am to 5pm, and work each evening (often into the early hours of the morning!) with speakers and designated assistants on these research projects. Students not assigned to these *research project groups* have the option to join one of two *problem sessions* devoted to solving advanced exercises related to the lecture series, or one of four *study groups* which focus on understanding the course lecture notes in detail; these additional activities are supervised by young researchers and allow students not assigned to one of the research projects to meaningfully engage with the workshop material on many levels. On the last day, the students from each research project group present their work to the whole school. The result is an extremely focused and immersive five days of mathematical activity for all.

In 2016, we began using specialized educational software expressly made for recording lectures together with dual HD web-cameras (one for the speaker and one for the primary document camera). The result has been a marked improvement



in the quality and availability of the lecture videos. The lectures are live-streamed with a rich viewing experience where the user can toggle between the speaker feed and the document camera, or view both simultaneously in several different formats. This enhancement allows remote participation in the workshop which lowers the barrier for entry to the AWS for students from a broad variety of circumstances. At AWS 2022, we offered problem sessions and study groups both in the virtual format thereby allowing virtual participation in the evening sessions.

This volume is comprised of the lecture notes which were prepared for the twenty-fifth Arizona Winter School on “Automorphic Forms beyond  $GL(2)$ ,” held March 5–9, 2022 at the University of Arizona in Tucson. The speakers were Ellen Eischen, Wee Teck Gan, Aaron Pollack, and Zhiwei Yun. We are greatly indebted to these authors for their hard work in making both the twenty-fifth AWS and this proceedings volume a reality. Akshay Venkatesh gave opening and closing lectures at the Winter School, which discussed the analogy between automorphic forms and topological field theories. We thank the National Science Foundation (NSF) for their longstanding and continued support of the Arizona Winter School, the Clay Mathematics Institute for their partnership in organizing the 2022 AWS, and the University of Arizona Department of Mathematics for their support. The editor is partially supported by the National Science Foundation (grant DMS #1901862) while editing this volume. Finally, we owe a great deal to the other members of the Southwest Center, both past and present, for their effort, perseverance, and vision in running the Arizona Winter School for more than twenty years, and for helping it to become the one-of-a-kind workshop that it is.

Hang Xue



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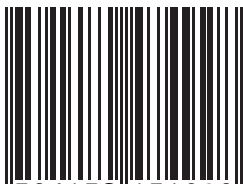
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The Langlands program has been a very active and central field in mathematics ever since its conception over 50 years ago. It connects number theory, representation theory and arithmetic geometry, and other fields in a profound way. There are nevertheless very few expository accounts beyond the  $GL_2$  case. This book features expository accounts of several topics on automorphic forms on higher rank groups, including rationality questions on unitary group, theta lifts and their applications to Arthur's conjectures, quaternionic modular forms, and automorphic forms over function fields and their applications to inverse Galois problems. It is based on the lecture notes prepared for the twenty-fifth Arizona Winter School on "Automorphic Forms Beyond  $GL_2$ ", held March 5–9, 2022, at the University of Arizona in Tucson. The speakers were Ellen Eischen, Wee Teck Gan, Aaron Pollack, and Zhiwei Yun.

The exposition of the book is in a style accessible to students entering the field. Advanced graduate students as well as researchers will find this a valuable introduction to various important and very active research areas.

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