Complex Numbers and Geometry
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Complex Numbers and Geometry

Liang-shin Hahn
To my parents

Shyr-Chyuan Hahn, M.D., Ph.D.
Shiu-Luan Tsung Hahn

And to my wife

Hwei-Shien Lee Hahn, M.D.
Preface

The shortest path between two truths in the real domain passes through the complex domain. — J. Hadamard

This book is the outcome of lectures that I gave to prospective high-school teachers at the University of New Mexico during the Spring semester of 1991. I believe that while the axiomatic approach is very important, too much emphasis on it in a beginning course in geometry turns off students’ interest in this subject, and the chance for them to appreciate the beauty and excitement of geometry may be forever lost. In our high schools the complex numbers are introduced in order to solve quadratic equations, and then no more is said about them. Students are left with the impression that complex numbers are artificial and not really useful and that they were invented for the sole purpose of being able to claim that we can solve every quadratic equation. In reality, the study of complex numbers is an ideal subject for prospective high-school teachers or students to pursue in depth. The study of complex numbers gives students a chance to review number systems, vectors, trigonometry, geometry, and many other topics that are discussed in high school, not to mention an introduction to a unified view of elementary functions that one encounters in calculus.

Unfortunately, complex numbers and geometry are almost totally neglected in our high-school mathematics curriculum. The purpose
of the book is to demonstrate that these two subjects can be blended together beautifully, resulting in easy proofs and natural generalizations of many theorems in plane geometry—such as the Napoleon theorem, the Simson theorem, and the Morley theorem. In fact, one of my students told me that she can not imagine that anyone who fails to become excited about the material in this book could ever become interested in mathematics.

The book is self-contained—no background in complex numbers is assumed—and can be covered at a leisurely pace in a one-semester course. Chapters 2 and 3 can be read independently. There are over 100 exercises, ranging from muscle exercises to brain exercises and readers are strongly urged to try at least half of these exercises. All the elementary geometry one needs to read this book can be found in Appendix A. The most sophisticated tools used in the book are the addition formulas for the sine and cosine functions and determinants of order 3. On several occasions matrices are mentioned, but these are supplementary in nature and those readers who are unfamiliar with matrices may safely skip these paragraphs. It is my belief that the book can be used profitably by high-school students as enrichment reading.

It is my pleasure to express heartfelt appreciation to my colleagues and friends, Professors Jeff Davis, Bernard Epstein, Reuben Hersh, Frank Kelly, and Ms. Moira Robertson, all of whom helped me with my awkward English on numerous occasions. (English is not my mother tongue.) Also, I want to express gratitude to my three sons, Shin-Yi, Shin-Jen and Shin-Hong, who read the entire manuscript in spite of their own very heavy schedules, corrected my English grammar, and made comments from quite different perspectives, which resulted in considerable improvement. Furthermore, I want to thank Ms. Linda Cicarella and Ms. Gloria Lopez, who helped me with \LaTeX, which is used to type the manuscript. Linda also prepared the index of the book. Last but not least, I am deeply grateful to Professor Roger Horn, the chair of the Spectrum Editorial Board, for his patience in correcting my English, and for his very efficient handling of my manuscript.

L.-s. H.
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APPENDIX A

Preliminaries in Geometry

A.1 Centers of a Triangle

A.1.1 The Centroid.

Lemma A.1.1. Let $D, E$ be the midpoints of the sides $AB, AC$ of $\triangle ABC$. Then

$$DE \parallel BC \quad \text{and} \quad \overline{DE} = \frac{1}{2} \overline{BC}.$$  

Proof. Extend $DE$ to $F$ so that $\overline{DE} = \overline{EF}$. Then in $\triangle ADE$ and $\triangle CFE$,

$$AE = CE, \quad DE = FE, \quad \angle AED = \angle CEF;$$

$$\therefore \triangle ADE \cong \triangle CFE.$$  

It follows that

$$CF = AD = DB,$$

and

$$\angle CFE = \angle ADE. \quad \therefore CF \parallel BD.$$
Thus the quadrangle $BCFD$ is a parallelogram.

$$
\therefore \frac{DE}{DF} = \frac{1}{2} \frac{DF}{BC} = \frac{1}{2} \frac{BC}{BC}, \quad \text{and} \quad DE \parallel BC.
$$

Actually, this lemma is a particular case of the following theorem.

**Theorem A.1.2.** Suppose $D, E$ are points on the sides $AB, AC$ of $\triangle ABC$ such that $DE \parallel BC$. Then

$$
\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}.
$$

The converse is also true.

**Proof.** $\triangle ADE \sim \triangle ABC$. 

**Theorem A.1.3.** The three medians of a triangle meet at a point. This point is called the centroid of the triangle.

**Proof.** Let $G$ be the intersection of the medians $BD$ and $CE$ of $\triangle ABC$. Extend $AG$ to $F$ so that $GF = AG$. Then in $\triangle ABF$, $E$ and $G$ are the midpoints of sides $AB$ and $AF$, respectively. Hence, by the previous lemma,

$$
BF \parallel EG \parallel GC.
$$
Similarly, \( CF \parallel GB \). Therefore, the quadrangle \( BFCG \) is a parallelogram, and so its two diagonals bisect each other, say at \( M \). We have shown that the extension of \( AG \) passes through the midpoint \( M \) of the side \( BC \); i.e., the three medians of a triangle intersect at a point. \( \square \)

Note that from our proof,

\[
AG = GF = 2GM, \quad CG = FB = 2GE,
\]

and similarly,

\[
BG = 2GD.
\]

A.1.2 The Circumcenter.

**Lemma A.1.4.** Suppose \( A \) and \( B \) are two fixed points. Then a point \( P \) is on the perpendicular bisector of the line segment \( AB \) if and only if \( PA = PB \).

**Proof.** Suppose \( P \) is on the perpendicular bisector of the line segment \( AB \). Join the point \( P \) and the midpoint \( M \) of \( AB \). Then

\[
\triangle PAM \cong \triangle PBM \quad \text{(by SAS)},
\]
and so $\overline{PA} = \overline{PB}$.

Conversely, if $\overline{PA} = \overline{PB}$, then

$$\triangle PAM \cong \triangle PBM \quad \text{(by SSS)},$$

where $M$ is the midpoint of the line segment $AB$. Therefore,

$$\angle AMP = \angle BMP = \frac{\pi}{2}.$$
Note that since the distances from the point $O$ to the three vertices are equal, if we draw a circle with $O$ as the center and $OA$ as its radius, we obtain a circumcircle of $\triangle ABC$.

### A.1.3 The Orthocenter.

**Theorem A.1.6.** The three perpendiculars from the vertices to the opposite sides of a triangle meet at a point. This point is called the **orthocenter** of the triangle.

**Proof.** Through each vertex of $\triangle ABC$, draw a line parallel to the opposite side, obtaining $\triangle A'B'C'$. Then the quadrangles $ABCB'$ and $ACBC'$ are parallelograms, and so $AB' = BC = C'A$. Since $B'C' \parallel BC$, the perpendicular from the vertex $A$ to the side $BC$ is the perpendicular bisector of the line segment $B'C'$. In other words, the three perpendiculars from the vertices of $\triangle ABC$ to the opposite sides are the perpendicular bisectors of the three sides of $\triangle A'B'C'$. Hence, by the previous theorem, these three lines meet at a point. $\square$

**Figure A.4**

*Alternate Proof.* Let $P$, $Q$, $R$ be the feet of the perpendiculars from the vertices $A$, $B$, $C$ to the respective opposite sides $BC$, $CA$, $AB$ of $\triangle ABC$. Observe that

$$\angle BQC = \angle BRC \left(= \frac{\pi}{2}\right),$$

and so, by Corollary A.2.2 below, $B$, $R$, $Q$, $C$ are cocyclic. Similarly, $C$, $P$, $R$, $A$ are cocyclic, so are $A$, $Q$, $P$, $B$. Therefore, by Lemma A.2.1
below,
\[ \angle APQ = \angle ABQ = \angle QCR = \angle APR. \]

Similarly,
\[ \angle BQR = \angle BQP, \quad \angle CRP = \angle CRQ. \]

We have shown that the three perpendiculars of \( \triangle ABC \) are the three angle bisectors of the pedal triangle \( PQR \). Hence, they meet at the incenter of \( \triangle PQR \), by Theorem A.1.8 below. \( \square \)

### A.1.4 The Incenter and the Three Excenters.

**Lemma A.1.7.** Let \( P \) be a point inside \( \angle BAC \). Then \( P \) is on the bisector of \( \angle BAC \) if and only if the distances from the point \( P \) to the sides \( AB \) and \( AC \) are equal.

**Proof.** Let \( P \) be an arbitrary point on the bisector of \( \angle BAC \), and \( D, E \) the feet of the perpendiculars from \( P \) to \( AB \) and \( AC \), respectively. Then in \( \triangle APD \) and \( \triangle APE \), two pairs of corresponding angles are equal and so these two triangles are similar. Moreover, they have a corresponding side \( AP \) in common, hence

\[ \triangle APD \cong \triangle APE. \quad \therefore \overline{PD} = \overline{PE}. \]

Conversely, suppose \( P \) is a point inside \( \angle BAC \) such that \( \overline{PD} = \overline{PE} \), where \( D \) and \( E \) are the feet of the perpendiculars from the point \( P \) to \( AB \) and \( AC \), respectively. Then, by the Pythagorean theorem, three pairs of corresponding sides of \( \triangle APD \) and \( \triangle APE \) are equal, and so

\[ \triangle APD \cong \triangle APE. \quad \therefore \angle PAD = \angle PAE. \]

\( \square \)

**Theorem A.1.8.** The three bisectors of the (interior) angles of a triangle meet at a point. This point is called the incenter of the triangle.

**Proof.** Let \( I \) be the intersection of the bisectors of the angles at the vertices \( B \) and \( C \) of \( \triangle ABC \), and \( D, E, F \) the feet of the perpendiculars from \( I \) to the three sides \( BC, CA, AB \), respectively. Then, since \( I \) is on
the bisector of $\angle ABC$, by the first part of the lemma, we have $\overline{TI} = \overline{TD}$. Similarly, since $I$ is also on the bisector of $\angle ACB$, we have $\overline{TD} = \overline{TE}$. Therefore, $\overline{TE} = \overline{TI}$. But then, by the second part of the lemma, $I$ must be on the bisector of $\angle BAC$.

Since the distances from the incenter $I$ to the three sides of a triangle are all equal, if we draw a circle with center at $I$ and use the distance from $I$ to a side as the radius, we obtain the circle tangent to all three sides of the triangle. This circle is called the incircle of the triangle.

**Theorem A.1.9.** The bisectors of two exterior angles and that of the remaining interior angle of a triangle meet at a point. This point is called an excenter of the triangle, and is the center of an excircle that is tangent to extensions of two sides and the remaining side of a triangle. A triangle has three excircles and three excircles. (See Figure A.6.)

**Proof.** The proof is essentially the same as that for the incenter (and the incircle).

**A.1.5 Theorems of Ceva and Menelaus.** Each centroid, orthocenter, incenter, and excenter is the intersection of three lines passing through the vertices of a triangle. For this type of problem, the following theorem of G. Ceva (1647–1734) is very effective.

**Theorem A.1.10.** Let $P$, $Q$, $R$ be points on (the extensions of) the respective sides $BC$, $CA$, $AB$ of $\triangle ABC$. Then the lines $AP$, $BQ$, $CR$ meet at
a point if and only if

\[
\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = 1.
\]
For example, to prove that the three medians of a triangle meet at a point, using previous notation, we have

\[
\frac{BM}{MC} = \frac{CD}{DA} = \frac{AE}{EB} = 1,
\]

and so the condition in the Ceva theorem is clearly satisfied.

In the case of the three perpendiculars, let

\[
a = \overline{BC}, \quad b = \overline{CA}, \quad c = \overline{AB}, \quad \alpha = \angle A, \quad \beta = \angle B, \quad \gamma = \angle C,
\]

and let \(P, Q, R\) be the feet of the perpendiculars from the vertices \(A, B, C\) to the respective opposite sides, then \(\overline{BP} = c\cos\beta\), etc., and so

\[
\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RA} = \frac{c\cos\beta}{b\cos\gamma} \cdot \frac{a\cos\gamma}{c\cos\alpha} \cdot \frac{b\cos\alpha}{a\cos\beta} = 1,
\]

and we are done.

To prove that the three angle bisectors meet at a point, let \(U, V, W\) be the intersections of the angle bisectors at \(A, B, C\) and the respective opposite sides. Then, by Lemma A.4.1 to the Apollonius circle below, we have \(\overline{BU}/\overline{UC} = b/c\), etc.,

\[
\frac{BU}{UC} \cdot \frac{CV}{VA} \cdot \frac{AW}{WB} = \frac{b}{c} \cdot \frac{c}{a} \cdot \frac{a}{b} = 1.
\]

The case of an excenter is essentially the same as that of the incenter, and so is left for the reader.

It remains to prove the Ceva theorem itself. Suppose \(AP, BQ, CR\) meet at a point, say \(T\). Draw the line passing through the point \(A\) parallel to the side \(BC\) meeting (the extensions of) \(BQ, CR\) at \(B', C'\), respectively. Since

\[
\triangle BPT \sim \triangle B'AT, \quad \triangle CPT \sim \triangle C'AT,
\]
we have
\[ \frac{BP}{PT} = \frac{B'A}{AT}, \quad \frac{PT}{PC} = \frac{AT}{AC'}, \quad \therefore \frac{BP}{PC} = \frac{B'A}{AC'}. \]

Similarly,
\[ \frac{CQ}{QA} = \frac{CB}{AB'}, \quad \text{and} \quad \frac{AR}{RB} = \frac{AC'}{BC}. \]

Hence multiplying the last three equalities together, we get the desired equality.

To prove the converse, let \( T \) be the intersection of (the extensions of) \( BQ \) and \( CR \), and \( P' \) the intersection of (the extensions of) \( AT \) and \( BC \). Then, by what we have shown,
\[ \frac{BP'}{P'C} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = 1. \]

On the other hand, by assumption, we also have
\[ \frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = 1. \quad \therefore \quad \frac{BP'}{P'C} = \frac{BP}{PC}. \]

Now adding 1 to both sides, we get
\[ \frac{BP' + P'C}{P'C} = \frac{BP + PC}{PC}. \quad \therefore \quad \frac{BC}{P'C} = \frac{BC}{PC}. \]

It follows that \( P'C = PC \), and so \( P' = P \).

(We have deliberately suppressed an accompanying configuration, giving a reader a chance to check that the proof works for all cases.)

As the careful reader will notice, the converse holds if and only if the line segments are considered as directed: Namely, \( \frac{BP}{PC} > 0 \) if \( \overrightarrow{BP} \) and \( \overrightarrow{PC} \) are in the same direction, and \( \frac{BP}{PC} < 0 \) if \( \overrightarrow{BP} \) and \( \overrightarrow{PC} \) are in the opposite direction. Similar considerations naturally apply to the other ratios.

The following theorem, closely associated with that of Ceva, was discovered by Menelaus of Alexandria (ca. 98):
**Theorem A.1.11.** Points $P$, $Q$, $R$ on (the extensions) of the respective sides $BC$, $CA$, $AB$ of $\triangle ABC$ are collinear if and only if

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = -1.$$

**Proof.** Since we shall not need this theorem, we merely sketch a proof, and leave the details for the reader. To prove that the condition is necessary, draw a line passing through the vertex $A$ parallel to the line determined by the points $P$, $Q$, $R$, meeting (the extension of) the side $BC$ at $A'$. Now express all the ratios involved in terms of those of the segments on the line $BC$. To prove sufficiency, imitate the proof of the Ceva theorem. \qed

**A.2 Angles Subtended by an Arc**

**Lemma A.2.1.** An angle subtended by an arc is equal to one half of its central angle. In particular, all the angles subtended by the same arc are equal.

**Proof.** Let $A$, $B$, $C$ be points on a circle $O$.

Case 1. Suppose the center $O$ is either on the segment $AC$ or on $BC$. To fix our notation, let the center $O$ be on the chord $AC$. Then,
since $\triangle OBC$ is an isosceles triangle, we have $\angle OCB = \angle OBC$. But $\angle AOB$ is an exterior angle of $\triangle OBC$,

\[
\therefore \angle AOB = \angle OBC + \angle OCB = 2\angle ACB.
\]

Case 2. Suppose the center $O$ is inside $\angle ACB$. Let $CD$ be a diameter. Then from Case 1, we have

\[
\angle ACB = \angle ACD + \angle DCB = \frac{1}{2}(\angle AOD + \angle DOB) = \frac{1}{2}\angle AOB.
\]

Case 3. Suppose the center $O$ is outside $\angle ACB$. As before, let $CD$ be a diameter. Then, again from Case 1, we have

\[
\angle ACB = \angle BCD - \angle ACD = \frac{1}{2}(\angle BOD - \angle AOD) = \frac{1}{2}\angle AOB.
\]

\[\square\]

**Theorem A.2.2.** Suppose points $C$ and $D$ are on the same side of a line $AB$. Then the points $A$, $B$, $C$, $D$ are cocyclic if and only if $\angle ACB = \angle ADB$. 

Proof. It remains to prove the converse. Draw the circle passing through the points $A$, $B$, and $C$. Suppose the point $D$ is inside this circle. Let $D'$ be the intersection of the circle and the extension of $AD$, then

$$\angle ADB = \angle AD'B + \angle DBD' > \angle AD'B = \angle ACB.$$ 

Now if the point $D$ is outside of this circle, let $D'$ be the intersection of the circle and $AD$. Then

$$\angle ADB < \angle ADB + \angle DBD' = \angle AD'B = \angle ACB.$$ 

Hence, if $\angle ADB = \angle ACB$, then the point $D$ must be on the circle passing through the points $A$, $B$, $C$ (and in this case, the equality clearly holds, by the previous lemma). \qed
Corollary A.2.3. Suppose $C$ and $D$ are on opposite sides of a line $AB$. Then the points $A, B, C, D$ are cocyclic if and only if

$$\angle ACB + \angle ADB = \pi.$$

Corollary A.2.4. The angle between a tangent to a circle and a chord is equal to angles subtended by the arc inside this angle.

Proof. Pictures are worth a thousand words. □

A.3 The Napoleon Theorem

Though the Napoleon theorem is not part of our needed background, the following is an elegant simplification by Kay Hashimoto (a 10th grader at Lakeside School, Seattle) in May 1992, of the proof of Ross Honsberger [Mathematical Gems, Mathematical Association of America, Washington, D.C., 1973, pp. 34–36].

Theorem A.3.1. On each side of an arbitrary triangle, draw an exterior equilateral triangle. Then the centroids of these three equilateral triangles are the vertices of a fourth equilateral triangle.

Proof. Given $\triangle ABC$, let $X, Y, Z$ be the centers of the circumcircles of the exterior equilateral triangles on the sides $BC, CA, AB$, respectively, and $O$ the intersection of the circles $Y$ and $Z$ (other than $A$). Then, by Corollary A.2.3 in the previous section, we have $\angle AOB = \frac{2\pi}{3} = \pi$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{napoleon_triangle}
\caption{Figure A.11}
\end{figure}
Proof Let the interior and exterior angle bisectors at the vertex \(A\) intersect the side \(BC\) of \(\triangle ABC\) at \(D\) and at \(E\), respectively. Choose Figure A.12.

\[ \angle AOC. \text{ Therefore, } \angle BOC = \frac{2\pi}{3}. \] It follows that the circle \(X\) also passes through the point \(O\) (again, by Corollary A.2.3). We have shown that the three circumcircles meet at the point \(O\).

Now, \(XY\), the line joining the two centers, is perpendicular to the common chord \(OC\). Similarly, \(XZ\) is perpendicular to \(OB\). But \[ \angle BOC = \frac{2\pi}{3}, \] and so \[ \angle X = \frac{\pi}{3}. \] Similarly, \[ \angle Y = \frac{\pi}{3} = \angle Z, \] and we are done.

\[ \square \]

A.4 The Apollonius Circle

Lemma A.4.1. The interior and the exterior bisectors of an angle at a vertex of a triangle divide the opposite side into the ratio of the lengths of the two remaining sides.

Proof. Let the interior and exterior angle bisectors at the vertex \(A\) intersect the side \(BC\) of \(\triangle ABC\) at \(D\) and at \(E\), respectively. Choose
the point $F$ on the extension of the side $AB$ such that $CF \parallel AD$. Then

$$\angle AFC = \angle BAD = \angle DAC = \angle ACF.$$  

Therefore, $\triangle ACF$ is an isosceles triangle. It follows that

$$BD : DC = BA : AF = BA : AC.$$  

Similarly, choose the point $G$ on $AB$ such that $CG \parallel AE$. Then

$$\angle AGC = \angle FAE = \angle EAC = \angle ACG.$$  

Therefore, $\triangle ACG$ is an isosceles triangle. It follows that

$$BE : EC = BA : AG = BA : AC.$$  

$\square$

Alternate Proof. We use the same notation as in the above proof. Since $D$ is on the bisector of $\angle ABC$, the perpendiculars from $D$ to $AB$ and $AC$ have the same length (by Lemma A.1.7). Therefore, the ratio of the areas of $\triangle ABD$ and $\triangle ACD$ is $AB : AC$. On the other hand, these two triangles have common height from the vertex $A$. Therefore, the ratio
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of the areas of these two triangles is also equal to $\frac{BD}{CD}$.

\[ \therefore \frac{BD}{CD} = \frac{AB}{AC}. \]

As for the exterior angle bisector $AE$ at the vertex $A$, consider $\triangle ABE$ and $\triangle ACE$, and carry out the same argument. \hfill \square

The converse of the lemma follows from the uniqueness of the point dividing the side of a triangle into a fixed ratio.

**Corollary A.4.2.** Let $D$, $E$ be the points on (the extension of) the side $BC$ of $\triangle ABC$ such that

\[ \frac{BD}{DC} = \frac{AB}{AC} = \frac{BE}{EC}. \]

Then $AD$ and $AE$ are the bisectors of the interior and exterior angles at the vertex $A$.

**Theorem A.4.3 (Apollonius).** Consider a pair of points $A$, $B$ and a fixed ratio $m : n$. Suppose $C$ and $D$ are the points on the line $AB$ such that

\[ \frac{CA}{CB} = \frac{DA}{DB} = \frac{m}{n}. \]

Then a point $P$ is on the circle having $CD$ as its diameter if and only if

\[ \frac{PA}{PB} = \frac{m}{n}. \]

**Proof.** Suppose $P$ is a point satisfying the condition

\[ \frac{PA}{PB} = \frac{CA}{CB} \left( = \frac{DA}{DB} \right). \]

Then, by Corollary A.4.2, $PC$, $PD$ are the bisectors of the interior and the exterior angles at the vertex $P$ of $\triangle PAB$. Hence $\angle CPD = \frac{\pi}{2}$, so the point $P$ is on the circle having $CD$ as its diameter.

Conversely, suppose $P$ is an arbitrary point on the circle with $CD$ as its diameter. Choose the points $E$, $F$ on (the extension of) $AP$ such that $BE \parallel CP$, $BF \parallel DP$. Then

\[ \frac{AP}{PE} = \frac{AC}{CB} = \frac{m}{n}, \]
and
\[ \overline{AP} : \overline{PF} = \overline{AD} : \overline{DB} = m : n. \]

Therefore, \( \overline{PE} = \overline{PF} \). Since \( BE \parallel CP \), \( BF \parallel DP \), and \( \angle CPD = \frac{\pi}{2} \), we have \( \angle EBF = \frac{\pi}{2} \). Hence \( P \) is the midpoint of the hypotenuse of the right triangle \( BEF \). It follows that \( \overline{PB} = \overline{PE} \). Therefore, \( \overline{AP} : \overline{PB} = m : n. \) \( \square \)
APPENDIX B
New Year Puzzles

The author has been sending New Year puzzles as season's greetings for the past several years. As the purpose is to popularize mathematics, these puzzles are not intended to be hard (except possibly in 1986). Since these puzzles are gaining popularity among the author's friends, we publish them here hoping readers will do the same.

1985

\[
\begin{align*}
0 &= \ (1 - 9 + 8) \times 5 \\
1 &= \ 1 - \sqrt{9} + 8 - 5 \\
2 &= \ 1 + (-\sqrt{9} + 8)/5 \\
3 &= \ -1 - 9 + 8 + 5 \\
4 &= \ 1 \times (-9 + 8) + 5 \\
5 &= \ 1 - 9 + 8 + 5 \\
6 &= \ 1 \times (9 - 8) + 5 \\
7 &= \ 1 + 9 - 8 + 5 \\
8 &= \ ? \\
9 &= \ \sqrt{-1 + 9 + 8 + 5} \\
10 &= \ (1 + 9 - 8) \times 5
\end{align*}
\]

Can you find a similar expression for 8? (Only additions, subtractions, multiplications, divisions, square roots, and parentheses are permitted. The solution is not unique.)

2. \[\square\square\square\square \times \square = \sqrt{\square} \cdot 19 \square\square 85 \square.\]

3. (a) The square of an integer \(n\) starts from 1985:

\[n^2 = 1985 \ldots\]
Find the smallest such positive integer $n$.

(b) Is there an integer whose square ends with 1985?

\begin{align*}
\text{HAPPY} & \quad - \quad \text{TIGER} \\
& \quad \quad \quad \quad \text{YEAR}
\end{align*}

under the conditions that

1. $TIGER$ being the third in the order of 12 animals (rat, ox, tiger, rabbit, dragon, snake, horse, ram, monkey, cock, dog, boar), the number represented by $TIGER$ divided by 12 gives a remainder 3;

$$TIGER \equiv 3 \pmod{12}; \quad \text{and}$$

2. as there are 10 possible digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 to fill in the 9 letters that appear in this alphametic problem, there is bound to be one digit missing. However, the missing digit turns out to be the remainder if the number represented by $YEAR$ is divided by 12.

Fill in the blanks with digits other than 1, 9, 8, 7 so that the equality becomes valid:

\[
\begin{array}{cccc}
\_ & 1 & \_ & 9 \\
\_ & \_ & \_ & \_ \\
& & & 87
\end{array}
\]

1988

1. 

$$1988 = 12^2 + 20^2 + 38^2 = 8^2 + 30^2 + 32^2$$
\[
8^2 + 18^2 + 40^2 = 4^2 + 26^2 + 36^2 \\
= 4^2 + 6^2 + 44^2 = \underline{2}^2 + \underline{2}^2 + \underline{2}^2;
\]

i.e., find another expression of 1988 as a sum of squares of three positive integers.

2. Show that 1988 cannot be expressed as a sum of squares of two positive integers.

1989

Observe that

\[
1989 = (1 + 2 + 3 + 4 + 5)^2 + (3 + 4 + 5 + 6 + 7 + 8 + 9)^2.
\]

Find 4 consecutive natural numbers \(p, q, r, s\), and 6 consecutive natural numbers \(u, v, w, x, y, z\), such that

\[
1989 = (p + q + r + s)^2 + (u + v + w + x + y + z)^2.
\]

1990

Let

\[
P_n = 2191^n - 803^n + 608^n - 11^n + 7^n - 2^n.
\]

Then

\[
P_1 = 1990, \quad P_2 = 4525260 = 1990 \cdot 2274.
\]

Prove that \(P_n\) is divisible by 1990 for every natural number \(n\).

1991

1. In a magic square, the sum of each row, column and diagonal is the same. For example, Fig 1 is a magic square with the magic sum 34. Fill in the blanks in Fig 2 to make it a magic square.
2. Can an integer with 2 or more digits, and all of whose digits are either 1, 3, 5, 7, or 9 (for example, 1991, 17, 731591375179, 753 are such integers) be the square of an integer?

1992

Choose any five numbers in Fig 1 so that no two of them are in the same row nor the same column, then *add* these five numbers, you will always get 1992. For example,

\[ 199 + 92 + 177 + 979 + 545 = 1992. \]

Fill in nine distinct positive integers into Fig 2 such that if you choose any three numbers, no two of them are in the same row, nor the same column, and *multiply* them together, then you will always get 1992. How many essentially different solutions can you find? [Two solutions are considered to be the same if one can be obtained from other by some or all of the following: (a) rotations, (b) reflections, (c) rearrangement of the order of the rows, (d) rearrangement of the order of the columns.]

1993

Let

\[ Q_n = 12^n + 43^n + 1950^n + 1981^n. \]
Then

\[ Q_1 = 12 + 43 + 1950 + 1981 = 1993 \cdot 2, \]
\[ Q_2 = 144 + 1849 + 3802500 + 3924361 = 7728584 = 1993 \cdot 3878, \]
\[ Q_3 = 1728 + 79507 + 741487500 + 7774159141 = 1518915376 = 1993 \cdot 7621232. \]

Determine all the positive integers \( n \) for which \( Q_n \) are divisible by 1993.

1994

We have a sequence of numbers which are reciprocals of the squares of integers 19 through 94:

\[ \frac{1}{19^2}, \frac{1}{20^2}, \frac{1}{21^2}, \ldots, \frac{1}{93^2}, \frac{1}{94^2}. \]

Suppose any pair, \( a \) and \( b \), of these numbers may be replaced by \( a + b - ab \). For example, two numbers \( \frac{1}{32^2} \) and \( \frac{1}{66^2} \) may be replaced by a single number \( \frac{163}{135168} \), because

\[ \frac{1}{32^2} + \frac{1}{66^2} - \frac{1}{32^2} \cdot \frac{1}{66^2} = \frac{163}{135168}. \]

Repeat this procedure until only one number is left. Show that the final number is independent of the way and the order the numbers are paired and replaced. What is the final number?
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Liang-shin Hahn was born in Tainan, Taiwan. He received his B.S. from the National Taiwan University and his PhD from Stanford University. After a brief period of teaching at the Johns Hopkins University, he moved to the University of New Mexico where he has been ever since. He has held visiting positions at the University of Washington (Seattle), the National Taiwan University (Taipei), the University of Tokyo, the International Christian University (Tokyo) and Sophia University (Tokyo), giving him the distinction of teaching mathematics in three countries, and in three languages.

As an unabashed admirer of the late Professor George Pólya, the author is fond of telling students: “The trick in teaching mathematics is that I do the easy part and you do the hard part,” because the author’s motto in heuristic teaching is: “Don’t try to teach everything. Teach the basic ideas, then use questions to guide students to explore and discover for themselves.”

The author has posed many interesting problems in The American Mathematical Monthly, and his conjecture on Egyptian fractions is widely cited. He has been solely responsible for composing the New Mexico Mathematics Contest problems since 1990. He is also the co-author, with Bernard Epstein, of Classical Complex Analysis.

He enjoys playing ping-pong, cultivating roses, listening to classical music and solving as well as creating mathematical puzzles.

The purpose of this book is to demonstrate that complex numbers and geometry can be blended together beautifully, resulting in easy proofs and natural generalizations of many theorems in plane geometry such as the theorems of Napoleon, Simson, Cantor and Morley.

Beginning with a construction of complex numbers, readers are taken on a guided tour that includes something for everyone, even veteran professional mathematicians. Yet, the entire book is accessible to students at the high school level.

The book is self-contained—no background in complex numbers is assumed—and it can be covered at a leisurely pace in a one-semester course. Over 100 exercises are included. The book would be suitable as a text for a geometry course, for a problem solving seminar, or as enrichment for students who are interested in mathematics as part of culture.