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Contents

Preface	ix
0 A History of Fourier Series	1
1 Heat Conduction and Fourier Series	13
1.1 The Laplace equation in two dimensions	13
1.2 Solutions of the Laplace equation	15
1.3 The complete solution of the Laplace equation	19
2 Convergence of Fourier Series	27
2.1 Abel summability and Cesàro summability	28
2.2 The Dirichlet and the Fejér kernels	29
2.3 Pointwise convergence of Fourier series	35
2.4 Term by term integration and differentiation	44
2.5 Divergence of Fourier series	46
3 Odds and Ends	51
3.1 Sine and cosine series	51
3.2 Functions with arbitrary periods	53
3.3 Some simple examples	55
3.4 Infinite products	61
3.5 π and infinite series	63

3.6	Bernoulli numbers	65
3.7	$\sin x/x$	67
3.8	The Gibbs phenomenon	70
3.9	Exercises	72
3.10	A historical digression	75
4	Convergence in L_2 and L_1	79
4.1	L_2 convergence of Fourier series	79
4.2	Fourier coefficients of L_1 functions	85
5	Some Applications	95
5.1	An ergodic theorem and number theory	95
5.2	The isoperimetric problem	98
5.3	The vibrating string	101
5.4	Band matrices	105
A	A Note on Normalisation	111
B	A Brief Bibliography	113
	Index	117

Preface

These notes provide a quick and brief introduction to Fourier Series. The emphasis is not only on the mathematics but also on the history of the subject, its importance, its applications and its place in the rest of science.

I first learnt about Fourier Series as a student of physics. Together with several other assorted topics, they formed a ragbag course called Mathematical Physics from which, when the time came, *real* physics courses would pick what they wanted. A little later, as a student of mathematics I came across Fourier Series in the middle of a course on Mathematical Analysis. On each occasion, my teachers and my books (all good) managed to keep a secret which I learnt later. Fourier Series are not just tools for the physicist and examples for the mathematician. They are directly responsible for the development of nearly one half of mathematical analysis over the last two centuries.

These notes have been consciously designed to reveal this aspect of the subject and something more. The development of Fourier Series is illustrative of a recurrent pattern in modern science. I hope the reader will see this pattern emerge from our discussion.

This book can be used by a variety of students. Mathematics students at the third year undergraduate level should be able to follow *most* of the discussion. Typically, such students may have had their first course in Analysis (corresponding to Chapters 1–8 of *Principles of Mathematical Analysis* by W. Rudin) and have a good working knowledge of complex numbers and basic differential equations. Such students can learn about Fourier Series from this book and, at the same time, reinforce their understanding of the analysis topics mentioned above. More preparation is required for reading Chapter 4, a part of Section 2.5, Sections 5.2 and 5.4. These parts presuppose familiarity with Lebesgue spaces and elements of Functional Analysis usually taught in the fourth year of an undergraduate or the first year of a graduate program. At many places in the book the reader will see a statement like “*Let f be a continuous or, more generally, an integrable function.*” Here

there is a choice. If it appears easier to handle continuous functions, the reader need not be worried about discontinuous ones at this stage.

Thus the material in this book can be used either to augment an Analysis course or to serve as the beginning of a special course leading to more advanced topics in Harmonic Analysis. It can also be used for a reading project. Some readers may be happy reading just Chapter 0 outlining the history of the subject; in some sense that captures the spirit of this book. Others may enjoy the several tidbits offered in Chapter 3.

Two editions of this book have appeared in India before this *Classroom Resource Materials* edition. I am much obliged to the editors of this series and to colleagues and friends H. Helson, A. I. Singh, S. K. Gupta, S. Serra and R. Horn for their comments and advice. The computer drawings were made by a former student S. Guha. I am thankful to him and to A. Shukla for preparing the electronic files.

Index

- Abel limit, 28
- Abel summability, 28
- algebra, 89
- approximate identity, 21

- Banach algebra, 89
- Banach space, 48
- band matrices, 105
- Basel problem, 75
- Bernoulli numbers, 65
- Bessel's inequality, 81
- boundary value problem, 15

- Cauchy–Schwarz inequality, 81
- Cesàro summability, 28
- convolution, 20, 85
 - and Fourier coefficients, 89
 - and smoothness, 44, 90
 - lack of an identity for, 90
- cotangent, 62

- Dido's problem, 100
- Dirac family, 22
- Dirac sequence, 21
- Dirichlet kernel, 29
- Dirichlet problem, 15
 - solution, 23
 - uniqueness of solution, 24
- Dirichlet's theorem, 43

- ergodic principle, 95
- Euler constant, 33
- exponential polynomials, 33

- Fejér kernel, 31
- Fejér's theorem, 31
- Fourier coefficients
 - of L_1 functions, 85
 - of L_2 functions, 81
 - uniqueness of, 24
 - with respect to an orthonormal system, 81
 - convolution, 88
 - order of magnitude of, 40
 - rate of decay, 91
 - relation with smoothness, 43
- Fourier series, 19, 28, 31
 - L_2 convergence of, 79
 - and closest approximation, 82
 - examples, 55
 - cosine series, 51, 52
 - definition, 19
 - divergence of, 46
 - exponential form, 52
 - pointwise convergence of, 35
 - termwise integration, 45
 - trigonometric form, 52
- Fubini's Theorem, 86
- function
 - on T , 19

- function (*continued*)
 - absolutely continuous, 44
 - bounded variation, 41
 - even and odd, 51
 - integrable, 26
 - Lipschitz continuous, 38
 - periodic, 19
 - piecewise C^1 , 36
- Gibbs phenomenon, 70
- harmonic function, 19
- Hausdorff moment theorem, 53
- Heaviside function, 57
- Hilbert space, 79
- infinite products, 61
- integrable function, 25
- isoperimetric problem, 98
- Jordan's theorem, 43
- Korovkin's theorem, 74
- Laplace equation
 - solution of, 19
- Lebesgue constants, 33
- Legendre polynomials, 83
- Newton's law of cooling, 13
- Newton's Second Law, 101
- norm, 105
- Parseval's relations, 83
- permutation matrix, 106
- pinching, 107
- Plancherel's theorem, 82
- Poisson integral, 20
- Poisson kernel, 20
- Poisson's theorem, 23
- positive definite sequence, 73
- positive operator, 74
- principle of localisation, 38
- pulse function, 55
- rapidly decreasing, 84
- Riemann–Lebesgue Lemma, 36, 88
- Riesz–Fischer theorem, 83
- saw-tooth curve, 57
- separation method, 17
- steady flow of heat, 13
- Tchebychev polynomial, 53
- temperature
 - maximum and minimum, 25
 - mean value property, 25
- Tonelli's theorem, 86
- triangular truncation operator, 109
- triangular wave, 57
- uniform boundedness principle, 48
- unitary matrix, 105
- vibrating string, 101
- Wallis formula, 65
- wave equation, 101
- Weierstrass approximation theorem, 25, 53
- Weyl's equidistribution theorem, 97
- Weyl's theorem, 96
- Wirtinger's inequality, 85
- zeta function, 67

Notation C^1 , 25 C^2 , 13 D , 15 $D_N(t)$, 29 $F_n(t)$, 31 $L_1(I)$, 80 $L_2(I)$, 80 L_n , 33 $O(\frac{1}{n})$, 40 $P(r, \theta)$, 20 P_r , 23 $S_N(f; \theta)$, 27 T , 19 $\text{Si}(x)$, 67 σ_n , 28 $\zeta(s)$, 67 $f * g$, 20 l_2 , 79 $o(\frac{1}{n})$, 44 s_N , 28 $\hat{f}(n)$, 19

About the Author

Rajendra Bhatia received his PhD from the Indian Statistical Institute in New Delhi, India. He has held visiting appointments at various universities across the world—Sapporo, Kuwait, Ljubljana, Pisa, Bielefeld, Lisbon, Toronto, and Berkeley, among others.

He is a member of the Indian Mathematical Society, the MAA, the AMS, and the International Linear Algebra Society (on whose board of directors he has served). He has been on the editorial boards of several mathematics journals such as *Linear Algebra and Its Applications*, *SIAM Journal on Matrix Analysis*, and *Linear and Multilinear Algebra*. He has been a recipient of the Indian National Science Academy Medal for Young Scientists and Shanti Swarup Bhatnagar Prize of the Indian Council for Scientific and Industrial Research. He is a Fellow of the Indian Academy of Sciences.

His previous widely acclaimed books include: *Perturbation Bounds for Matrix Eigenvalues*, and *Matrix Analysis*.

This is a concise introduction to Fourier series covering history, major themes, theorems, examples and applications. It can be used to learn the subject, and also to supplement, enhance and embellish undergraduate courses on mathematical analysis. The book begins with a brief summary of the rich history of Fourier series over three centuries. The subject is presented in a way that enables the reader to appreciate how a mathematical theory develops in stages from a practical problem (such as conduction of heat) to an abstract theory dealing with concepts such as sets, functions, infinity and convergence. The abstract theory then provides unforeseen applications in diverse areas. The author starts out with a description of the problem that led Fourier to introduce his famous series. The mathematical problems this leads to are then discussed rigorously. Examples, exercises and directions for further reading and research are provided, along with a chapter that provides materials at a more advanced level suitable for graduate students. The author demonstrates applications of the theory to a broad range of problems. The exercises of varying levels of difficulty that are scattered throughout the book will help readers test their understanding of the material.