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# Fourier Series

Rajendra Bhatia

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# Fourier Series

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# Preface

These notes provide a quick and brief introduction to Fourier Series. The emphasis is not only on the mathematics but also on the history of the subject, its importance, its applications and its place in the rest of science.

I first learnt about Fourier Series as a student of physics. Together with several other assorted topics, they formed a ragbag course called Mathematical Physics from which, when the time came, *real* physics courses would pick what they wanted. A little later, as a student of mathematics I came across Fourier Series in the middle of a course on Mathematical Analysis. On each occasion, my teachers and my books (all good) managed to keep a secret which I learnt later. Fourier Series are not just tools for the physicist and examples for the mathematician. They are directly responsible for the development of nearly one half of mathematical analysis over the last two centuries.

These notes have been consciously designed to reveal this aspect of the subject and something more. The development of Fourier Series is illustrative of a recurrent pattern in modern science. I hope the reader will see this pattern emerge from our discussion.

This book can be used by a variety of students. Mathematics students at the third year undergraduate level should be able to follow *most* of the discussion. Typically, such students may have had their first course in Analysis (corresponding to Chapters 1–8 of *Principles of Mathematical Analysis* by W. Rudin) and have a good working knowledge of complex numbers and basic differential equations. Such students can learn about Fourier Series from this book and, at the same time, reinforce their understanding of the analysis topics mentioned above. More preparation is required for reading Chapter 4, a part of Section 2.5, Sections 5.2 and 5.4. These parts presuppose familiarity with Lebesgue spaces and elements of Functional Analysis usually taught in the fourth year of an undergraduate or the first year of a graduate program. At many places in the book the reader will see a statement like "*Let $f$ be a continuous or, more generally, an integrable function.*" Here there is a choice. If it appears easier to handle continuous functions, the reader need not be worried about discontinuous ones at this stage.

Thus the material in this book can be used either to augment an Analysis course or to serve as the beginning of a special course leading to more advanced topics in Harmonic Analysis. It can also be used for a reading project. Some readers may be happy reading just Chapter 0 outlining the history of the subject; in some sense that captures the spirit of this book. Others may enjoy the several tidbits offered in Chapter 3.

Two editions of this book have appeared in India before this *Classroom Resource Materials* edition. I am much obliged to the editors of this series and to colleagues and friends H. Helson, A. I. Singh, S. K. Gupta, S. Serra and R. Horn for their comments and advice. The computer drawings were made by a former student S. Guha. I am thankful to him and to A. Shukla for preparing the electronic files.

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## Notation

## About the Author

**Rajendra Bhatia** received his PhD from the Indian Statistical Institute in New Delhi, India. He has held visiting appointments at various universities across the world—Sapporo, Kuwait, Ljubljana, Pisa, Bielefeld, Lisbon, Toronto, and Berkeley, among others.

He is a member of the Indian Mathematical Society, the MAA, the AMS, and the International Linear Algebra Society (on whose board of directors he has served). He has been on the editorial boards of several mathematics journals such as *Linear Algebra and Its Applications*, *SIAM Journal on Matrix Analysis*, and *Linear and Multilinear Algebra*. He has been a recipient of the Indian National Science Academy Medal for Young Scientists and Shanti Swarup Bhatnagar Prize of the Indian Council for Scientific and Industrial Research. He is a Fellow of the Indian Academy of Sciences.

His previous widely acclaimed books include: *Perturbation Bounds for Matrix Eigenvalues*, and *Matrix Analysis*.

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This is a concise introduction to Fourier series covering history, major themes, theorems, examples and applications. It can be used to learn the subject, and also to supplement, enhance and embellish undergraduate courses on mathematical analysis.The book begins with a brief summary of the rich history of Fourier series over three centuries. The subject is presented in a way that enables the reader to appreciate how a mathematical theory develops in stages from a practical problem (such as conduction of heat) to an abstract theory dealing with concepts such as sets, functions, infinity and convergence. The abstract theory then provides unforeseen applications in diverse areas.The author starts out with a description of the problem that led Fourier to introduce his famous series. The mathematical problems this leads to are then discussed rigorously. Examples, exercises and directions for further reading and research are provided, along with a chapter that provides materials at a more advanced level suitable for graduate students. The author demonstrates applications of the theory to a broad range of problems.The exercises of varying levels of difficulty that are scattered throughout the book will help readers test their understanding of the material.

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